Journal of Mathematical Research with Applications
May, 2018, Vol. 38, No. 3, pp. 259–275
DOI:10.3770/j.issn:2095-2651.2018.03.005
Http://jmre.dlut.edu.cn

Dynamics of An Almost Periodic Solutions for A Non-Autonomous Discrete Competitive System with Feedback Controls

Zhouhong LI[1,*], **Li WU**[2]

1. *Department of Mathematics, Yuxi Normal University, Yunnan* 653100*, P. R. China;*
2. *College of Resource and Environment, Yuxi Normal University, Yunnan* 653100*, P. R. China*

Abstract This paper investigates the persistence and existence of almost periodic solutions for a discrete competitive system with feedback controls based on the comparison theorem of the difference equation and constructing appropriate Lyapunov function, and several sufficient conditions for the existence of positive almost periodic solutions for the model are obtained. Finally, a numerical example is given to illustrate effectiveness of our main results.

Keywords persistence; almost periodic solution; discrete competitive system; feedback controls

MR(2010) Subject Classification 34K14; 37N25; 92D25

1. Introduction

In the past ten years, the study on the dynamics of dynamic population has attracted many researcher's attention [1–11]. The simple two competitive species model can be written as follows [3]:

$$\begin{cases} \dot{x}_1(t) = x_1(t)[a_1 - b_1x_1(t) - c_1x_2(t) - d_1x_1^2(t)], \\ \dot{x}_2(t) = x_2(t)[a_2 - b_2x_2(t) - c_2x_1(t) - d_2x_2^2(t)], \end{cases}$$

where $x_1(t)$, $x_2(t)$ can be interpreted as the density of two competing species at time t, respectively. a_1 and a_2 stand for the intrinsic growth rates of two species, b_1, d_1, b_2 and d_2 represent the effects of intra-specific competition, c_1 and c_2 are the effects of inter-specific competition. However, realistic models require the inclusion of the effect of changing environment. Tang et al. [10] discussed the following non-autonomous competition system with impulsive perturbations:

$$\begin{cases} \dot{x}_1(t) = x_1(t)[a_1(t) - b_1(t)x_1(t) - c_1(t)x_2(t) - d_1(t)x_1^2(t)], \quad t \neq \tau_k, \\ \dot{x}_2(t) = x_2(t)[a_2(t) - b_2(t)x_2(t) - c_2(t)x_1(t) - d_2(t)x_2^2(t)], \quad t \neq \tau_k, \\ x_1(\tau_k^+) = (1+\gamma_{1k})x_1(\tau_k), \quad t = \tau_k, \\ x_2(\tau_k^+) = (1+\gamma_{2k})x_2(\tau_k), \quad t = \tau_k,\ k \in \mathbb{N}. \end{cases}$$

Received April 21, 2017; Accepted December 16, 2017
Supported by the National Natural Science Foundation of China (Grant No. 11561070) and the Natural Scientific Research Fund Project of Yunnan Province (Grant No. 2014FD049).
* Corresponding author
E-mail address: zhouhli@yeah.net (Zhouhong LI); liwu2017@yeah.net (Li WU)

Some conditions were derived that guarantee the sufficient conditions for the uniformly asymptotic stability of a unique positive almost periodic solution. Moreover, as we know, since the discrete time models can also provide efficient computational models of continuous models for numerical simulations, it is reasonable to study discrete time models governed by difference equations.

Tang et al. [11] considered the following periodic discrete competitive system subject to feedback controls:

$$\begin{cases} y_1(k+1) = y_1(k)\exp\{a_1(k) - b_1(k)y_1(n) - c_1(k)y_2(k) - d_1(k)x_1^2(k) - e_1(k)v_1(n)\}, \\ y_2(k+1) = y_2(k)\exp\{a_2(k) - b_2(k)y_2(k) - c_2(k)y_1(n) - d_2(k)y_2^2(k) - e_2(k)v_2(n)\}, \\ \Delta v_1 = h_1(k) - f_1(k)v_1(k) + g_1(k)y_1(k), \\ \Delta v_2 = h_2(k) - f_2(k)v_2(n) + g_2(k)y_2(n), \quad k \in \mathbb{Z}^+. \end{cases} \tag{1.1}$$

Sufficient conditions which guarantee the persistence of system (1.1) are studied. Moreover, assuming that the coefficients in the system are periodic sequences, they obtained the sufficient conditions which guarantee the existence of a globally asymptotically stable periodic solution of system (1.1).

As we well know, systems without feedback controls are very important in the models of competitive populations dynamics. However, we note that ecosystems in the real world are continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. In 1993, Gopalsamy and Weng [1] introduced a feedback control variable into the delayed logistic model and discussed the asymptotic behavior of solutions in logistic models with feedback controls, in which the control variables satisfy certain differential equation. In fact, feedback control is the basic mechanism by which models, whether mechanical, electrical, or biological, maintain their equilibrium or homeostasis. During the last decade, a series of mathematical models have been established to describe the dynamics of feedback control systems, we refer to [2,5–7,11–13]. Furthermore, more and more dynamics of the competition system has important significance, for example, see [3,4], [9,10] and the references therein for details. Moreover, many results about the existence of almost periodic solutions of a continuous time system with impulsive effects, we can refer to [10] and the references cited therein. There are few works that consider the existence of almost periodic solutions for discrete time population dynamic model with feedback controls. However, on the other hand, studies on competitive dynamical systems not only involve stability and periodicity, but also involve other dynamic behaviors such as almost periodicity, chaos and bifurcation. In reality, almost periodicity is universal than periodicity.

Stimulated by the above reason, we consider a non-autonomous discrete competitive system with feedback controls:

$$\begin{cases} x_1(n+1) = x_1(n)\exp\{a_1(n) - b_1(n)x_1(n) - c_1(n)x_2(n) - d_1(n)x_1^2(n) - e_1(n)u_1(n)\}, \\ x_2(n+1) = x_2(n)\exp\{a_2(n) - b_2(n)x_2(n) - c_2(n)x_1(n) - d_2(n)x_2^2(n) - e_2(n)u_2(n)\}, \\ \Delta u_1(n) = h_1(n) - f_1(n)u_1(n) + g_1(n)x_1(n), \\ \Delta u_2(n) = h_2(n) - f_2(n)u_2(n) + g_2(n)x_2(n), \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.2}$$

where $\Delta u_i(n) = u_i(n+1) - u_i(n)$ $(i = 1,2)$ are the first-order forward difference operators, $x_i(n)$ $(i = 1,2)$ stand for the densities of species x_i at the nth generation, $a_i(n)$ represent the natural growth rates of species x_i at the nth generation, $b_i(n)$ and $d_i(n)$ stand for the intraspecific effects of the nth generation of species x_i on own population, and $c_i(n)$ measure the interspecific effects of the nth generation of species x_j on species x_i. The coefficients $\{a_i(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}, \{e_i(n)\}, \{f_i(n)\}, \{g_i(n)\}$ and $h_i(n)$ are all almost ω-periodic sequences with $0 < f_i(n) < 1$, $i, j = 1, 2, i \neq j$, $\mathbb{Z}^+$ is the set of nonnegative integers.

To the best of our knowledge, though many works have been done for the population dynamic systems with feedback controls, through most of the works deal with the continuous time models. On the existence and stability of almost periodic sequence solutions for the discrete biological models, some results are found in the literature, we refer to [1,7,8,12,13,15]. In the present paper we will study the existence and uniqueness of almost periodic solutions for the system (1.2).

Remark 1.1 Let $x_i(n) = y_i(k)$, $u_i(n) = v_i(k)$ $(i = 1,2)$, system (1.2) reduces to the system (1.1). As we know, ecosystems in the real world are often distributed by unpredictable forces which can result in changes in biological parameters such as survival rates, so it is necessary to study models with control variables which are so-called feedback control. Moreover, it is more realistic to consider almost periodic systems than periodic systems.

Throughout this paper, we always assume that

(H$_1$) $\{a_i(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}, \{e_i(n)\}, \{f_i(n)\}, \{g_i(n)\}$ and $h_i(n)$ for $i = 1, 2$ are bounded nonnegative almost periodic sequences such that

$$\begin{aligned}
&0 < a_i^L < a_i(n) < a_i^M, \quad 0 < b_i^L < b_i(n) < b_i^M, \quad 0 < c_i^L < c_i(n) < c_i^M, \\
&0 < d_i^L < d_i(n) < d_i^M, \quad 0 < e_i^L < e_i(n) < e_i^M, \quad 0 < f_i^L < f_i(n) < f_i^M < 1, \\
&0 < g_i^L < g_i(n) < g_i^M, \quad 0 < h_i^L < h_i(n) < h_i^M.
\end{aligned}$$

Here, for any bounded sequence $\{\theta(n)\}, \theta^M = \sup_{n\in N}\{\theta(n)\}$ and $\theta^L = \inf_{n\in N}\{\theta(n)\}$. Furthermore, denote $x_1^* = \frac{\exp(a_1^M - 1)}{b_1^L}$, $x_2^* = \frac{\exp(a_2^M - 1)}{b_2^L}$, $u_i^* = \frac{h_i^M + g_i^M x_i^*}{f_i^L}$ $(i = 1,2)$, we need the following assumptions:

(H$_2$) $a_1^L > c_1^M x_2^* + e_1^M u_1^*$,

(H$_3$) $a_2^L > c_2^M x_1^* + e_2^M u_2^*$.

