Journal of Mathematical Research with Applications
May, 2018, Vol. 38, No. 3, pp. 293–302
DOI:10.3770/j.issn:2095-2651.2018.03.008
Http://jmre.dlut.edu.cn

A New Location Invariant Moment-Type Estimator and Its Asymptotic Normality

Weiqi LIU[1,3,*], **Shanshan LIANG**[2]

1. *Research Center for Management and Decision Making, Shanxi University, Shanxi* 030006*, P. R. China;*
2. *School of Mathematical Sciences, Shanxi University, Shanxi* 030006*, P. R. China;*
3. *Faculty of Finance and Banking, Shanxi University of Finance and Economics, Shanxi* 030006*, P. R. China*

Abstract The moment estimator has been widely used in extreme value theory in order to estimate the extreme value index, however it is not location invariant. In this paper, based on the moment-type estimator, we propose a new location invariant moment-type estimator, and discuss its asymptotic normality under the second order regular variation. Finally, a simulation is presented to compare this new estimator with another location invariant moment-type estimator $\hat{\gamma}_n^M(k_0,k)$ proposed by Ling, which indicates that the new estimator has good performances.

Keywords extreme value index; moment-type estimator; regular variation; location invariant; asymptotic normality

MR(2010) Subject Classification 62F12; 62G32; 65C05

1. Introduction

Suppose $X_1, X_2, \ldots, X_n$ are independent identically distributed (i.i.d.) random variables with common distribution function (d.f.) F, and let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the associated increasing order statistics. If F satisfies (1.1), we say that F belongs to the domain of attraction of an extreme value distribution G_γ, denoted by $F \in D(G_\gamma)$, i.e., there exist real numbers $a_n > 0$ and $b_n \in R$ such that

$$P(\frac{X_{n,n}-b_n}{a_n} \le x) = F^n(a_n x + b_n) \to G_\gamma(x), \tag{1.1}$$

as $n \to \infty$, where

$$G_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-\frac{1}{\gamma}}\}, & \text{for } 1+\gamma x > 0, \text{ if } \gamma \ne 0; \\ \exp\{-\exp(-x)\}, & \text{for } x \in R, \text{ if } \gamma = 0. \end{cases}$$

The shape parameter γ plays here a central role, which measures the weight of the right tail function defined as $\overline{F} = 1 - F$. As a result, in applications of extreme value theory, dealing with economic problems, such as risk management [1] and currency issue [2], the problem of

Received March 16, 2017; Accepted March 1, 2018
Supported by the National Social Science Foundation of China (Grant No. 15BJY164).
* Corresponding author
E-mail address: liuwq@sxu.edu.cn (Weiqi LIU); liangss0226@foxmail.com (Shanshan LIANG)

estimating the tail index has received much attention by many researchers in statistics. There are several common estimators of extreme value index.

When $\gamma > 0$, the most famous estimator is the Hill [3] estimator:

$$\hat{\gamma}_n^H(k) = \frac{1}{k}\sum_{i=0}^{k-1}(\log X_{n-i,n} - \log X_{n-k,n}) \tag{1.2}$$

with an intermediate integer sequence $k = k_n$ satisfying $k \longrightarrow \infty$, $k/n \longrightarrow 0$ as $n \longrightarrow \infty$. For general $\gamma \in R$, a well-known estimator is the moment estimator proposed by Dekkers et al.[4]

$$\hat{\gamma}_n^M(k) = M_n^{(1)} + 1 - \frac{1}{2}(1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}})^{-1}, \tag{1.3}$$

where

$$M_n^{(j)} = \frac{1}{k}\sum_{i=0}^{k-1}(\log X_{n-i,n} - \log X_{n-k,n})^j, \quad j = 1, 2.$$

Note that $M_n^{(1)}$ is Hill estimator. The weak consistency and asymptotic normality of $\hat{\gamma}_n^M(k)$ are proved. Simulation and empirical analysis show that it is sensitive to the linear transform of data set and the choice of threshold k, thus location invariant to estimator is a basic requirement.

The earliest location invariant estimator is Pickands [5] estimator defined as:

$$\hat{\gamma}_n^p(k) = \frac{1}{\log 2}\log\frac{X_{n-k+1,n} - X_{n-2k+1,n}}{X_{n-2k+1,n} - X_{n-4k+1,n}}, \tag{1.4}$$

although it is location invariant, it has a significant larger variance. Alves [6] proposed a location invariant Hill-type estimator:

$$\hat{\gamma}_n^H(k_0, k) = \frac{1}{k_0}\sum_{i=0}^{k_0-1}\log\frac{X_{n-i,n} - X_{n-k,n}}{X_{n-k_0,n} - X_{n-k,n}}, \quad \gamma > 0, \tag{1.5}$$

here $k \to \infty, k_0 \to \infty, k/n \to 0, k_0/k \to 0$ as $n \to \infty$. Ling et al. [7] transformed (1.3) into a location invariant estimator as follows:

$$\hat{\gamma}_n^M(k_0, k) = M_n^{(1)}(k_0, k) + 1 - \frac{1}{2}(1 - \frac{(M_n^{(1)}(k_0, k))^2}{M_n^{(2)}(k_0, k)})^{-1}, \tag{1.6}$$

where

$$M_n^{(j)}(k_0, k) = \frac{1}{k_0}\sum_{i=0}^{k_0-1}(\log\frac{X_{n-i,n} - X_{n-k,n}}{X_{n-k_0,n} - X_{n-k,n}})^j, \quad j = 1, 2,$$

and derived its asymptotic normality in [8]. Wang [9] proposed a new moment-type estimator

$$\hat{\gamma}_n^{(1)}(k) = (\frac{1}{2}M_n^{(2)})^{\frac{1}{2}} + 1 - \frac{1}{2}(1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}})^{-1}. \tag{1.7}$$

