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The Norm of a Class of Singular Integral Operators from L^∞ onto Bloch-Type Spaces

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Abstract In this paper, we obtain the exact norm of a class of singular integral operators $Q_\alpha, \alpha > 0$, defined by

$$Q_\alpha f(z) = \alpha \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^{\alpha+1}} \mathrm{d}A(w),$$

from $L^\infty(\mathbb{D})$ onto Bloch-type space $\mathcal{B}_\alpha$ over the unit disk $\mathbb{D}$, which is an extension of the Bergman projection P. We also consider the norm for this operator from $C(\overline{\mathbb{D}})$ onto the little Bloch-type space $\mathcal{B}_{\alpha,0}$.

Keywords operator norm; singular integral operator; Bloch-type space

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1. Introduction and main result

Let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. For each $\alpha > 0$, the Bloch-type space $\mathcal{B}_\alpha$ denotes the space of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty, \tag{1.1}$$

where $||f||_{\mathcal{B}_\alpha}$ is a semi-norm, and $\mathcal{B}_\alpha$ is a Banach space [1] with the corresponding norm

$$\|f\|_\alpha = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|. \tag{1.2}$$

In the case for the semi-norm $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$, Perälä [2] determined the norm of the Bergman projection P from $L^\infty(\mathbb{D})$ onto Bloch space $\mathcal{B}$, he obtained that $\|P\|_{\mathcal{B}} = \frac{8}{\pi}$. A generalization of this result in the unit ball $\mathbb{B}_n$ was done by Kalaj and Marković [3]. Later in [4], Perälä completed his earlier result [2] by finding the norm of the Bergman projection with the norm of (1.2), where the author got $\|P\| = 1 + \frac{8}{\pi}$. Recently, Kalaj and Vujadinović extended this result to the case of unit ball $\mathbb{B}_n$, interesting readers can see [5].

This paper is devoted to study the norm of a class of singular integral operator $Q_\alpha : L^\infty(\mathbb{D}) \to \mathcal{B}_\alpha$, $\alpha > 0$, which is another extension of the Bergman projection, defined as

$$Q_\alpha f(z) = \alpha \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^{\alpha+1}} \mathrm{d}A(w), \tag{1.3}$$

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where $\mathrm{d}A$ denotes the normalized Lebesgue measure over the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. This operator was introduced by Zhu [1], and the author had proved that Q_α maps $L^\infty(\mathbb{D})$ boundedly onto $\mathcal{B}_\alpha$.

Now, we state the main result of this paper.

Theorem 1.1 *Let $0<\alpha<\infty$. Then the norm of operator $Q_\alpha : L^\infty(\mathbb{D}) \to \mathcal{B}_\alpha$ is*

$$\|Q_\alpha\| = \alpha + \alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}.$$

If $\alpha = 1$, Q_α is the well-known Bergman projection P, and the norm $P : L^\infty(\mathbb{D}) \to \mathcal{B}$ is $\|P\| = 1 + \frac{8}{\pi}$, which coincides with the main result of Perälä in [4].

2. Preliminaries

Before the proof of Theorem 1.1, we present some lemmas.

Lemma 2.1 ([6]) *Let $s>-1, t\in\mathbb{R}$, and $J_{s,t}(z)$ defined as*

$$J_{s,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-z\bar{w}|^{2+s+t}}\mathrm{d}A(w).$$

Then

(i) If $t<0$, then for all $z\in\mathbb{D}$,

$$\frac{\Gamma(1+s)}{\Gamma(2+s)} \le J_{s,t}(z) \le \frac{\Gamma(1+s)\Gamma(-t)}{\Gamma^2(\frac{2+s-t}{2})}.$$

(ii) If $t>0$, then for all $z\in\mathbb{D}$,

$$\frac{\Gamma(1+s)}{\Gamma(2+s)} \le (1-|z|^2)J_{s,t}(z) \le \frac{\Gamma(1+s)\Gamma(t)}{\Gamma^2(\frac{2+s+t}{2})}.$$

(iii) Finally, $t=0$, then for all $z\in\mathbb{D}$,

$$\frac{\Gamma(1+s)}{\Gamma^2(1+\frac{s}{2})} \le |z|^2\Big(\log\frac{1}{1-|z|^2}\Big)^{-1} J_{s,0}(z) \le \frac{1}{1+s}.$$

In fact, Lemma 2.1 is the case of $n=1$ of [6, Theorem 1.3]. Moreover, the following integral equality is also needed.

Lemma 2.2 *If $\alpha>0$, we have*

$$\int_{\mathbb{D}} \frac{1}{|1-w|^{2-\alpha}}\mathrm{d}A(w) = \frac{\Gamma(\alpha)}{\Gamma^2(\frac{2+\alpha}{2})}.$$

It seems that Lemma 2.2 may be known, but we cannot find any relative reference.