By the biological meaning, we focus our discussion on the positive solution of the model (1.2). So it is assumed that the initial conditions of model (1.2) are of the form

$$x_i(0) > 0, \quad u_i(0) > 0, \quad i = 1, 2. \tag{1.3}$$

One can easily show that all the solutions of model (1.2) with the initial condition (1.3) are defined and remain positive for all $n \in \mathbb{Z}^+$.

The rest of this paper is organized as follows. Next section, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, the persistence of model (1.2) is established. In Section 4, based on the persistence result, we show the existence and uniformly asymptotical stability of an almost periodic solution to model (1.2). An example is given in Section 5.

2. Preliminaries

In order to obtain the main results, we give the definitions and lemmas of the involved terminologies.

Definition 2.1 ([14]) *A sequence $x : \mathbb{Z} \to \mathbb{R}$ is called an almost periodic sequence if the ϵ-translation number set of x*

$$E\{\epsilon, x\} = \{\tau \in \mathbb{Z} : |x(n+\tau) - x(n)| < \epsilon, \quad \forall n \in \mathbb{Z}\}$$

is a relatively dense set in $\mathbb{Z}$ for all $\epsilon > 0$; that is, for any given $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer $\tau \in E\{\epsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \epsilon, \quad \forall n \in \mathbb{Z},$$

τ is called the ϵ-translation number of $x(n)$.

Definition 2.2 ([12]) *Let $f : \mathbb{Z} \times \mathbb{D} \to \mathbb{R}^k$, where $\mathbb{D}$ is an open set in $\mathbb{R}^k$, $f(n,x)$ is said to be almost periodic in n uniformly for $x \in \mathbb{D}$, or uniformly almost periodic for short, if for any $\epsilon > 0$ and any compact set $\mathbb{S}$ in $\mathbb{D}$, there exists a positive integer $l(\epsilon, \mathbb{S})$ such that any interval of length $l(\epsilon, \mathbb{S})$ contains an integer τ for which*

$$|f(n+\tau, x) - f(n,x)| < \epsilon, \quad \forall n \in \mathbb{Z}, \ x \in \mathbb{S}.$$

τ is called the ϵ-translation number of $f(n,x)$.

Lemma 2.3 ([15]) *$\{x(n)\}$ is an almost periodic sequence if and only if for any sequence $\{h'_k\} \subset \mathbb{Z}$ there exists a subsequence $\{h_k\} \subset \{h'_k\}$ such that $x(n+h_k)$ converges uniformly on $n \in \mathbb{Z}$ as $k \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

Zhang and Zheng [14] consider the following almost periodic delay difference system

$$x(n+1) = f(n, x_n), \quad n \in \mathbb{Z}^+, \tag{2.1}$$

where $f : \mathbb{Z}^+ \times C_B \to \mathbb{R}$, $C_B = \{\phi \in C : \|\phi\| < B\}$, $C = \{\phi : [-\tau, 0]_{\mathbb{Z}} \to \mathbb{R}\}$ with $\|\phi\| = \sup_{s\in[-\tau,0]_{\mathbb{Z}}} |\phi(s)|$, $f(n,\phi)$ is almost periodic in n uniformly for $\phi \in C_B$ and is continuous in ϕ, while $x_n \in C_B$ is defined as $x_n(s) = x(n+s)$ for all $s \in [-\tau, 0]_{\mathbb{Z}}$.

The product system of (1.2) is in the form of

$$x(n+1) = f(n, x_n), \quad y(n+1) = f(n, y_n). \tag{2.2}$$

A discrete Lyapunov functional of (1.2) is a functional $V : \mathbb{Z}^+ \times C_B \times C_B \to \mathbb{R}^+$ which is continuous in its second and third variables. Define the difference of V along the solution of system (1.2) by

$$\Delta V_{(1.2)}(n, \phi, \psi) = V(n+1, x_{n+1}(n,\phi), y_{n+1}(n,\psi)) - V(n, \phi, \psi),$$

where $(x(n,\phi), y(n,\psi))$ is a solution of system (1.2) through $(n, (\phi, \psi))$, $\phi, \psi \in C_B$.

Lemma 2.4 ([14]) *Suppose that there exists a Lyapunov functional $V(n, \phi, \psi)$ satisfying the following conditions:*

(1) $a(|\phi(0)-\psi(0)|)\le V(n,\phi,\psi)\le b(\|\phi-\psi\|)$, *where* a, $b\in\mathcal{P}$ *with* $\mathcal{P}=\{a:[0,\infty)\to[0,\infty)|a(0)=0$ *and* $a(u)$ *is continuous, increasing in* $u\}$.

(2) $|V(n,\phi_1,\psi_1)-V(n,\phi_2,\psi_2)|\le L(\|\phi_1-\phi_2\|+\|\psi_1-\psi_2\|)$, *where* $L>0$ *is a constant.*

(3) $\Delta V_{(1.2)}(n,\phi,\psi)\le-\gamma V(n,\phi,\psi)$, *where* $0<\gamma<1$ *is a constant.*

Moreover, if there exists a solution $x(n)$ *of (1.2) such that* $\|x_n\|\le B^*<B$ *for all* $n\in\mathbb{Z}^+$, *then there exists a unique uniformly asymptotically stable almost periodic solution* $p(n)$ *of (1.2) which satisfies* $|p(n)|\le B^*$ *for all* $n\in\mathbb{I}$. *In particular, if* $f(n,\phi)$ *is periodic of period* ω, *then (1.2) has a unique uniformly asymptotically stable periodic solution of period* ω.

3. Persistence

In this section, we establish a persistence result for system (1.2).

Proposition 3.1 *Assume that* (H$_1$) *holds. For every solution* $(x_1(n),x_2(n),u_1(n),u_2(n))^T$ *of system (1.2),*

$$\limsup_{n\to\infty}x_i(n)<x_i^*,\quad \limsup_{n\to\infty}u_i(n)<u_i^*,\quad i=1,2. \tag{3.1}$$

Proof We first present two cases to prove that

$$\limsup_{n\to\infty}x_1(n)<x_1^*. \tag{3.2}$$

Case 1 By the first equation of system (1.2), from (H$_1$) and (1.3), we have

$$\begin{aligned}x_1(n+1)&=x_1(n)\exp\{a_1(n)-b_1(n)x_1(n)-c_1(n)x_2(n)-d_1(n)x_1^2(n)-e_1(n)u_1(n)\}\\&<x_1(n)\exp\{a_1(n)-b_1(n)x_1(n)\}\\&=x_1(n)\exp\{a_1(n)[1-\frac{b_1(n)}{a_1(n)}x_1(n)]\}.\end{aligned}$$

Then there exists $l_0\in\mathbb{N}$ such that $x_1(l_0+1)\ge x_1(l_0)$. So, $1-\frac{b_1(l_0)x_1(l_0)}{a_1(l_0)}\ge0$. Hence,

$$\begin{aligned}x_1(l_0+1)&<x_1(l_0)\exp\{a_1(l_0)-b_1(l_0)x_1(l_0)\}\\&\le x_1(l_0)\exp\left\{a_1^M[1-\frac{b_1(l_0)x_1(l_0)}{a_1(l_0)}]\right\}\\&\le\frac{\exp(a_1^M-1)}{b_1^L}:=x_1^*.\end{aligned}\tag{3.3}$$

Here we used $\max_{x\in\mathbb{R}^+}x\exp(a-bx)=\exp(a-1)/b$ for $a,b>0$ and $\mathbb{R}^+$ is the set of all positive real numbers. We claim that $x_1(n)\le x_1^*$ for $n\ge l_0$.

In fact, if there exists an integer $m\ge n_0+2$ such that $x_1(m)>x_1^*$, and let m_1 be the least integer between n_0 and m such that $x_1(m)=\max_{n_0\le n\le m-1}\{x_1(n)\}$, then $m_1\ge n_0+2$ and $x_1(m_1)>x_1(m_1-1)$ which implies $x_1(m_1)<x_1^*<x_1(m)$. This is impossible. The claim is proved.

Case 2 $x_1(n)\ge x_1(n{+}1)$ for $n\in\mathbb{N}$. In particular, $\lim_{n\to\infty}x_1(n)$ exists, denoted by $\bar{x}_1$. We claim that $\bar{x}_1<x_1^*$. By way of contradiction, assume that $\bar{x}_1>x_1^*$. Take $\lim_{n\to\infty}(1-\frac{a(n)x_1(n)}{b(n)})=0$.

Noting that $\frac{b_1^L}{a_1^M} \leq x_1^*$, we have

$$1 - \frac{b_1(n)x_1(n)}{a_1(n)} \leq 1 - \frac{b_1^L \bar{x}_1}{a_1^M} < 0, \tag{3.4}$$

for $n \in \mathbb{N}$, which is a contradiction. This proves the claim.