Combining with $\hat{\gamma}_n^M(k_0, k)$ in (1.6) and $\hat{\gamma}_n^{(2)}(k)$ in (1.8) proposed by Ferreira [10] as follows:

$$\hat{\gamma}_n^{(2)}(k) = (\frac{1}{2}M_n^{(2)})^{\frac{1}{2}} + 1 - \frac{2}{3}(1 - \frac{M_n^{(1)}M_n^{(2)}}{M_n^{(3)}})^{-1}, \tag{1.8}$$

we propose a new location invariant moment-type estimator defined as:

$$\hat{\gamma}_n^N(k_0,k) = M_n^{(1)}(k_0,k) + 1 - \frac{2}{3}(1 - \frac{M_n^{(1)}(k_0,k)M_n^{(2)}(k_0,k)}{M_n^{(3)}(k_0,k)})^{-1}, \tag{1.9}$$

and discuss asymptotic properties under second order regular variation. And use Monte-Carlo method to compare it with $\hat{\gamma}_n^M(k_0,k)$ defined in (1.6) above.

2. The main results

Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. random variables with common Pareto distribution $F_Y(y) = 1 - y^{-1}, y \geq 1$ and $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$ be the associated increasing order statistics.

Let $E_1, E_2, \ldots, E_n$ be i.i.d. random variables with common standard exponential distribution $F_X(x) = 1 - e^x, x \geq 0$ and $E_{1,n} \leq E_{2,n} \leq \cdots \leq E_{n,n}$ be the associated increasing order statistics.

Let '$\stackrel{d}{=}$','$\stackrel{d}{\longrightarrow}$' and '$\stackrel{p}{\longrightarrow}$' represent identically distributed, convergence in distribution and convergence in probability, respectively.

We assume that $F \in D(G_\gamma)$, which is equivalent to suppose that $U = (1/(1-F))^{\leftarrow}$ is regularly varying with index γ ($U \in RV_\gamma$), notice that the following relations are true:

$$\begin{aligned} &X_{i,n} \stackrel{d}{=} U(Y_{i,n}), \\ &\{\frac{Y_{n-i,n}}{Y_{n-k_0,n}}\}_{i=0}^{k_0-1} \stackrel{d}{=} \{Y_{k_0-i,k_0}\}_{i=0}^{k_0-1}, \\ &\log Y_{i,n} \stackrel{d}{=} E_{i,n}, \end{aligned} \tag{2.1}$$

where sequences k and k_0 satisfy

$$\begin{gathered} k = k_n = o(n), \quad k_0 = o(k_n), \quad k_n \to \infty, \quad k_0 \to \infty, \quad \frac{k_0}{k}, \frac{k}{n} \to 0, \\ k_0/\log k, k/\log n \to \infty, \quad n \to \infty, \\ \frac{k}{n} Y_{n-k,n} \stackrel{p}{\longrightarrow} 1. \end{gathered} \tag{2.2}$$

The following conditions are equivalent

$$F \in D(G_\gamma) \Longleftrightarrow 1 - F \in RV_{-1/\gamma} \Longleftrightarrow U \in RV_\gamma, \quad \gamma > 0, \tag{2.3}$$

if and only if there exists a function $a(t) > 0$, we say $F \in D(G_\gamma)$, such that

$$\frac{U(tx) - U(t)}{a(t)} \to \frac{x^\gamma - 1}{\gamma}, \quad \text{if } \gamma = 0, \; \frac{U(tx) - U(t)}{a(t)} \to \log x \tag{2.4}$$

as $t \to \infty$, where $U = (\frac{1}{1-F})^{\leftarrow}$, as the inverse function of $\frac{1}{1-F}$ (see [11]). It is equivalent to

$$\frac{U(tx) - U(t)}{U(ty) - U(t)} \to \frac{x^\gamma - 1}{y^\gamma - 1}, \quad \text{if } \gamma = 0, \; \frac{U(tx) - U(t)}{U(ty) - U(t)} \to \frac{\log x}{\log y}. \tag{2.5}$$

We say $\{k(n)\}$ is intermediate rank sequence if it satisfies

$$k(n) \to 0, \; \frac{k(n)}{n} \to 0, \quad n \to \infty. \tag{2.6}$$

If there exist function $a(t) > 0$ and $A(t) > 0$ satisfying $A(t) \to 0, t \to \infty$ such that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - D_\gamma(x)}{A(t)} = H(x), \tag{2.7}$$

where

$$D_\gamma(x) = \begin{cases} \frac{x^\gamma - 1}{\gamma}, & \gamma \neq 0; \\ \ln x, & \gamma = 0. \end{cases}$$