Proof We use the classical notation ${}_2F_1[a,b,c,\lambda]$ to denote hyper geometric function

$${}_2F_1[a,b,c,\lambda] \triangleq \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{\lambda^n}{n!}, \tag{2.1}$$

with $c\neq 0,-1,-2,\ldots$, where $(a)_n = a(a+1)\cdots(a+n-1)$, $n\ge 1$.

This series gives an analytic function for $|\lambda|<1$, called the Gauss hypergeometric function associated to (a,b,c). We refer to [7, Chapter II] for one property which will be used.

$$ {}_2F_1[a,b,c,1^-]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \mathrm{Re}(c-a-b)>0. \tag{2.2}$$

Using Taylor series expansion of $(1-w)^t$, where $t=\frac{\alpha-2}{2}, \alpha\neq 2$, and combining (2.1), (2.2), we obtain

$$\begin{aligned}
\int_{\mathbb{D}}\frac{1}{|1-w|^{2-\alpha}}\mathrm{d}A(w)&=\int_{\mathbb{D}}|1-w|^{\alpha-2}\mathrm{d}A(w)=\int_{\mathbb{D}}(1-w)^t(1-\bar{w})^t\mathrm{d}A(w),\quad t=\frac{\alpha-2}{2}\\
&=\sum_{n=0}^{\infty}\frac{[t(t-1)\cdots(t-n+1)]^2}{(n!)^2}\int_{\mathbb{D}}|w|^{2n}\mathrm{d}A(w)\\
&=\sum_{n=0}^{\infty}\frac{[(n-1-t)(n-2-t)\cdots(-t)]^2}{(n+1)!}\frac{1}{n!}\\
&=\sum_{n=0}^{\infty}\frac{(-t)_n(-t)_n}{(2)_n}\frac{1^n}{n!}={}_2F_1[\frac{2-\alpha}{2},\frac{2-\alpha}{2},2,1^-]\\
&=\frac{\Gamma(2)\Gamma(2-\frac{2-\alpha}{2}-\frac{2-\alpha}{2})}{\Gamma(2-\frac{2-\alpha}{2})\Gamma(2-\frac{2-\alpha}{2})}=\frac{\Gamma(\alpha)}{\Gamma^2(\frac{2+\alpha}{2})}.
\end{aligned}$$

The above formula is obviously true for $\alpha=2$. $\square$

3. Norm of $Q_\alpha: L^\infty(\mathbb{D})\to\mathcal{B}_\alpha$

This section is devoted to the proof of Theorem 1.1, based on the Section 2, we can now prove the main result of this paper.

Proof of Theorem 1.1 The proof is divided into three steps.

Step 1. We will prove that $\|Q_\alpha\|\le\alpha+\alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}$.

For any $g(z)\in L^\infty(\mathbb{D})$, we have

$$f(z)=Q_\alpha g(z)=\alpha\int_{\mathbb{D}}\frac{g(w)}{(1-z\bar{w})^{\alpha+1}}\mathrm{d}A(w).$$

Differentiating on the two sides of the above equality, we get

$$f'(z)=\alpha(\alpha+1)\int_{\mathbb{D}}\frac{\bar{w}g(w)}{(1-z\bar{w})^{\alpha+2}}\mathrm{d}A(w),$$

thus, Lemma 2.1 implies

$$\begin{aligned}
|f'(z)|&\le\alpha(\alpha+1)||g||_\infty\int_{\mathbb{D}}\frac{1}{|1-z\bar{w}|^{\alpha+2}}\mathrm{d}A(w)\\
&\le\alpha(\alpha+1)||g||_\infty\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}(1-|z|^2)^{-\alpha}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|Q_\alpha g\|_\alpha=\|f\|_\alpha&=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|\\
&\le\big(\alpha+\alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}\big)\|g\|_\infty,
\end{aligned}$$

it follows that

$$\|Q_\alpha\| \le \alpha + \alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}. \tag{3.1}$$

In the next two steps, we will give function sequences to show that the equality can be achieved. In detail, in Step 2, we construct a function $g_z \in L^\infty(\mathbb{D})$ for any fixed $z \in \mathbb{D}$, such that $\|Q_\alpha g_z\|_{\mathcal{B}_\alpha} \to \alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}\|g_z\|_\infty$ as $|z| \to 1^-$. Based on Step 2, we will give test functions $g_z^r \in L^\infty(\mathbb{D})$, such that $\|Q_\alpha\| \ge \alpha + \alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}$.