Similarly to the above analysis, it is not difficult to get $\limsup_{n\to\infty} x_2(n) < x_2^*$, where

$$x_2^* = \frac{\exp(a_2^M - 1)}{b_2^L}.$$

In the following, for all $i = 1, 2$, we prove that $\lim_{n\to+\infty} u_i(n) \leq u_i^*$. For any $\epsilon > 0$, there exits an integer $n_0 \in \mathbb{Z}^+$ such that $x_i(n) \leq x_i^* + \epsilon$ for all $n \geq n_0$. By the third and fourth equations of system (1.2), we can get

$$\begin{aligned}
u_i(n) &= \prod_{i=0}^{n-1}(1 - f_i(i))\Big[u_i(0) + \sum_{i=0}^{n-1}\frac{h_i(i) + g_i(i)x_i(i)}{\prod_{j=0}^{i}(1 - f_i(j))}\Big] \\
&\leq (1 - f_i^L)^n\Big[u_i(0) + \sum_{i=0}^{n_0-1}\frac{h_i(i) + g_i(i)x_i(i)}{\prod_{j=0}^{i}(1 - f_i(j))}\Big] + [h_i^M + g_i^M(x_i^* + \epsilon)]\sum_{i=n_0}^{n-1}\prod_{j=i+1}^{n-1}(1 - f_i(j)) \\
&\leq (1 - f_i^L)^n\Big[u_i(0) + \sum_{i=0}^{n_0-1}\frac{h_i(i) + g_i(i)x_i(i)}{\prod_{j=0}^{i}(1 - f_i(j))}\Big] + [h_i^M + g_i^M(x_i^* + \epsilon)]\sum_{i=n_0}^{n-1}(1 - f_i^L)^{n-i-1}.
\end{aligned}$$

Since $0 < f_i^L < 1$, we can find two positive numbers Λ_i such that $1 - f_i^L = e^{-\Lambda_i}$, using Stolz's theorem, we have

$$\lim_{n\to\infty}\sum_{i=n_0}^{n-1}(1 - f_i^L)^{n-i-1} = \lim_{n\to\infty}\frac{\sum_{i=n_0}^{n-1} e^{\Lambda_i(i+1)}}{e^{\Lambda_i n}} = \frac{1}{1 - e^{-\Lambda_i}} = \frac{1}{f_i^L}.$$

Hence $\limsup_{n\to\infty} u_i(n) \leq \frac{h_i^M + g_i^M(x_i^* + \epsilon)}{f_i^L}$. Since ϵ is arbitrary, let $\epsilon \to 0$, we obtain that

$$\limsup_{n\to\infty} u_i(n) \leq \frac{h_i^M + g_i^M x_i^*}{f_i^L} := u_i^*, \quad i = 1, 2. \tag{3.5}$$

Then $\limsup_{n\to\infty} u_i(n) \leq u_i^*$ is valid. So the proof of Proposition 3.1 is completed. □

Proposition 3.2 *Assume that (H_1)–(H_3) hold, where x_i^* and u_i^* ($i = 1, 2$) are the same in Proposition 3.1. Then*

$$\liminf_{n\to\infty} x_i(n) > x_{i*}, \ \liminf_{n\to\infty} u_i(n) > u_{i*}, \quad i = 1, 2, \tag{3.6}$$

where

$$\begin{aligned}
x_{1*} &= \Delta_1 \exp\{a_1^L - b_1^M x_1^* - c_1^M x_2^* - d_1^M x_1^{*2} - e_1^M u_1^*\}, \\
x_{2*} &= \Delta_2 \exp\{a_2^L - b_2^M x_2^* - c_2^M x_1^* - d_2^M x_2^{*2} - e_2^M u_2^*\}, \\
u_{i*} &= \frac{h_i^L + g_i^L x_{i*}}{f_i^M}, \quad i = 1, 2.
\end{aligned}$$

Proof Firstly, we also present two cases to prove that

$$\liminf_{n\to\infty} x_1(n) \geq x_{1*}.$$

For any $\epsilon > 0$, according to Proposition 3.1, there exists $n_0 \in \mathbb{N}$ such that

$$x_i(n) \le x_i^* + \epsilon, \quad u_i(n) \le u_i^* + \epsilon, \quad i = 1, 2, \tag{3.7}$$

for $n \ge n_0$.

Case 1 There exists a positive integer $l_0 \ge n_0$ such that $x_1(l_0 + 1) \le x_1(l_0)$. Note that for $n \ge n_0$, we have

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\{a_1(n) - b_1(n)x_1(n) - c_1(n)x_2(n) - d_1(n)x_1^2(n) - e_1(n)u_1(n)\} \\ &\ge x_1(n)\exp\{a_1(n)[1 - \frac{b_1(n)x_1(n)}{a_1(n)} - \frac{c_1(n)x_2(n)}{a_1(n)} - \frac{d_1(n)x_1^2(n)}{a_1(n)} - \frac{e_1(n)u_1(n)}{a_1(n)}]\}. \end{aligned}$$

In particular, with $n = l_0$, we obtain

$$1 - \frac{b_1^M x_1(l_0)}{a_1^L} - \frac{c_1^M(x_2^* + \epsilon)}{a_1^L} - \frac{d_1^M x_1^2(l_0)}{a_1^L} - \frac{e_1^M(u_1^* + \epsilon)}{a_1^L} \le 0,$$

which implies that

$$x_1(l_0) \ge \frac{-b_1^M + \sqrt{b_1^{M2} - 4d_1^M(c_1^M(x_2^* + \epsilon) - a_1^M + e_1^M(u_1^* + \epsilon))}}{2d_1^M} := \Delta_{1\epsilon}^+,$$

$$x_1(l_0) \ge \frac{-b_1^M - \sqrt{b_1^{M2} - 4d_1^M(c_1^M(x_2^* + \epsilon) - a_1^M + e_1^M(u_1^* + \epsilon))}}{2d_1^M} := \Delta_{1\epsilon}^-.$$

Since $-b_1^M - \sqrt{b_1^{M2} - 4d_1^M(c_1^M(x_2^* + \epsilon) - a_1^M + e_1^M(u_1^* + \epsilon))} < 0$, it follows from (H_1) $\Delta_{1\epsilon}^- < 0$ and $x_1(l_0) < 0$. From system (1.3), we get

$$\begin{aligned} x_1(l_0+1) &> \Delta_{1\epsilon}^+ \exp\{a_1^L[1 - \frac{b_1^M(x_1^* + \epsilon)}{a_1^L} - \frac{c_1^M(x_2^* + \epsilon)}{a_1^L} - \frac{d_1^M(x_1^* + \epsilon)^2}{a_1^L} - \frac{e_1^M(u_1^* + \epsilon)}{a_1^L}]\} \\ &:= x_{1\epsilon}. \end{aligned} \tag{3.8}$$

We claim that $x_1(n) \ge x_{1\epsilon}$ for $n \ge l_0$.

By way of contradiction, assume that there exists $p_0 \ge l_0$ such that $x_1(p_0) < x_{1\epsilon}$. Then $p_0 \ge l_0 + 2$, let $p_1 \ge l_0 + 2$ be the smallest integer such that $x_1(p_1) < x_{1\epsilon}$. Then $x(p_1 - 1) < x(p_1)$. The above argument produces that $x_1(p_1) \ge x_{1\epsilon}$, a contradiction. This proves the claim.

Case 2 We assume that $x_1(n+1) \ge x_1(n)$ for all large $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_1(n)$ exists, denoted by $\underline{x}_1$. We claim that $\underline{x}_1 \ge \Delta_{1\epsilon}^+$. By way of contradiction, assume that $\underline{x}_1 < \Delta_{1\epsilon}^+$. Take

$$\lim_{n\to\infty}(1 - \frac{b_1(n)x_1(n)}{a_1(n)} - \frac{c_1(n)x_2(n)}{a_1(n)} - \frac{d_1(n)x_1^2(n)}{a_1(n)} - \frac{e_1(n)u_1(n)}{a_1(n)}) = 0,$$

which is a contradiction, since

$$\begin{aligned} &\lim_{n\to\infty}(1 - \frac{b_1(n)x_1(n)}{a_1(n)} - \frac{c_1(n)x_2(n)}{a_1(n)} - \frac{d_1(n)x_1^2(n)}{a_1(n)} - \frac{e_1(n)u_1(n)}{a_1(n)}) \\ &\quad \ge 1 - \frac{b_1^M x_1(n)}{a_1^L} - \frac{c_1^M(x_2^* + \epsilon)}{a_1^L} - \frac{d_1^M(x_1^* + \epsilon)^2}{a_1^L} - \frac{e_1^M(u_1^* + \epsilon)}{a_1^L} > 0. \end{aligned}$$

Note that $x_1^* \ge a_1^M \ge a_1^L$, we see that $\Delta_{1\epsilon}^+ \ge x_{1\epsilon}$, and $\lim_{\epsilon\to 0} x_{1\epsilon} = x_{1*}$. We can easily see that $\liminf_{n\to\infty} x_1(n) \ge x_{1*}$ holds. Similarly, we can prove that $\liminf_{n\to\infty} x_2(n) \ge x_{2*}$. Thus for any $\epsilon > 0$ small enough, there exists a positive integer n_0, such that $x_i(n) \geq x_{i*} - \epsilon > 0$ for all $\epsilon > 0$.