$H(x)$ is not times of $D_\gamma(x)$, we say that $U(t)$ satisfies second order regular variation. From de Haan and Ferreira [12], we know

$$H(x) = \begin{cases} \frac{1}{\rho}(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}), & \rho < 0,\ \gamma \neq 0; \\ \frac{1}{\rho}(\frac{x^\rho - 1}{\rho} - \ln x), & \rho < 0,\ \gamma = 0; \\ \frac{1}{\gamma}(x^\gamma \ln x - \frac{x^\gamma - 1}{\gamma}), & \rho = 0,\ \gamma \neq 0; \\ \frac{1}{2}(\ln x)^2, & \rho = \gamma = 0. \end{cases} \tag{2.8}$$

3. Asymptotic property of estimator $\hat{\gamma}_n^N(k_0, k)$

Lemma 3.1 *If second order regular variation (2.5) holds when $\rho < 0$, then for arbitrary $\varepsilon, \delta > 0$, there exists $t_0 = t_0(\varepsilon, \delta) > 0$ satisfying*

$$\Big|\frac{\ln \frac{U(tx)-U(t)}{a(t)} - \ln D_\gamma(x)}{A(t)} - B_{\gamma,\rho}(x)\Big| \leq \varepsilon(1 + \frac{x^{\gamma+\rho+\delta}}{D_\gamma(x)}) \tag{3.1}$$

when $t > t_0, x > 1$, where

$$B_{\gamma,\rho}(x) = \begin{cases} \frac{\gamma}{\gamma+\rho} \frac{x^{\gamma+\rho}-1}{x^\gamma - 1}, & \gamma+\rho \neq 0; \\ \frac{\gamma}{x^\gamma - 1} \ln x, & \gamma + \rho = 0. \end{cases}$$

.

The proof easily follows from [13, Lemma 4.1].

Lemma 3.2 *If second order regular variation (2.5) holds when $\rho < 0$, then*

(i) For all $x > 1, y > 1$

$$\lim_{t\to\infty} \frac{\ln \frac{U(tx)-U(t)}{U(ty)-U(t)} - \ln \frac{D_\gamma(x)}{D_\gamma(y)}}{A(t)} = F_{\gamma,\rho}(x, y) \tag{3.2}$$

holds, where

$$F_{\gamma,\rho}(x,y) = \begin{cases} \frac{1}{D_\gamma(x)} \frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{1}{D_\gamma(y)} \frac{y^{\gamma+\rho}-1}{\gamma+\rho}, & \gamma+\rho \neq 0; \\ \frac{1}{D_\gamma(x)} \ln x - \frac{1}{D_\gamma(y)} \ln y, & \gamma+\rho = 0. \end{cases}$$

(ii) For arbitrary $\varepsilon, \delta > 0$, there exists $t_0 = t_0(\varepsilon, \delta) > 0$ satisfying

$$\Big|\frac{\ln \frac{U(tx)-U(t)}{U(ty)-U(t)} - \ln \frac{D_\gamma(x)}{D_\gamma(y)}}{A(t)} - F_{\gamma,\rho}(x,y)\Big| \leq T_{\gamma,\rho}(x,y), \tag{3.3}$$

when $t > t_0, x > y > 1$, where

$$T_{\gamma,\rho}(x,y) = \varepsilon(2 + \frac{x^{\gamma+\rho+\delta}}{D_\gamma(x)} + \frac{y^{\gamma+\rho+\delta}}{D_\gamma(y)}).$$

The proof easily follows from [13, Lemma 4.1].

Theorem 3.3 *If second order regular variation (2.5) holds, and $k(n), k_0(n)$ are intermediate rank sequences, then for sufficiently large n,*

(i) When $\gamma > 0$,

(a) if $\gamma + \rho \neq 0$, then

$$\begin{aligned}\hat{\gamma}_n^N(k_0,k) =&\gamma + \frac{1}{12}P_3 - \frac{1}{4}P_2 + (\gamma - \frac{1}{2})P_1 + d(\frac{k}{k_0})^{-\gamma} + tA(\frac{n}{k})(\frac{k}{k_0})^{\rho} + \\ & o_p((\frac{k_0}{k})^{\gamma}) + o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho}) + o_p(\frac{1}{\sqrt{k_0}}),\end{aligned} \tag{3.4}$$

(b) if $\gamma + \rho = 0$, then

$$\begin{aligned}\hat{\gamma}_n^N(k_0,k) =&\gamma + \frac{1}{12}P_3 - \frac{1}{4}P_2 + (\gamma - \frac{1}{2})P_1 + d(\frac{k}{k_0})^{-\gamma} + tA(\frac{n}{k})(\frac{k}{k_0})^{-\gamma}\ln(\frac{k_0}{k}) + \\ & o_p((\frac{k_0}{k})^{\gamma}) + o_p(A(\frac{n}{k})(\frac{k_0}{k})^{\gamma}\ln(\frac{k_0}{k})) + o_p(\frac{1}{\sqrt{k_0}}),\end{aligned} \tag{3.5}$$

where $P_j = \frac{1}{k_0}\sum_{i=1}^{k_0}(\ln Y_i)^j - \mu_j$, $\mu_j = \Gamma(j+1), j = 1,2,3$ and d, t are constants only related to α, γ, ρ;

(ii) When $\gamma = 0$,

$$\begin{aligned}\hat{\gamma}_n^N(k_0,k) =&\frac{1}{12}P_3 - \frac{1}{2}P_1 - \frac{1}{4}P_2 + t(\frac{k}{k_0})^{\rho}A(\frac{n}{k}) + (\ln(\frac{k}{k_0}))^{-1} + \\ & o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho}) + o_p(\frac{1}{\sqrt{k_0}}),\end{aligned} \tag{3.6}$$

where μ_j, P_j are defined as above;