Step 2. For any $z \in \mathbb{D}$, let

$$g_z(w) = \frac{w|1-z\bar{w}|^{2+\alpha}}{|w|(1-\bar{z}w)^{2+\alpha}}. \tag{3.2}$$

Then $\|g_z\|_\infty = 1$, and

$$Q_\alpha g_z(w) = \alpha\int_{\mathbb{D}} \frac{g_z(u)}{(1-w\bar{u})^{\alpha+1}}\mathrm{d}A(u) = \alpha\int_{\mathbb{D}} \frac{u|1-z\bar{u}|^{2+\alpha}}{|u|(1-\bar{z}u)^{2+\alpha}(1-w\bar{u})^{\alpha+1}}\mathrm{d}A(u).$$

Taking the derivative of this on the two sides with respect to w, we obtain

$$\begin{aligned}(Q_\alpha g_z)'(w) &= \alpha(\alpha+1)\int_{\mathbb{D}} \frac{u|1-z\bar{u}|^{2+\alpha}\bar{u}}{|u|(1-\bar{z}u)^{2+\alpha}(1-w\bar{u})^{\alpha+2}}\mathrm{d}A(u)\\ &= \alpha(\alpha+1)\int_{\mathbb{D}} \frac{|u||1-z\bar{u}|^{2+\alpha}}{(1-\bar{z}u)^{2+\alpha}(1-w\bar{u})^{\alpha+2}}\mathrm{d}A(u).\end{aligned}$$

Then

$$\begin{aligned}(1-|z|^2)^\alpha|(Q_\alpha g_z)'(z)| =&\alpha(\alpha+1)\int_{\mathbb{D}} \frac{|u|(1-|z|^2)^\alpha}{|1-z\bar{u}|^{\alpha+2}}\mathrm{d}A(u)\\ =&\alpha(\alpha+1)\int_{\mathbb{D}} \frac{|\varphi_z(w)|(1-|z|^2)^\alpha}{|1-z\overline{\varphi_z(w)}|^{\alpha+2}}J_z\mathrm{d}A(w) \quad (3.3)\\ =&\alpha(\alpha+1)\int_{\mathbb{D}} \frac{|z-w|}{|1-z\bar{w}|^{3-\alpha}}\mathrm{d}A(w), \quad (3.4)\end{aligned}$$

where $J_z = \frac{(1-|z|^2)^2}{|1-z\bar{w}|^4}$ in (3.3) is the Jacobi of the mobius transformation. Let

$$F(z) = \int_{\mathbb{D}} \frac{|z-w|}{|1-z\bar{w}|^{3-\alpha}}\mathrm{d}A(w).$$

Claim The function $F(z)$ is radial, i.e., $F(z) = F(|z|)$ for every $z \in \mathbb{D}$.

In fact, suppose $z = re^{i\theta}$, and make a change of variables $w \to e^{i\theta}w$, then

$$\begin{aligned}F(z) &= \int_{\mathbb{D}} \frac{|z-w|}{|1-z\bar{w}|^{3-\alpha}}\mathrm{d}A(w) = \int_{\mathbb{D}} \frac{|re^{i\theta}-we^{i\theta}|}{|1-re^{i\theta}\bar{w}e^{-i\theta}|^{3-\alpha}}\mathrm{d}A(w)\\ &= \int_{\mathbb{D}} \frac{|r-w|}{|1-r\bar{w}|^{3-\alpha}}\mathrm{d}A(w) = F(r) = F(|z|),\end{aligned}$$

and so $F(z)$ is radial.

By the claim, we set

$$F(1) = \lim_{r\to 1^-} F(r) = \int_{\mathbb{D}} \frac{|1-w|}{|1-\bar{w}|^{3-\alpha}}\mathrm{d}A(w) = \int_{\mathbb{D}} \frac{1}{|1-w|^{2-\alpha}}\mathrm{d}A(w), \tag{3.5}$$

thus we have extended the function F to a continuous function on $\overline{\mathbb{D}}$, and it is not difficult to prove F is subharmonic, so F obtains its maximum on the boundary by maximum principle.