Next, we will prove that $\liminf_{n\to\infty} u_i(n) \geq u_{i*}$ for all $i = 1, 2$. For any $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{Z}^+$ such that $x_i(n) \geq x_{i*} - \epsilon$ for $n \geq n_0$. We have from the third and fourth equations of system (1.2) that

$$\begin{aligned} u_i(n) &= \prod_{i=0}^{n-1}(1-f_i(i))\Big[u_i(0) + \sum_{i=0}^{n-1}\frac{h_i(i)+g_i(i)x_i(i)}{\prod_{j=0}^{i}(1-f_i(j))}\Big] \\ &\geq (1-f_i^M)^n\Big[u_i(0) + \sum_{i=0}^{n_0-1}\frac{h_i(i)+g_i(i)x_i(i)}{\prod_{j=0}^{i}(1-f_i(j))}\Big] + [h_i^L + g_i^L(x_{i*}-\epsilon)]\sum_{i=n_0}^{n-1}\prod_{j=i+1}^{n-1}(1-f_i(j)) \\ &\geq (1-f_i^M)^n\Big[u_i(0) + \sum_{i=0}^{n_0-1}\frac{h_i(i)+g_i(i)x_i(i)}{\prod_{j=0}^{i}(1-f_i(j))}\Big] + [h_i^L + g_i^L(x_{i*}-\epsilon)]\sum_{i=n_0}^{n-1}(1-f_i^M)^{n-i-1}. \end{aligned}$$

Since $0 < f_i^M < 1$, we can find two positive numbers Γ_i such that $1 - f_i^M = e^{-\Gamma_i}$. Using Stolz's theorem, we have

$$\lim_{n\to\infty}\sum_{i=n_0}^{n-1}(1-f_i^M)^{n-i-1} = \lim_{n\to\infty}\frac{\sum_{i=n_0}^{n-1}e^{\Gamma_i(i+1)}}{e^{\Gamma_i n}} \to \frac{1}{1-e^{-\Gamma_i}} = \frac{1}{f_i^M}.$$

Hence $\liminf_{n\to\infty} u_i(n) \geq \frac{h_i^L + g_i^L(x_{i*}-\epsilon)}{f_i^M}$. Since ϵ is arbitrary, let $\epsilon \to 0$, we obtain that

$$\liminf_{n\to\infty} u_i(n) \geq \frac{h_i^L + g_i^L x_{i*}}{f_i^M} := u_{i*}. \tag{3.9}$$

So the proof of Proposition 3.2 is completed. □

Remark 3.3 Similar results have been obtained by Tan [11, Lemmas 1 and 2].

Now the main result of this section is obtained as follows.

Theorem 3.4 *Assume that (H$_1$)–(H$_3$) hold. Then system (1.2) is persistent.*

4. Main result

According to Lemma 2.4, we first prove that there exists a bounded solution of system (1.2), and then construct an adaptive Lyapunov functional for system (1.2).

The next result tells that there exists a bounded solution of system (1.2).

We denote by Ω the set of all solutions $X(n) = (x_1(n), x_2(n), u_1(n), u_2(n))^T$ of system (1.2) satisfying $x_{i*} \leq x_i(n) \leq x_i^*, u_{i*} \leq u_i(n) \leq u_i^*$ for all $n \in \mathbb{Z}^+$.

Proposition 4.1 *Assume that (H$_1$)–(H$_3$) hold. Then $\Omega \neq \varnothing$.*

Proof It is now possible to show by an inductive argument that system (1.2) leads to

$$\begin{cases} x_1(n) = x_1(0)\exp\sum\limits_{l=0}^{n-1}\{a_1(l) - b_1(l)x_1(l) - c_1(l)x_2(l) - d_1(l)x_1^2(l) - e_1(l)u_1(l)\}, \\ x_2(n) = x_2(0)\exp\sum\limits_{l=0}^{n-1}\{a_2(l) - b_2(l)x_2(l) - c_2(l)x_1(l) - d_2(l)x_2^2(l) - e_2(l)u_2(l)\}, \\ u_1(n) = u_1(0) - \sum\limits_{l=0}^{n-1}\{-h_1(l) + f_1(l)u_1(l) - g_1(l)x_1(l)\}, \\ u_2(n) = u_2(0) - \sum\limits_{l=0}^{n-1}\{-h_2(l) + f_2(l)u_1(l) - g_2(l)x_1(l)\}. \end{cases} \tag{4.1}$$

From Propositions 3.1 and 3.2, any solution $X(n) = (x_1(n), x_2(n), u_1(n), u_2(n))^T$ of system (1.2) with initial condition (1.3) satisfies system (4.1). Hence, for any $\epsilon > 0$, there exists n_0 such that if n_0 is sufficiently large, we have

$$x_{i*} - \epsilon \le x_i(n) \le x_i^* + \epsilon, \quad u_{i*} - \epsilon \le u_i(n) \le u_i^* + \epsilon, \ \ \forall n \ge n_0, \ i = 1, 2. \tag{4.2}$$

Let $\{t_n\}$ be any integer-valued sequence such that $t_n \to \infty$ as $n \to \infty$. We claim that there exists a subsequence of $\{t_n\}$, we still denote it by $\{t_n\}$ such that

$$x_i(n + t_n) \to x_i^*(n) \tag{4.3}$$

uniformly in n on any finite subset $\mathbf{B}$ of $\mathbb{Z}$ as $n \to \infty$, where $\mathbf{B} = \{a_1, a_2, \dots, \alpha_m\}$, $a_h \in \mathbb{Z}$ $(h = 1, 2, \dots, m)$ and m is a finite number.

In fact, for any finite subset $\mathbf{B} \subset \mathbb{Z}$, when a is large enough, $t_n + a_h > n_0$, $h = 1, 2, \dots, m$. So

$$x_{i*} - \epsilon \le x_i(n + t_n) \le x_i^* + \epsilon, \quad u_{i*} - \epsilon \le u_i(n + t_n) \le u_i^* + \epsilon. \tag{4.4}$$

That is, $\{x_i(n + t_n)\}, \{u_i(n + t_n)\}$ are uniformly bounded for large enough n.

Similarly, for $a_2 \in \mathbf{B}$, we can choose a subsequence $\{t_n^2\}$ of $\{t_n^1\}$ such that $\{x_i(a_2 + t_n^2)\}, \{u_i(a_2 + t_n^2)\}$ uniformly converges on $\mathbb{Z}^+$ for n large enough.

Repeating this procedure, for $a_m \in \mathbf{B}$, we obtain a subsequence $\{t_n^m\}$ of $\{t_n^{m-1}\}$ such that $\{x_i(a_m + t_n^m)\}, \{u_i(a_m + t_n^m)\}$ uniformly converges on $\mathbb{Z}^+$ for n large enough.

Now pick the sequence $\{t_n^m\}$ which is a subsequence of $\{t_n\}$, we still denote it by $\{t_n\}$, then for all $n \in \mathbf{B}$, we have $x_i(n + t_n) \to x_i^*(n), u_i(n + t_n) \to u_i^*(n)$ uniformly in $n \in \mathbf{B}$ as $p \to \infty$.

By the arbitrariness of $\mathbf{B}$, the conclusion is valid.

Since $\{a_i(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}, \{e_i(n)\}, \{f_i(n)\}, \{g_i(n)\}$ and $\{h_i(n)\}$ $(i = 1, 2)$ are almost periodic sequences, for the above sequence $\{\tau_p\}, \tau_p \to \infty$ as $p \to \infty$, there exists a subsequence still denoted by $\{\tau_p\}$ (if necessary, we take a subsequence) such that

$$\begin{aligned} &a_i(n + \tau_p) \to a_i(n), \quad b_i(n + \tau_p) \to b_i(n), \quad c_i(n + \tau_p) \to c_i(n), \\ &d_i(n + \tau_p) \to d_i(n), \quad e_i(n + \tau_p) \to e_i(n), \quad f_i(n + \tau_p) \to f_i(n), \\ &g_i(n + \tau_p) \to g_i(n), \quad h_i(n + \tau_p) \to h_i(n), \quad \text{for all} \ \ i = 1, 2, \end{aligned}$$

as $p \to \infty$ uniformly on $\mathbb{Z}^+$. For any $\sigma \in \mathbb{Z}$, we can assume that $\tau_p + \sigma \ge n_0$ for p large enough. Let $n \ge 0$ and $n \in \mathbb{Z}^+$. Using an inductive argument of system (1.2) from $\tau_p + \sigma$ to $n + \tau_p + \sigma$

leads to

$$\begin{cases} x_1(n+\tau_p+\sigma) = x_1(\tau_p+\sigma)\exp\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{a_1(l)-b_1(l)x_1(l)-c_1(l)x_2(l)- \\ \qquad\qquad d_1(l)x_1^2(l)-e_1(l)u_1(l)\big\}, \\ x_2(n+\tau_p+\sigma) = x_2(\tau_p+\sigma)\exp\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{a_2(l)-b_2(l)x_2(l)-c_2(l)x_1(l)- \\ \qquad\qquad d_2(l)x_2^2(l)-e_2(l)u_2(l)\big\}, \\ u_1(n+\tau_p+\sigma) = u_1(\tau_p+\sigma)-\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{-h_1(l)+f_1(l)u_1(l)-g_1(l)x_1(l)\big\}, \\ u_2(n+\tau_p+\sigma) = u_2(\tau_p+\sigma)-\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{-h_2(l)+f_2(l)u_2(l)-g_2(l)x_2(l)\big\}. \end{cases} \tag{4.5}$$