(iii) When $\gamma < 0$,

$$\begin{aligned}\hat{\gamma}_n^N(k_0,k) =&\gamma - \frac{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}{6\gamma^3}(\frac{1-3\gamma}{2}P_3 - \frac{3\gamma^2(1-\gamma)}{(1-2\gamma)(1-3\gamma)}P_1 + \frac{3\gamma}{2}P_2) + \\ & tA(\frac{n}{k})(\frac{k}{k_0})^{\rho} - \frac{\gamma}{1-\gamma}(\frac{k}{k_0})^{\gamma} + o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho}) + o_p(\frac{1}{\sqrt{k_0}}),\end{aligned} \tag{3.7}$$

where $P_j = \frac{1}{k_0}\sum_{i=1}^{k_0}(1 - Y_i^{\gamma})^j - \mu_j, \mu_j = \frac{(-1)^j j!\gamma^j}{(1-\gamma)\cdots(1-j\gamma)}$, $j = 1,2,3$ and d, t are constants only related to α, γ, ρ.

Proof (i) If $\gamma < 0$, when $n \to \infty$, $(\frac{Y_{n-i,n}}{Y_{n-k,n}})^{\gamma} < (\frac{Y_{n-k_0,n}}{Y_{n-k,n}})^{\gamma} \xrightarrow{p} 0$ satisfying for all $i = 1, 2, \ldots, k_0$. Thus, from (2.1) and Lemma 3.2, one has

$$\begin{aligned}M_n^{(1)}(k_0,k) &\stackrel{d}{=} \frac{1}{k_0}\sum_{i=1}^{k_0}\big(\ln\frac{U(Y_{n-i+1,n}) - U(Y_{n-k,n})}{U(Y_{n-k_0,n}) - U(Y_{n-k,n})}\big) \\ &= \frac{1}{k_0}\sum_{i=1}^{k_0}\Big(\ln\frac{(\frac{Y_{n-i+1,n}}{Y_{n-k,n}})^{\gamma} - 1}{(\frac{Y_{n-k_0,n}}{Y_{n-k,n}})^{\gamma} - 1} + \frac{\gamma}{\gamma+\rho}A(\frac{n}{k})(\frac{(\frac{Y_{n-i+1,n}}{Y_{n-k,n}})^{\gamma+\rho} - 1}{(\frac{Y_{n-i+1,n}}{Y_{n-k,n}})^{\gamma} - 1} - \frac{(\frac{Y_{n-k_0,n}}{Y_{n-k,n}})^{\gamma+\rho} - 1}{(\frac{Y_{n-k_0,n}}{Y_{n-k,n}})^{\gamma} - 1})(1 + o_p(1))\Big) \\ &= (\frac{k}{k_0})^{\gamma}\frac{1}{k_0}\sum_{i=1}^{k_0}\big(1 - (\frac{Y_{n-i+1,n}}{Y_{n-k_0,n}})^{\gamma}\big)(1 + o_p(1)) +\end{aligned}$$

$$\frac{\gamma}{\gamma+\rho}A(\frac{n}{k})(\frac{k}{k_0})^{\gamma+\rho}\frac{1}{k_0}\sum_{i=1}^{k_0}(1-(\frac{Y_{n-i+1,n}}{Y_{n-k_0,n}})^{\gamma+\rho})(1+o_p(1))$$
$$=(\frac{k}{k_0})^{\gamma}\Big\{\frac{1}{k_0}\sum_{i=1}^{k_0}(1-Y_i^{\gamma})+\frac{\gamma}{\gamma+\rho}A(\frac{n}{k})(\frac{k}{k_0})^{\rho}\frac{1}{k_0}\sum_{i=1}^{k_0}(1-Y_i^{\gamma+\rho})+$$
$$o_p(\frac{1}{\sqrt{k_0}})+o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})\Big\}.$$

Take advantage of Large Number Theorem, when $n\to\infty$,

$$\frac{1}{k_0}\sum_{i=1}^{k_0}(1-Y_i^{\gamma})\xrightarrow{p}\frac{\gamma}{\gamma-1},\quad \frac{1}{k_0}\sum_{i=1}^{k_0}(1-Y_i^{\gamma+\rho})\xrightarrow{p}\frac{\gamma+\rho}{\gamma+\rho-1}.$$

So

$$M_n^{(j)}=(\frac{k}{k_0})^{j\gamma}\{\frac{(-1)^j j!\gamma^j}{(1-\gamma)\cdots(1-j\gamma)}+P_j+t_jA(\frac{n}{k})(\frac{k}{k_0})^{\rho}+o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})+o_p(\frac{1}{\sqrt{k_0}})\}$$

satisfying for all $j=1,2,3$, where

$$t_1=\frac{\gamma}{\gamma+\rho-1},t_2=\frac{2\gamma}{\gamma+\rho}(1-\frac{1}{1-\gamma}-\frac{1}{1-\gamma-\rho}+\frac{1}{1-2\gamma-\rho}),$$
$$t_3=\frac{3\gamma}{\gamma+\rho}(\frac{2\gamma^2}{(1-\gamma)(1-2\gamma)}-\frac{1}{1-\gamma-\rho}+\frac{2}{1-2\gamma-\rho}-\frac{1}{1-3\gamma-\rho}).$$