Based on (3.4), (3.5) and the Claim, Lemma 2.2 implies

$$\begin{aligned}\|Q_\alpha g_z\|_{\mathcal{B}_\alpha} &= \sup_{w\in\mathbb{D}}(1-|w|^2)^\alpha|(Q_\alpha g_z)'(w)| \\ &\geq (1-|z|^2)^\alpha|(Q_\alpha g_z)'(z)| \\ &\to \alpha(\alpha+1)F(1) \\ &= \alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{2+\alpha}{2})}, \quad \text{as} \quad |z|\to 1^-. \end{aligned} \tag{3.6}$$

Step 3. Now, we define new test functions g_z^r with $||g_z^r||_\infty = 1$ as follows:

$$g_z^r(w) = \begin{cases} g_z(w), & \text{if } |w|\geq r; \\ 1, & \text{if } |w|\leq r^2, \end{cases}$$

and define g_z^r on $r^2<|w|<r$ so that g_z^r is continuous on $\overline{\mathbb{D}}$, where $g_z(w)$ is as in (3.2). So

$$\begin{aligned}|Q_\alpha g_z^r(0)| =&\Big|\alpha\int_{\mathbb{D}} g_z^r(w)\mathrm{d}A(w)\Big| \\ \geq&\alpha\Big(\int_{B(0,r^2)}\mathrm{d}A(w)-\int_{\mathbb{D}/B(0,r^2)}|g_z^r(w)|\mathrm{d}A(w)\Big) \\ \geq&\alpha\Big(r^4-\int_{\mathbb{D}/B(0,r^2)}\mathrm{d}A(w)\Big) \\ =&\alpha(2r^4-1)\to\alpha, \quad \text{as} \quad r\to 1^-. \end{aligned} \tag{3.7}$$

By the definition of $g_z^r(w)$, it is easy to find that $|g_z(w)-g_z^r(w)|\leq 2$ on $\mathbb{D}$, and $|g_z(w)-g_z^r(w)|=0$ on $|w|\geq r$, so

$$\begin{aligned}(1-|z|^2)^\alpha|(Q_\alpha(g_z-g_z^r))'(z)| =&(1-|z|^2)^\alpha\alpha(\alpha+1)\Big|\int_{\mathbb{D}}\frac{\bar{w}(g_z(w)-g_z^r(w))}{(1-z\bar{w})^{\alpha+2}}\mathrm{d}A(w)\Big| \\ \leq&\alpha(\alpha+1)\int_{B(0,r)}\frac{2|w|(1-|z|^2)^\alpha}{|1-z\bar{w}|^{\alpha+2}}\mathrm{d}A(w)\to 0, \quad \text{as} \quad |z|\to 1^-. \end{aligned} \tag{3.8}$$

For any $\epsilon>0$, from (3.7) we may pick $r>0$ such that

$$|Q_\alpha g_z^r(0)|>\alpha-\frac{\epsilon}{2}. \tag{3.9}$$

Fix such r, combining (3.6) and (3.8), we can pick $z\in\mathbb{D}$, such that

$$(1-|z|^2)^\alpha|(Q_\alpha g_z^r)'(z)|>\alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}-\frac{\epsilon}{2}.$$

Thus

$$\begin{aligned}\|Q_\alpha\|\geq\|Q_\alpha g_z^r\|_\alpha &= |Q_\alpha g_z^r(0)|+\sup_{w\in\mathbb{D}}(1-|w|^2)^\alpha|(Q_\alpha g_z^r)'(w)| \\ &>\alpha-\frac{\epsilon}{2}+(1-|z|^2)^\alpha|(Q_\alpha g_z^r)'(z)| \\ &>\alpha-\frac{\epsilon}{2}+\alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}-\frac{\epsilon}{2} \\ &=\alpha+\alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{2+\alpha}{2})}-\epsilon. \end{aligned}$$

Therefore, $\|Q_\alpha\| \geq \alpha + \alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{2+\alpha}{2})}$, which together with (3.1) in Step 1 proves Theorem 1.1. $\Box$

Let $\mathcal{B}_{\alpha,0}$ stand for little Bloch-type space, consisting of analytic functions f with $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha|f'(z)| = 0$. We equip $\mathcal{B}_{\alpha,0}$ also with the norm $||f||_\alpha$ used for the Bloch-type space in (1.2). It is known that $\mathcal{B}_{\alpha,0}$ is a closed subspace of $\mathcal{B}_\alpha$, i.e., $\mathcal{B}_{\alpha,0}$ is the closure in $\mathcal{B}_\alpha$ of the polynomials.

Zhu [1] proved that $Q_\alpha : C(\overline{\mathbb{D}}) \to \mathcal{B}_{\alpha,0}$ is bounded and onto, where $C(\overline{\mathbb{D}})$ stands for the functions continuous on $\overline{\mathbb{D}}$. Notice that $g_z^r(w) \in C(\overline{\mathbb{D}})$, which shows the following corollary.

Corollary 3.1 *Let $0 < \alpha < \infty$. Then the norm of operator $Q_\alpha : C(\overline{\mathbb{D}}) \to \mathcal{B}_{\alpha,0}$ is*

$$\|Q_\alpha\| = \alpha + \alpha(\alpha+1)\frac{\Gamma(\alpha)}{\Gamma^2(\frac{\alpha+2}{2})}.$$

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