Then, for $i=1,2$, we have

$$\begin{cases} x_1(n+\tau_p+\sigma) = x_1(\tau_p+\sigma)\exp\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{a_1(l+\tau_p)-b_1(l+\tau_p)x_1(l+\tau_p)- \\ \qquad\qquad c_1(l+\tau_p)x_2(l+\tau_p)-d_1(l+\tau_p)x_1^2(l+\tau_p)-e_1(l+\tau_p)u_1(l+\tau_p)\big\}, \\ x_2(n+\tau_p+\sigma) = x_2(\tau_p+\sigma)\exp\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{a_2(l+\tau_p)-b_2(l+\tau_p)x_2(l+\tau_p)- \\ \qquad\qquad c_2(l+\tau_p)x_1(l+\tau_p)-d_2(l+\tau_p)x_2^2(l+\tau_p)-e_2(l+\tau_p)u_2(l+\tau_p)\big\}, \\ u_1(n+\tau_p+\sigma) = u_1(\tau_p+\sigma)-\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{-h_1(l+\tau_p)+f_1(l+\tau_p)u_1(l)- \\ \qquad\qquad g_1(l+\tau_p)x_1(l+\tau_p)\big\}, \\ u_2(n+\tau_p+\sigma) = u_2(\tau_p+\sigma)-\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{-h_2(l+\tau_p)+f_2(l+\tau_p)u_2(l+\tau_p)- \\ \qquad\qquad g_2(l+\tau_p)x_2(l+\tau_p)\big\}. \end{cases} \tag{4.6}$$

Let $p\to\infty$, for any $n\ge 0$,

$$\begin{cases} x_1^*(n+\tau_p+\sigma) = x_1^*(\tau_p+\sigma)\exp\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{a_1(l)-b_1(l)x_1^*(l)-c_1(l)x_2^*(l)- \\ \qquad\qquad d_1(l){x_1^*}^2(l)-e_1(l)u_1^*(l)\big\}, \\ x_2^*(n+\tau_p+\sigma) = x_2^*(\tau_p+\sigma)\exp\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{a_2(l)-b_2(l)x_2^*(l)-c_2(l)x_1^*(l)- \\ \qquad\qquad d_2(l){x_2^*}^2(l)-e_2(l)u_2^*(l)\big\}, \\ u_1^*(n+\tau_p+\sigma) = u_1^*(\tau_p+\sigma)-\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{-h_1(l)+f_1(l)u_1^*(l)-g_1(l)x_1^*(l)\big\}, \\ u_2^*(n+\tau_p+\sigma) = u_2^*(\tau_p+\sigma)-\sum\limits_{l=\tau_p+\sigma}^{n+\tau_p+\sigma-1}\big\{-h_2(l)+f_2(l)u_2^*(l)-g_2(l)x_2^*(l)\big\}. \end{cases} \tag{4.7}$$

By the arbitrariness of $\sigma, X^*=(x_1^*(n),x_2^*(n),u_1^*(n),u_2^*(n))^T$ is a solution of system (1.2) on $\mathbb{Z}^+$. It is clear that $0<x_{i*}\le x_i^*(n)\le x_i^*, 0<u_{i*}\le u_i^*(n)\le u_i^*$, for all $n\in\mathbb{Z}^+, i=1,2$. So $\Omega\ne\varnothing$. Proposition 4.1 is valid. $\Box$

The main results of the following theorem concern the existence of a uniformly asymptotically stable almost periodic sequence solution of system (1.2).

Theorem 4.2 *Assume that* (H_1)*–*(H_3) *hold. Suppose further that* (H_4)*:* $0 < \Theta < 1$*, where*

$$\begin{aligned}
\Theta_1 &= 1 - W_1 - \Omega_1 - g_1^M \xi_1(n)(2 - f_1^l),\\
\Theta_2 &= 1 - W_2 - \Omega_2 - g_2^M \xi_2(n)(2 - f_2^l),\\
\Theta_3 &= 2f_1^l - f_1^{l2} - g_1^M \xi_1(n)(1 - f_1^l) - W_3,\\
\Theta_4 &= 2f_2^l - f_2^{l2} - g_2^M \xi_2(n)(1 - f_2^l) - \Omega_3,\\
\Theta &= \min\{\Theta_1, \Theta_2, \Theta_3, \Theta_4\},
\end{aligned}$$

then there exists a unique uniformly asymptotically stable almost periodic solution $X(n) = (x_1(n), x_2(n), u_1(n), u_2(n))^T$ *of system (1.2) which is bounded by* Ω *for all* $n \in \mathbb{Z}^+$*.*

Proof Let $p_i(n) = \ln x_i(n)$ $(i = 1, 2)$. From (1.2), we have

$$\begin{aligned}
p_1(n+1) &= p_1(n) + a_1(n) - b_1(n)e^{p_1(n)} - c_1(n)e^{p_2(n)} - d_1(n)e^{2p_1(n)} - e_1(n)u_1(n),\\
p_2(n+1) &= p_2(n) + a_2(n) - b_2(n)e^{p_2(n)} - c_2(n)e^{p_1(n)} - d_2(n)e^{2p_2(n)} - e_2(n)u_2(n),\\
\Delta u_1(n) &= h_1(n) - f_1(n)u_1(n) + g_1(n)e^{p_1(n)},\\
\Delta u_2(n) &= h_2(n) - f_2(n)u_2(n) + g_2(n)e^{p_2(n)}, \quad n \in \mathbb{Z}^+.
\end{aligned} \tag{4.8}$$

From Proposition 4.1, we know that system (4.8) has a bounded solution $Y(n) = (p_1(n), p_2(n), u_1(n), u_2(n))^T$ satisfying

$$\ln x_{i*} \le p_i(n) \le \ln x_i^*, \quad u_{i*} \le u_i(n) \le u_i^*, \quad i = 1, 2, n \in \mathbb{Z}^+. \tag{4.9}$$

Hence, $|p_i(n)| \le A_i, |u_i(n)| \le B_i$, where $A_i = \max\{|\ln x_{i*}|, \ln x_i^*\}, B_i = \max\{u_{i*}, u_i^*\}, i = 1, 2$.

For $(X, U) \in \mathbb{R}^{2+2}$, we define the norm $\|(X, U)\| = \sum_{i=1}^2 |x_i| + \sum_{i=1}^2 |u_i|$.

Consider the product system of system (4.8)

$$\begin{cases}
p_1(n+1) = p_1(n) + a_1(n) - b_1(n)e^{p_1(n)} - c_1(n)e^{p_2(n)} - d_1(n)e^{2p_1(n)} - e_1(n)u_1(n),\\
p_2(n+1) = p_2(n) + a_2(n) - b_2(n)e^{p_2(n)} - c_2(n)e^{p_1(n)} - d_2(n)e^{2p_2(n)} - e_2(n)u_2(n),\\
\Delta u_i(n) = h_i - f_i(n)u_i(n) + g_i(n)e^{p_i(n)}, \quad i = 1, 2,\\
q_1(n+1) = q_1(n) + a_1(n) - b_1(n)e^{q_1(n)} - c_1(n)e^{q_2(n)} - d_1(n)e^{2q_1(n)} - e_1(n)v_1(n),\\
q_2(n+1) = q_2(n) + a_2(n) - b_2(n)e^{q_2(n)} - c_2(n)e^{q_1(n)} - d_2(n)e^{2q_2(n)} - e_2(n)v_2(n),\\
\Delta v_i(n) = h_i - f_i(n)v_i(n) + g_i(n)e^{q_i(n)}, \quad i = 1, 2.
\end{cases} \tag{4.10}$$

Suppose that $P = (p_1(n), p_2(n), u_1(n), u_2(n))^T, Q = (q_1(n), q_2(n), v_1(n), v_2(n))^T$ are any two solutions of system (4.10) defined on $\mathbb{Z}^+ \times S^* \times S^*$. Then $\|P\| \le B, \|Q\| \le B$, where

$$\begin{aligned}
B &= \sum_{i=1}^2 \{A_i + B_i\},\\
S^* &= \{(p_1(n), p_2(n), u_1(n), u_2(n)) : \ln x_{i*} \le p_i(n) \le \ln x_i^*, \quad u_{i*} \le u_i(n) \le u_i^*,\\
&\qquad i = 1, 2, n \in \mathbb{Z}^+\}.
\end{aligned} \tag{4.11}$$