Thus,

$$M_n^{(1)}(k_0,k)=(\frac{k}{k_0})^{\gamma}\{\frac{-\gamma}{1-\gamma}+\frac{\gamma}{\gamma+\rho-1}A(\frac{n}{k})(\frac{k}{k_0})^{\rho}+o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})+o_p(\frac{1}{\sqrt{k_0}})\},$$

$$M_n^{(3)}(k_0,k)-M_n^{(1)}(k_0,k)M_n^{(2)}(k_0,k)$$
$$=(\frac{k}{k_0})^{3\gamma}\Big\{\frac{-4\gamma^3}{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}+P_3-\frac{2\gamma^2}{(1-2\gamma)(1-3\gamma)}P_1-\frac{\gamma}{\gamma-1}P_2+t_4A(\frac{n}{k})(\frac{k}{k_0})^{\rho}+$$
$$o_p(A(\frac{n}{k})\left(\frac{k}{k_0}\right)^{\rho})+o_p(\frac{1}{\sqrt{k_0}})\Big\},$$

where $t_4=t_3-\frac{\gamma}{\gamma-1}t_2-\frac{2\gamma^2}{(1-\gamma)(1-2\gamma)}$. By Taylor Expansion, one has

$$(M_n^{(3)}(k_0,k)-M_n^{(1)}(k_0,k)M_n^{(2)}(k_0,k))^{-1}$$
$$=(\frac{k}{k_0})^{-3\gamma}\Big\{\frac{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}{-4\gamma^3}-(\frac{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}{4\gamma^3})^2$$
$$\big(P_3-\frac{2\gamma^2}{(1-2\gamma)(1-3\gamma)}P_1-\frac{\gamma}{\gamma-1}P_2\big)+t_5A(\frac{n}{k})(\frac{k}{k_0})^{\rho}+o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})+o_p(\frac{1}{\sqrt{k_0}})\Big\},$$

where $t_5=-(\frac{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}{4\gamma^3})^2t_4$, so

$$\hat{\gamma}_n^N(k_0,k)=\gamma-\frac{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}{6\gamma^3}\big(\frac{1-3\gamma}{2}P_3-\frac{3\gamma^2(1-\gamma)}{(1-2\gamma)(1-3\gamma)}P_1+\frac{3\gamma}{2}P_2\big)+$$
$$tA(\frac{n}{k})(\frac{k}{k_0})^{\rho}-\frac{\gamma}{1-\gamma}(\frac{k}{k_0})^{\gamma}+o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})+o_p(\frac{1}{\sqrt{k_0}}),$$

$t=-\frac{(1-\gamma)^2(1-2\gamma)(1-3\gamma)}{4\gamma^3}t_3-\frac{6\gamma^3}{(1-\gamma)(1-2\gamma)(1-3\gamma)}t_5$.

(ii) If $\gamma = 0$, from second order regular variation, we have

$$\frac{U(tx)-U(t)}{U(ty)-U(t)} = 1 + \frac{\ln\frac{x}{y} + A(t)y^{\rho}\frac{(\frac{x}{y})^{\rho}-1}{\rho}(1+o_p(1))}{\ln y + A(t)\frac{y^{\rho}-1}{\rho}(1+o_p(1))}.$$

Note that $(\frac{Y_{n-i+1,n}}{Y_{n-k,n}})^{\rho} < (\frac{Y_{n-k_0,n}}{Y_{n-k,n}})^{\rho} \xrightarrow{p} 0$ holds for all $i = 1, 2, \ldots, k_0$ as $n \to \infty$, if we use $\frac{Y_{n-i+1,n}}{Y_{n-k,n}}$ and $\frac{Y_{n-k_0,n}}{Y_{n-k,n}}$ to replace x and y, respectively, then