Choose Lyapunov function defined on $\mathbb{Z}^+ \times S^* \times S^*$ as follows:

$$V(n,P,Q)=\sum_{i=1}^{2}\{(p_i(n)-q_i(n))^2+(u_i(n)-v_i(n))^2\}. \tag{4.12}$$

It is easy to see that the norm $\|P-Q\|=\sum_{i=1}^{2}\{|p_i(n)-q_i(n)|+|u_i(n)-v_i(n)|\}$ and the norm $\|P-Q\|_*=\{\sum_{i=1}^{2}\{(p_i(n)-q_i(n))^2+(u_i(n)-v_i(n))^2\}\}^{1/2}$ are equivalent, that is, there exist two constants $C_1>0, C_2>0$ such that

$$C_1\|P-Q\|\le \|P-Q\|_*\le C_2\|P-Q\|, \tag{4.13}$$

then

$$(C_1\|P-Q\|)^2\le \|P-Q\|_*\le (C_2\|P-Q\|)^2. \tag{4.14}$$

Let $a\in C(\mathbb{R}^+,\mathbb{R}^+), a(x)=C_1^2x^2, b\in C(\mathbb{R}^+,\mathbb{R}^+), b(x)=C_2^2x^2$. Thus condition (1) in Lemma 2.4 is satisfied.

In addition,

$$\begin{aligned}
&|V(n,P,Q)-V(n,\tilde{P},\tilde{Q})|\\
&=\Big|\sum_{i=1}^{2}\{(p_i(n)-q_i(n))^2+(u_i(n)-v_i(n))^2\}-\sum_{i=1}^{2}\{(\tilde{p}_i(n)-\tilde{q}_i(n))^2+(\tilde{u}_i(n)-\tilde{v}_i(n))^2\}\Big|\\
&\le\sum_{i=1}^{2}|((p_i(n)-q_i(n))^2+(u_i(n)-v_i(n))^2|+\sum_{i=1}^{2}|(\tilde{p}_i(n)-\tilde{q}_i(n))^2+(\tilde{u}_i(n)-\tilde{v}_i(n))^2|\\
&=\sum_{i=1}^{2}\{|(p_i(n)-q_i(n))+(\tilde{p}_i(n)-\tilde{q}_i(n))||(p_i(n)-q_i(n))-(\tilde{p}_i(n)-\tilde{q}_i(n))|\}\\
&\quad\sum_{i=1}^{2}\{|(u_i(n)-v_i(n))+(\tilde{u}_i(n)-\tilde{v}_i(n))|||(u_i(n)-v_i(n))-(\tilde{u}_i(n)-\tilde{v}_i(n))||\}\\
&\le\sum_{i=1}^{2}\{(|p_i(n)|+|q_i(n)|+|\tilde{p}_i(n)|+|\tilde{q}_i(n)|)(|p_i(n)-\tilde{p}_i(n)|+|q_i(n)-\tilde{q}_i(n)|)\}\\
&\quad\sum_{i=1}^{2}\{(|u_i(n)|+|v_i(n)|+|\tilde{u}_i(n)|+|\tilde{v}_i(n)|)(|u_i(n)-\tilde{u}_i(n)|+|v_i(n)-\tilde{v}_i(n)|)\}\\
&\le L\Big\{\sum_{i=1}^{2}\{|p_i(n)-\tilde{p}_i(n)|+|u_i(n)-\tilde{u}_i(n)|\}+\sum_{i=1}^{2}\{|q_i(n)-\tilde{q}_i(n)|+|v_i(n)-\tilde{v}_i(n)|\}\Big\}\\
&=L\{\|P-\tilde{P}\|+||Q-\tilde{Q}\|\},
\end{aligned} \tag{4.15}$$

where $L=4\max\{A_i,B_i\}$ $(i=1,2)$. Hence condition (2) of Lemma 2.4 is satisfied.

Finally, calculating ΔV of $V(n)$ along the solutions of (4.10), we can obtain

$$\begin{aligned}
\Delta V_{(4.10)}(n)=&V(n+1)-V(n)\\
=&\sum_{i=1}^{2}\{[p_i(n+1)-q_i(n+1)]^2+(u_i(n+1)-v_i(n+1))^2\}-\\
&\sum_{i=1}^{2}\{[p_i(n)-q_i(n)]^2+[u_i(n)-v_i(n)]^2\}
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=1}^{2}\{(p_i(n+1)-q_i(n+1))^2-(p_i(n)-q_i(n))^2+\\
&(u_i(n+1)-v_i(n+1))^2-(u_i(n)-v_i(n))^2\}\\
&=\sum_{i=1}^{2}\{[p_i(n+1)-q_i(n+1]^2-(p_i(n)-q_i(n))^2+[(1-f_i(n))(u_i(n)-\\
&v_i(n))+g_i(n)(e^{p_i(n)}-e^{q_i(n)})]^2-(u_i(n)-v_i(n))^2\}.
\end{aligned}
\tag{4.16}
$$

In view of system (4.1) and using the mean value theorem, we get

$$
e^{p_i(n)}-e^{q_i(n)}=\xi_i(n)(p_i(n)-q_i(n)),\quad i=1,2, \tag{4.17}
$$

where $\xi_i(n)$ lies between $e^{p_i(n)}$ and $e^{q_i(n)}$.

$$
\begin{aligned}
&[p_1(n+1)-q_1(n+1)]^2\\
&=\big[(p_1(n)-q_1(n))-b_1(n)(e^{p_1(n)}-e^{q_1(n)})-c_1(n)(e^{p_2(n)}-e^{q_2(n)})-\\
&\quad d_1(n)(e^{2p_1(n)}-e^{2q_1(n)})-e_1(n)(u_1(n)-v_1(n))\big]^2\\
&=[(p_1(n)-q_1(n))-b_1(n)\xi_1(n)(p_1(n)-q_1(n))-c_1(n)\xi_2(n)(p_2(n)-q_2(n))-\\
&\quad 2d_1(n)\xi_1(n)(p_1(n)-q_1(n))-e_1(n)(u_1(n)-v_1(n))]^2\\
&=(1-b_1(n)\xi_1(n)-2d_1(n)\xi_1(n))^2(p_1(n)-q_1(n))^2-2[(1-b_1(n)\xi_1(n)-\\
&\quad 2d_1(n)\xi_1(n))(p_1(n)-q_1(n))(c_1(n)\xi_2(n)(p_2(n)-q_2(n))+\\
&\quad e_1(n)(u_1(n)-v_1(n)))]+c_1^2(n)\xi_2^2(n)(p_2(n)-q_2(n))^2+\\
&\quad 2c_1(n)\xi_2(n)e_1(n)(p_2(n)-q_2(n))(u_1(n)-v_1(n))+e_1^2(n)(u_1(n)-v_1(n))^2\\
&\le(1-b_1^L\xi_1^L-2d_1^L\xi_1(n))^2[p_1(n)-q_1(n)]^2+c_1^{M2}\xi_2(n)^2[p_2(n)-q_2(n)]^2+\\
&\quad 2(b_1^M\xi_1(n)+2d_1^M\xi_1(n)-1)c_1^M\xi_2|p_1(n)-q_1(n)||p_2(n)-q_2(n)|+\\
&\quad 2(b_1^M\xi_1(n)+2d_1^M\xi_1(n)-1)e_1^M|p_1(n)-q_1(n)||u_1(n)-v_1(n)|+\\
&\quad e_1^{M2}(u_1(n)-v_1(n))^2+2c_1^M\xi_2(n)e_1^M|u_1(n)-v_1(n)||p_2(n)-q_2(n)|\\
&\le(1-b_1^L\xi_1(n)-2d_1^L\xi_1^l)^2[p_1(n)-q_1(n)]^2+c_1^{M2}\xi_2(n)^2[p_2(n)-q_2(n)]^2+\\
&\quad (b_1^M\xi_1(n)+2d_1^M\xi_1(n)-1)c_1^M\xi_2(n)[(p_1(n)-q_1(n))^2+(p_2(n)-q_2(n))^2]+\\
&\quad (b_1^M\xi_1(n)+2d_1^M\xi_1(n)-1)e_1^M[(p_1(n)-q_1(n))^2+(u_1(n)-v_1(n))^2]+\\
&\quad e_1^{M2}(u_1(n)-v_1(n))^2+c_1^M\xi_2(n)e_1^M[(u_1(n)-v_1(n))^2+(p_2(n)-q_2(n))^2]\\
&=W_1[p_1(n)-q_1(n)]^2+W_2[p_2(n)-q_2(n)]^2+W_3[u_1(n)-v_1(n)]^2,
\end{aligned}
\tag{4.18}
$$

where

$$
\begin{aligned}
W_1=&(1-b_1^L\xi_1(n)-2d_1^L\xi_1(n))^2+(b_1^M\xi_1(n)+2d_1^M\xi_1(n)-1)c_1^M\xi_2(n)+(b_1(n)\xi_1(n)+\\
&2d_1^M\xi_1(n)-1)e_1^M,\\
W_2=&c_1^{M2}\xi_2(n)^2+(b_1^M\xi_1(n)+2d_1^M\xi_1(n)-1)c_1^M\xi_2(n)+c_1^M\xi_2(n)e_1^M,\\
W_3=&e_1^{M2}+(b_1^M\xi_1(n)+2d_1^M\xi_1(n)-1)e_1^M+c_1^M\xi_2(n).
\end{aligned}
$$