$$\begin{aligned}
M_n^{(j)}(k_0,k) &\overset{d}{=} \frac{1}{k_0}\sum_{i=1}^{k_0}\Big(\ln\frac{U(Y_{n-i+1,n})-U(Y_{n-k,n})}{U(Y_{n-k_0,n})-U(Y_{n-k,n})}\Big)^j \\
&= \frac{1}{k_0}\sum_{i=1}^{k_0}\Bigg(\ln\Big(1+\frac{\ln\frac{\frac{Y_{n-i+1,n}}{Y_{n-k,n}}}{\frac{Y_{n-k_0,n}}{Y_{n-k,n}}} + A(\frac{n}{k})(\frac{Y_{n-k_0,n}}{Y_{n-k,n}})^{\rho}\frac{\Big(\frac{\frac{Y_{n-i+1,n}}{Y_{n-k,n}}}{\frac{Y_{n-k_0,n}}{Y_{n-k,n}}}\Big)^{\rho}-1}{\rho}(1+o_p(1))}{\ln\frac{Y_{n-k_0,n}}{Y_{n-k,n}} + A(\frac{n}{k})\frac{\Big(\frac{Y_{n-k_0,n}}{Y_{n-k,n}}\Big)^{\rho}-1}{\rho}(1+o_p(1))}\Big)\Bigg)^j \\
&\overset{d}{=} \frac{1}{k_0}\sum_{i=1}^{k_0}\Bigg(\ln\Big(1+\frac{\ln Y_{k_0-i+1,k_0} + A(\frac{n}{k})(Y_{k-k_0,k})^{\rho}\frac{(Y_{k_0-i+1,k_0})^{\rho}-1}{\rho}(1-o_p(1))}{\ln Y_{k-k_0,k} + A(\frac{n}{k})\frac{(Y_{k-k_0,k})^{\rho}-1}{\rho}(1+o_p(1))}\Big)\Bigg)^j \\
&\overset{d}{=} \frac{1}{k_0}\sum_{i=1}^{k_0}\Bigg(\ln(1+\frac{\ln Y_{k_0-i+1,k_0} + A(\frac{n}{k})(\frac{k}{k_0})^{\rho}\frac{(Y_{k_0-i+1,k_0})^{\rho}-1}{\rho}(1+o_p(1))}{\ln\frac{k}{k_0} + A(\frac{n}{k})\frac{(\frac{k}{k_0})^{\rho}-1}{\rho}(1+o_p(1))})\Bigg)^j \\
&= (\ln\frac{k}{k_0})^{-j}\frac{1}{k_0}\sum_{i=1}^{k_0}\Big(\ln Y_{k_0-i+1,k_0} + A(\frac{n}{k})\big(\frac{k}{k_0}\big)^{\rho}\frac{(Y_{k_0-i+1,k_0})^{\rho}-1}{\rho}(1+o_p(1))\Big)^j.
\end{aligned}$$

Thus,

$$\begin{aligned}
M_n^{(1)}(k_0,k) &\overset{d}{=} (\ln\frac{k}{k_0})^{-1}\big\{1+P_1+\frac{1}{1-\rho}A(\frac{n}{k})(\frac{k}{k_0})^{\rho} + o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})\big\}, \\
M_n^{(2)}(k_0,k) &\overset{d}{=} \big(\ln\frac{k}{k_0}\big)^{-2}\big\{2+P_2+\frac{2(2-\rho)}{(1-\rho)^2}A(\frac{n}{k})(\frac{k}{k_0})^{\rho} + o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})\big\}, \\
M_n^{(3)}(k_0,k) &\overset{d}{=} \big(\ln\frac{k}{k_0}\big)^{-3}\{6+P_3+\frac{18-6(1-\rho)^3}{\rho(1-\rho)^3}A(\frac{n}{k})(\frac{k}{k_0})^{\rho} + o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho})\},
\end{aligned}$$

where $P_j = \frac{1}{k_0}\sum_{i=1}^{k_0}[(\ln Y_i)^j - \Gamma(j+1)]$, $j = 1, 2, 3$. Thus,

$$\begin{aligned}
&M_n^{(3)}(k_0,k) - M_n^{(1)}(k_0,k)M_n^{(2)}(k_0,k) \\
&\quad= (\ln\frac{k}{k_0})^{-3}\{4+P_3-2P_1-P_2+t_1A(\frac{n}{k})(\frac{k}{k_0})^{\rho} + o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho}) + o_p(\frac{1}{\sqrt{k_0}})\},
\end{aligned}$$

$t_1 = \frac{18-6(1-\rho)^3}{\rho(1-\rho)^3} - \frac{2(2-\rho)}{(1-\rho)^2} - \frac{2}{1-\rho}$. Then make use of Taylor expansion, we know

$$\begin{aligned}
&\hat{\gamma}_n^N(k_0,k) \\
&\quad= \frac{1}{12}P_3 - \frac{1}{2}P_1 - \frac{1}{4}P_2 + tA(\frac{n}{k})(\frac{k}{k_0})^{\rho} + (\ln(\frac{k}{k_0}))^{-1} + o_p(A(\frac{n}{k})(\frac{k}{k_0})^{\rho}) + o_p(\frac{1}{\sqrt{k_0}})
\end{aligned}$$

where $t = -\frac{6-2(1-\rho)^3}{2\rho(1-\rho)^3} + \frac{1}{4}t_1$;

(iii) If $\gamma > 0$ and $\gamma + \rho \neq 0$, from Li [13] one has

$$M_n^{(\alpha)}(k_0,k) = \gamma^{\alpha}\Gamma(\alpha+1) + \gamma^{\alpha}P_{\alpha} + d_{\alpha}(\frac{k_0}{k})^{\gamma}(1+o_p(1)) + c_{\alpha}A(\frac{n}{k})(\frac{k_0}{k})^{-\rho}(1+o_p(1)),$$

where P_j is defined as above, and $d_\alpha = \alpha\gamma^\alpha\mu_\alpha(-\gamma)$, $c_\alpha = \frac{\alpha\gamma^\alpha\rho\mu_\alpha(\rho)}{\gamma+\rho}$, $\alpha = 1, 2, 3$. Using Taylor expansion we can derive

$$\begin{aligned}\hat{\gamma}_n^N(k_0, k) =&\gamma + \frac{1}{12}P_3 - \frac{1}{4}P_2 + (\gamma - \frac{1}{2})P_1 + d(\frac{k_0}{k})^\gamma + tA(\frac{n}{k})(\frac{k}{k_0})^\rho + \\ &o_p((\frac{k_0}{k})^\gamma) + o_p(A(\frac{n}{k})(\frac{k}{k_0})^\rho) + o_p(\frac{1}{\sqrt{k_0}}),\end{aligned}$$

where d, t are constants only related to α, γ, ρ. If $\gamma + \rho = 0$, the proof is similar to above. □

Theorem 3.4 *Suppose $Q_\alpha = \sqrt{k_0}P_\alpha$, and $k_0(n)$ is intermediate rank sequence, then*

$$(Q_1, Q_2, Q_3) \xrightarrow{d} N(0, \Sigma^2), \quad n \to \infty, \tag{3.8}$$

where $N(0, \Sigma^2)$ represents three dimensional normal distribution and the mean value is zero vector, and the variance is Σ^2, further more

$$\Sigma^2 = \begin{cases} \begin{pmatrix} 1 & 4 & 18 \\ 4 & 20 & 108 \\ 18 & 108 & 684 \end{pmatrix}, & \gamma \geq 0, \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, & \gamma < 0, \end{cases}$$

where

$$a_{11} = \frac{\gamma^2}{(1-\gamma)(1-2\gamma)}, \quad a_{12} = -\frac{4\gamma^3}{(1-\gamma)^2(1-2\gamma)(1-3\gamma)},$$

$$a_{13} = \frac{18\gamma^4}{(1-\gamma)^2(1-2\gamma)(1-3\gamma)(1-4\gamma)}, \quad a_{21} = -\frac{4\gamma^3}{(1-\gamma)^2(1-2\gamma)(1-3\gamma)},$$

$$a_{22} = \frac{4\gamma^4(5-11\gamma)}{(1-\gamma)^2(1-2\gamma)^2(1-3\gamma)(1-4\gamma)}, \quad a_{23} = -\frac{36\gamma^5(3-7\gamma)}{(1-\gamma)^2(1-2\gamma)^2(1-3\gamma)(1-4\gamma)(1-5\gamma)},$$

$$a_{31} = \frac{18\gamma^4}{(1-\gamma)^2(1-2\gamma)(1-3\gamma)(1-4\gamma)}, \quad a_{32} = -\frac{36\gamma^5(3-7\gamma)}{(1-\gamma)^2(1-2\gamma)^2(1-3\gamma)(1-4\gamma)(1-5\gamma)},$$

$$a_{33} = \frac{36\gamma^6(19-105\gamma-146\gamma^2)}{(1-\gamma)^2(1-2\gamma)^2(1-3\gamma)^2(1-4\gamma)(1-5\gamma)(1-6\gamma)}.$$

Proof Let $Q = \sqrt{k_0}(aP_1 + bP_2 + cP_3)$, $f(t)$ is characteristic function of it, then for arbitrary $a, b, c \in R$, when $\gamma > 0$, from the expression of P_j and Taylor expression, we have

$$\begin{aligned}f(t) &= E\exp\{it\sqrt{k_0}(aP_1 + bP_2 + cP_3)\} \\ &= E\exp\Big\{\frac{it}{\sqrt{k_0}}\sum_{j=1}^{k_0}[a(\ln Y_j - \Gamma(2)) + b((\ln Y_j)^2 - \Gamma(3)) + c((\ln Y_j)^3 - \Gamma(4))]\Big\} \\ &= \prod_{j=1}^{k_0} E\exp\Big\{\frac{it}{\sqrt{k_0}}[a(\ln Y_j - \Gamma(2)) + b((\ln Y_j)^2 - \Gamma(3)) + c((\ln Y_j)^3 - \Gamma(4))]\Big\} \\ &= \prod_{j=1}^{k_0}\Big\{1 - \frac{t^2}{2k_0}E[a(\ln Y_j - \Gamma(2)) + b((\ln Y_j)^2 - \Gamma(3)) + c((\ln Y_j)^3 - \Gamma(4))]^2 + o(\frac{1}{k_0})\Big\}\end{aligned}$$

$$= \{1 - \frac{t^2}{2k_0}L + o(\frac{1}{k_0})\}^{k_0} \to \exp(-\frac{t^2}{2}L),$$

where

$$\begin{aligned} L &= E\big[a(\ln Y_j - \Gamma(2)) + b((\ln Y_j)^2 - \Gamma(3)) + c((\ln Y_j)^3 - \Gamma(4))\big]^2 \\ &= a^2(\Gamma(3) - \Gamma(2)^2) + b^2(\Gamma(5) - \Gamma(3)^2) + c^2(\Gamma(7) - \Gamma(4)^2) \\ &= a^2 + 20b^2 + 684c^2, \end{aligned}$$

if $a = 1$, $b = c = 0$, then $L = 1$. And that of the other part $\gamma < 0$ is omitted as its proof is similar to the first part, therefore (3.8) can be derived. $\square$

4. Numerical simulation

In this section, we present the results of simulation study intended to compare the new estimator $\hat{\gamma}_n^N(k_0, k)$ with $\hat{\gamma}_n^M(k_0, k)$ defined in (1.6). We consider a random sample with size $n = 1000$ and the replications is $m = 1000$ from following two models:

(1) Fréchet(1) distribution: $F(x) = \exp(-x^{-1})$, $x > 0$, $\gamma = 1$;

(2) Burr(0.5, 1) distribution: $F(x) = 1 - (1 + x^{\frac{1}{2}})^{-1}$, $x > 0$, $\gamma = 2$.