Similarly, we also obtain

$$
\begin{aligned}
&[p_2(n+1)-q_2(n+1)]^2\\
&\quad=\big[(p_2(n)-q_2(n))-b_2(n)(e^{p_2(n)}-e^{q_2(n)})-c_2(n)(e^{p_1(n)}-e^{q_1(n)})-\\
&\qquad d_2(n)(e^{2p_2(n)}-e^{2q_2(n)})-e_2(n)(u_2(n)-v_2(n))\big]^2\\
&\quad=[(p_2(n)-q_2(n))-b_2(n)\xi_2(n)(p_2(n)-q_2(n))-c_2(n)\xi_1(n)(p_1(n)-q_1(n))-\\
&\qquad 2d_2(n)\xi_2(n)(p_2(n)-q_2(n))-e_2(n)(u_2(n)-v_2(n))]^2\\
&\quad=(1-b_2(n)\xi_2(n)-2d_2(n)\xi_2(n))^2(p_2(n)-q_2(n))^2-2[(1-b_2(n)\xi_2(n)-\\
&\qquad 2d_2(n)\xi_2(n))(p_2(n)-q_2(n))(c_2(n)\xi_1(n)(p_1(n)-q_1(n))+\\
&\qquad e_2(n)(u_2(n)-v_2(n)))]+c_2^2(n)\xi_1^2(n)(p_1(n)-q_1(n))^2+\\
&\qquad 2c_2(n)\xi_1(n)e_2(n)(p_1(n)-q_1(n))(u_2(n)-v_2(n))+e_2^2(n)(u_2(n)-v_2(n))^2\\
&\quad\le(1-b_2^L\xi_2(n)-2d_2^L\xi_2(n))^2[p_2(n)-q_2(n)]^2+c_2^{M2}\xi_1(n)^2[p_1(n)-q_1(n)]^2+\\
&\qquad 2(b_2^M\xi_2(n)+2d_2^M\xi_2(n)-1)c_2^M\xi_1(n)|p_1(n)-q_1(n)||p_2(n)-q_2(n)|+\\
&\qquad 2(b_2^M\xi_2(n)+2d_1^M\xi_1(n)-1)e_2^M|p_2(n)-q_2(n)||u_2(n)-v_2(n)|+\\
&\qquad e_2^{M2}(u_2(n)-v_2(n))^2+2c_2^M\xi_1(n)e_2^M|u_2(n)-v_2(n)||p_1(n)-q_1(n)|\\
&\quad\le(1-b_2^L\xi_2(n)-2d_2^L\xi_2(n))^2[p_2(n)-q_2(n)]^2+c_2^{M2}\xi_1(n)^2[p_1(n)-q_1(n)]^2+\\
&\qquad (b_2^M\xi_2(n)+2d_2^M\xi_2(n)-1)c_2^M\xi_1(n)[(p_1(n)-q_1(n))^2+(p_2(n)-q_2(n))^2]+\\
&\qquad (b_2^M\xi_2(n)+2d_1^M\xi_1(n)-1)e_2^M[(p_2(n)-q_2(n))^2+(u_2(n)-v_2(n))^2]+\\
&\qquad e_2^{M2}(u_2(n)-v_2(n))^2+c_2^M\xi_1(n)e_2^M[(u_2(n)-v_2(n))^2+(p_1(n)-q_1(n))^2]\\
&\quad=\Omega_1[p_1(n)-q_1(n)]^2+\Omega_2[p_2(n)-q_2(n)]^2+\Omega_3[u_2(n)-v_2(n)]^2,
\end{aligned}\tag{4.19}
$$

where

$$
\begin{aligned}
\Omega_1=&c_2^{M2}\xi_1(n)^2+(b_2^M\xi_2(n)+2d_2^M\xi_2(n)-1)c_2^M\xi_1(n)+c_2^M\xi_1(n)e_2^M,\\
\Omega_2=&(1-b_2^L\xi_2(n)-2d_2^L\xi_2(n))^2+(b_2^M\xi_2(n)+\\
&2d_2^M\xi_2(n)-1)c_2^M\xi_1(n)+(b_2^M\xi_2(n)+2d_1^M\xi_1(n)-1)e_2^M,\\
\Omega_3=&e_2^{M2}+(b_2^M\xi_2(n)+2d_1^M\xi_1(n)-1)e_2^M+c_2^M\xi_1(n)e_2^M.
\end{aligned}
$$

From system (4.10), we also obtain

$$
\begin{aligned}
&[u_i(n+1)-v_i(n+1)]^2-[u_i(n)-v_i(n)]^2\\
&\quad=[(1-f_i(n))^2-1](u_i(n)-v_i(n))^2+g_i^2(n)(e^{p_i(n)}-e^{q_i(n)})^2+\\
&\qquad 2g_i(n)(1-f_i(n))(u_i(n)-v_i(n))(e^{p_i(n)}-e^{q_i(n)})\\
&\quad\le(f_i^{L2}-2f_i^L)(u_i(n)-v_i(n))^2+g_i^M\xi_i(n)(p_i(n)-q_i(n))^2+\\
&\qquad 2g_i^M(1-f_i^L)\xi_i(n)|u_i(n)-v_i(n)||p_i(n)-q_i(n)|\\
&\quad\le(f_i^{L2}-2f_i^L)(u_i(n)-v_i(n))^2+g_i^M\xi_i(n)(p_i(n)-q_i(n))^2+\\
&\qquad g_i^M(1-f_i^L)\xi_i(n)[(u_i(n)-v_i(n))^2+(p_i(n)-q_i(n))^2]\\
&\quad=(f_i^{L2}-2f_i^L+g_i^M(1-f_i^L)\xi_i(n))(u_i(n)-v_i(n))^2+
\end{aligned}
$$

$$(g_i^M(1-f_i^L)\xi_i(n)+g_i^M\xi_i(n))(p_i(n)-q_i(n))^2,\ i=1,2. \tag{4.20}$$

From (4.16)–(4.19), we have

$$\begin{aligned}\Delta V_{(24)}(n)\leq&[W_1+\Omega_1+g_1^M\xi_1(n)(1-f_1^L)+g_1^M\xi_1(n)-1][p_1(n)-q_1(n)]^2+\\&[W_2+\Omega_2+g_2^M\xi_2(n)(1-f_2^L)+g_2^M\xi_2(n)-1][p_2(n)-q_2(n)]^2+\\&[W_3+f_1^{L2}-2f_1^L+g_1^M\xi_1(n)(1-f_1^L)][u_1(n)-v_1(n)]^2+\\&[\Omega_3+f_2^{L2}-2f_2^L+g_2^M\xi_2(n)(1-f_2^L)][u_2(n)-v_2(n)]^2\\=&-[1-W_1-\Omega_1-g_1^M\xi_1(n)(2-f_1^L)][p_1(n)-q_1(n)]^2-\\&[1-W_2-\Omega_2-g_2^M\xi_2(n)(2-f_2^L)][p_2(n)-q_2(n)]^2-\\&[2f_1^L+f_1^{L2}-g_1^M\xi_1(n)(1-f_1^L)-W_3][u_1(n)-v_1(n)]^2-\\&[2f_2^L+f_2^{L2}-g_2^M\xi_2(n)(1-f_2^L)-\Omega_3][u_2(n)-v_2(n)]^2\\\leq&-\Theta\sum_{i=1}^{2}\left\{(u_i(n)-v_i(n))^2+(p_i(n)-q_i(n))^2\right\}\\=&-\Theta V(n),\end{aligned}$$

where $\Theta=\min\{\Theta_1,\Theta_2,\Theta_3,\Theta_4\}$. That is, there exists a positive constant $0<\Theta<1$ such that $\Delta_{(24)}(n)\leq-\Theta V(n)$. From $0<\Theta<1$, condition (3) of Lemma 2.4 is satisfied. Hence, from Lemma 2.4, there exists a unique uniformly asymptotically stable almost periodic solution $X(n)=(x_1(n),x_2(n),u_1(n),u_2(n))$ of system (4.10) which is bounded by S^* for all $n\in\mathbb{Z}^+$, which means that there exists a uniqueness and global attraction of the almost periodic solution $X(n)=(x_1(n),x_2(n),u_1(n),u_2(n))^T$ of system (1.2) which is bounded by Ω for all $n\in\mathbb{Z}^+$. This completes the proof. □

5. Numerical example

In this section, we will present an example to illustrate the effectiveness of our theoretical results.