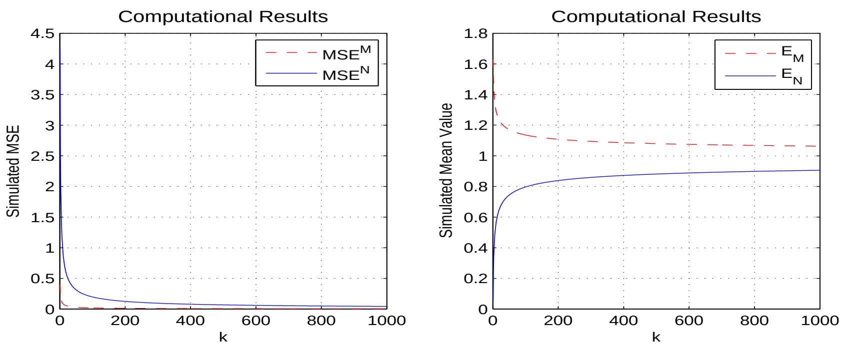

Figure 1 Simulated MSE and mean value of $\hat{\gamma}_n^M(k_0, k)$ and $\hat{\gamma}_n^N(k_0, k)$ for Fréchet(1) model with $\gamma = 1$

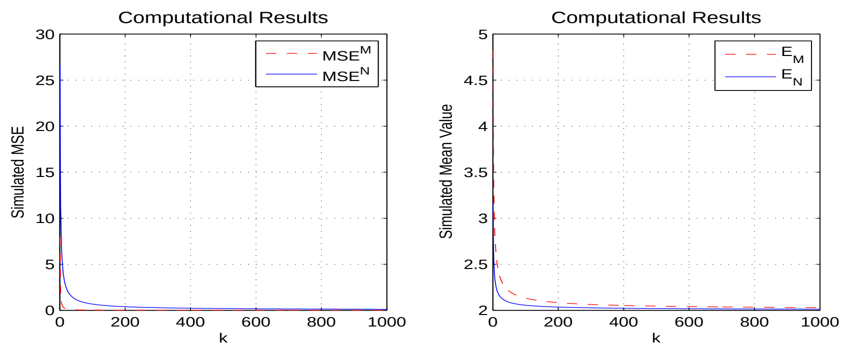

Figure 2 Simulated MSE and mean value of $\hat{\gamma}_n^M(k_0, k)$ and $\hat{\gamma}_n^N(k_0, k)$ for Burr(0.5, 1) model with $\gamma = 2$

These graphs show MSE and mean value of estimators $\hat{\gamma}_n^M(k_0, k)$ and $\hat{\gamma}_n^N(k_0, k)$, red lines represent $\hat{\gamma}_n^M(k_0, k)$ and blue lines represent $\hat{\gamma}_n^N(k_0, k)$. We see that $\hat{\gamma}_n^N(k_0, k)$ has comparable

MSE and mean value as $\hat{\gamma}_n^M(k_0, k)$ under the Fréchet(1) model, and $\hat{\gamma}_n^N(k_0, k)$ is much closer to the true value than $\hat{\gamma}_n^M(k_0, k)$ under Burr$(0.5, 1)$ model. Thus, the new estimator $\hat{\gamma}_n^N(k_0, k)$ has certain practicality in estimating unknown extreme value index.

References

[1] A. J. MCNEIL, E. ZENTRUM. *Extreme value theory for risk managers.* Departement Mathematik Eth Zentrum, 1999, 93–113.

[2] J. NICOLAU, P. M. M. RODRIGUES. *A New Regression-Based Tail Index Estimator: An Application to Exchange Rates.* Working Papers, 2015.

[3] B. M. HILL. *A simple approach to inference about the tail of a distribution.* Ann. Statist., 1975, **3**(5): 1163–1174.

[4] A. L. M. DEKKERS, J. H. J. EINMAHL, L. D. HAAN. *A moment estimator for the index of an extreme-value distribution.* Ann. Statist., 1989, **17**(4): 1833–1855.

[5] J. PICKANDS. *Statistical inference using extreme order statistics.* Ann. Statist., 1975, **3**(1): 119–131.

[6] M. I. F. ALVES. *A location invariant hill-type estimator.* Extremes, 2001, **4**(3): 199–217.

[7] Chengxiu LING, Zuoxiang PENG, S. NADARAJAH. *A location invariant moment-type estimator (I).* Theory Probab. Math. Statist., 2008, **76**: 23–31.

[8] Chengxiu LING, Zuoxiang PENG, S. NADARAJAH. *A location invariant moment-type estimator (II).* Theory Probab. Math. Statist., 2008, **77**: 177–189.

[9] Shuliang WANG. *A new kind of moment-type estimator.* J. Southwest University, 2007, **29**(5): 61–65.

[10] A. FERREIRA, L. D. HAAN, Liang PENG. *On optimising the estimation of high quantiles of a probability distribution.* Statistics, 2003, **37**(5): 401–434.

[11] S. I. RESNICK. *Extreme Values, Regular Variation, and Point Processes.* Springer-Verlag, New York, 1987.

[12] L. D. HAAN, A. FERREIRA. *Extreme value theory: an introduction.* Series in Operations Research and Financial Engineering, 2006, **60**(1): 1–20.

[13] Jiaona LI. *A class of unbiased location invariant hill-type estimators for heavy tailed distributions.* Electron. J. Stat., 2008, **2**(3): 829–847.