Example 5.1 Consider the following discrete competition system with feedback controls:

$$\begin{cases}x_1(n+1)=x_1(n)\exp\{0.7+0.03\sin(\sqrt{2}n)-(0.6-0.02\cos(\sqrt{2}n))x_1(n)-\\\qquad(0.02+0.01\sin(\sqrt{2}n))x_2(n)-(0.004-0.002\cos(\sqrt{2}n))x_1^2(n)-\\\qquad(0.004+0.001\sin(\sqrt{2}n))u_1(n)\},\\x_2(n+1)=x_2(n)\exp\{0.8-0.2\cos(\sqrt{2}n)-(0.8+0.06\cos(\sqrt{2}n))x_2(n)-\\\qquad(0.06+0.002\cos(\sqrt{2}n))x_1(n)-(0.02+0.001\sin(\sqrt{2}n))x_2^2(n)-\\\qquad(0.007+0.0001\sin(\sqrt{2}n))u_2(n)\},\\\Delta u_1(n)=0.002+0.0001\sin(\sqrt{3}n)-(0.4+0.001\cos(\sqrt{2}n))u_1(n)+\\\qquad(0.009+0.004\sin(\sqrt{2}n))x_1(n),\\\Delta u_2(n)=0.006+0.0002\sin(\sqrt{3}n)-(0.45+0.003\sin(\sqrt{2}n))u_2(n)+\\\qquad(0.004+0.0004\cos(\sqrt{2}n))x_2(n),n\in\mathbb{Z}^+.\end{cases} \tag{5.1}$$

Then system (5.1) is persistence and has a unique uniformly asymptotically stable almost periodic sequence solution.

Proof It is easy to see that $\{a_i(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}, \{e_i(n)\}, \{f_i(n)\}, \{g_i(n)\}$, and $\{h_i(n)\}$ for $i = 1, 2$ are bounded nonnegative almost periodic sequences. By calculation of Matlab software, we obtain

$$\begin{aligned}
&x_{1*} = 0.4651, \quad x_1^* = 1.6667, \quad x_{2*} = 0.7222, \quad x_2^* = 1.1748,\\
&u_{1*} = 0.0105, \quad u_1^* = 0.0596, \quad u_2^* = 0.0343, \quad u_{2*} = 0.0146,\\
&a_1^l - c_1^M x_2^* - e_1^M u_1^* = 0.3645 > 0,\\
&a_2^l - c_2^M x_1^* - e_2^M u_2^* = 0.5864 > 0,\\
&\Theta_1 \approx 0.2914, \ \Theta_2 \approx 0.7851, \ \Theta_3 \approx 0.5887, \ \Theta_4 \approx 0.6851,\\
&\Theta = \min\{\Theta_1, \Theta_3, \Theta_3, \Theta_4\} = 0.2914.
\end{aligned}$$

Then $0 < \Theta < 1$. So we can see that all conditions of Theorem 4.2 hold. According to Theorem 4.2, system (5.1) has a unique global attraction of the almost periodic solution which is bounded by Ω for all $n \in \mathbb{Z}^+$. In fact, by simulations, at least two trajectories with different initial sates have been tracked, and their dynamics are illustrated in Figure 1, which are confirmed by our theory. Figure 2 is dynamical behavior of system (5.1) with different initial state.

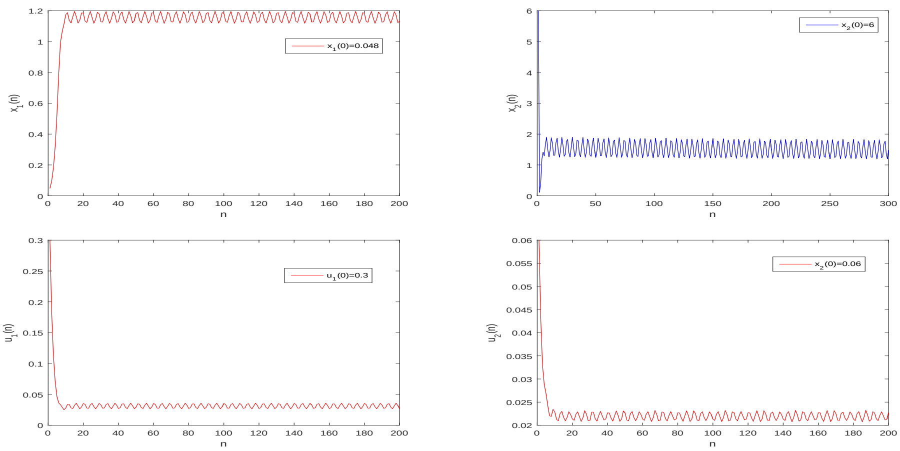

Figure 1 Time response of the states $x_1(t)$, $x_2(t)$ $u_1(t)$ and $u_2(t)$ of system (5.1)

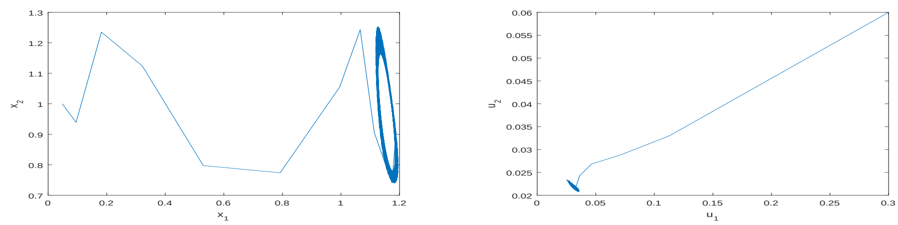

Figure 2 Dynamical behavior of system (5.1): two-dimensional phase portrait

Acknowledgements We thank the referees for their time and comments.

References

[1] K. GOPALSAMY, Peixuan WEN. *Feedback regulation of logistic growth.* Internat. J. Math. Math. Sci., 1993, **16**(1): 177–192.

[2] Fengde CHEN. *The permanence and global attractivity of Lotka-Volterra competition system with feedback controls.* Nonlinear Anal. Real World Appl., 2006, **7**(1): 133–143.

[3] K. GOPALSAMY. *Stability and Oscillations in Delay Differential Equations of Population Dynamics.* Kluwer Academic Publishers Group, Dordrecht, 1992.

[4] Zhijun LIU, Meng FAN, Lansun CHEN. *Globally asymptotic stability in two periodic delayed competitive systems.* Appl. Math. Comput., 2008, **197**(1): 271–287.

[5] Zhong LI, Maoan HAN, Fengde CHEN. *Influence of feedback controls on an autonomous Lotka-Volterra competitive system with infinite delays.* Nonlinear Anal. Real World Appl., 2013, **14**(1): 402–413.

[6] Zhijun LIU. *Persistence and periodic solution in two species competitive system with feedback controls.* J. Biomath., 2002, **17**(2): 251–255.

[7] Zhouhong LI. *Persistence and almost periodic solutions for a discrete ratio-dependent Leslie system with feedback control.* Adv. Difference Equ., 2014, **2014**(1): 1–19.

[8] Chengying NIU, Xiaoxing CHEN. *Almost periodic sequence solutions of a discrete Lotka-Volterra competitive system with feedback control.* Nonlinear Anal. Real World Appl., 2009, **10**(5): 3152–3161.

[9] G. P. SAMANTA. *A two-species competitive system under the influence of toxic substances.* Appl. Math. Comput., 2010, **216**: 291–299.

[10] Ronghua TAN, Weifang LIU, Qinglong WANG, et al. *Uniformly asymptotic stability of almost periodic solutions for a competitive system with impulsive perturbations.* Adv. Difference Equ., 2014, **2014**(2): 1–11.

[11] Ronghua TAN, Zuxiong LI, Qinglong WANG, et al. *Positive periodic solutions of a periodic discrete competitive system subject to feedback controls.* J. Appl. Math. 2014, Art. ID 927626, 10 pp.

[12] Zheng WANG, Yongkun LI. *Almost periodic solutions of a discrete mutualism model with feedback controls.* Discrete Dyn. Nat. Soc., 2010, Art. ID 286031, 18 pp.

[13] Tianwei ZHANG, Yongkun LI, Yuan YE. *Persistence and almost periodic solutions for a discrete fishing model with feedback control.* Commun. Nonlinear Sci. Numer. Simul., 2011, **16**(3): 1564–1573.

[14] Shunian ZHANG, Guang ZHENG. *Almost periodic solutions of delay difference systems.* Appl. Math. Comput., 2002, **131**(2-3): 497–516.

[15] Tianwei ZHANG, Xiaorong GAN. *Almost periodic solutions for a discrete fishing model with feedback control and time delays.* Commun. Nonlinear Sci. Numer. Simul., 2014, **19**(1): 150–163.

[16] D. WEXLER. *Solutions périodiques et presque-périodiques des systémes d'équations différentielles aux impulsions.* Rev. Roumaine Math. Pures Appl., 1965, **10**: 1163–1199. (in French)