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Strongly $\mathcal{W}$-Gorenstein Complexes

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Abstract The notion of strongly $\mathcal{W}$-Gorenstein complexes is introduced for a self-orthogonal class $\mathcal{W}$ of modules. We obtain a characterization of such complexes and apply to strongly Gorenstein injective complexes.

Keywords strongly $\mathcal{W}$-Gorenstein complex; strongly Gorenstein injective complex; Gorenstein injective module

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1. Introduction

The main idea of Gorenstein homological algebra is to replace projective modules by Gorenstein projective modules. These modules were introduced by Enochs and Jenda [1] as a generalization of finitely generated module of G-dimension zero over a two-sided noetherian ring, in the sense of Auslander and Bridger [2]. This subject in the category of complexes has been also considered by many authors and developed to an advanced level, for example [3–10].

A complex X is said to be Gorenstein injective if there exists an exact sequence of injective complexes

$$\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$$

with $C \cong \mathrm{Ker}(E^0 \to E^1)$ and which remains exact after applying $\mathrm{Hom}(I, -)$ for any injective complex I.

It is shown that a complex C is Gorenstein injective if and only if C_m is Gorenstein injective in R-Mod for each $m \in \mathbb{Z}$ (see [10, Proposition 2.8]). In spite of the fact that this result is a generalization of classical result: a complex C is injective if and only if C is exact with each $\mathrm{Z}_n(C)$ injective modules for $n \in \mathbb{Z}$, it seems that a better corresponding relationship has not yet been established between Gorenstein and classical version.

To this end, the aim of the present paper is to introduce and investigate so-called strongly $\mathcal{W}$-Gorenstein complexes, as a main result of this note, the following result is obtained (cf. Theorem 3.6):

Theorem 1.1 *Let G be a complex. Then the following statements are equivalent:*

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(1) G is strongly $\mathcal{W}$-Gorenstein.

(2) G is a complex of $\mathcal{W}$-Gorenstein R-modules and any morphism $f: V \longrightarrow G$ and $g: G \longrightarrow V$ are null homotopic whenever V is a CE $\mathcal{W}$ complex.

As applications to Theorem 1.1, we establish a relationship for strongly Gorentein injective complexes.

Corollary 1.2 *Let G be a complex. Then the following statements are equivalent:*

(1) G is strongly Gorenstein injective.

(2) G is exact with $\mathrm{Z}_n(G)$ Gorenstein injective R-modules for each $n \in \mathbb{Z}$.

Namely, for a complex C, the following diagram can be obtained:

$$\begin{tikzcd}
C \text{ is injective} \arrow[r, "\subseteq"] \arrow[d, "\Updownarrow"'] & C \text{ is strongly Gorenstein injective} \arrow[d, "\Updownarrow"] \\
C \text{ is exact and } \mathrm{Z}_n(C) \text{ is injective} \arrow[r, "\subseteq"] & C \text{ is exact and } \mathrm{Z}_n(C) \text{ is Gorenstein injective.}
\end{tikzcd}$$

Diagram 1 Object relation diagram

2. Preliminaries

Throughout this paper, R denotes a ring with unity. A complex

$$\cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of left R-modules will be denoted by (C, δ) or C. For a ring R, R-Mod denotes the category of left R-modules, unless stated otherwise, R-modules always denote left R-modules, $\mathscr{C}(R)$ denotes the abelian category of complexes of left R-modules.

We use both the superscript and subscript notations for complexes. When we use subscripts for a complex we will use superscripts to distinguish complexes. So if $\{C^i\}_{i\in I}$ is a family of complexes, C^i will be

$$\cdots \xrightarrow{\delta_2} C_1^i \xrightarrow{\delta_1} C_0^i \xrightarrow{\delta_0} C_{-1}^i \xrightarrow{\delta_{-1}} \cdots.$$

Given a left R-module M, we use the notation $D^m(M)$ to denote the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the mth and $(m-1)$th positions and set $\overline{M} = D^0(M)$. We also use the notation $S^m(M)$ to denote the complex with M in the mth place and 0 in the other places and set $\underline{M} = S^0(M)$.

Given a complex C and an integer m, $\Sigma^m C$ denotes the complex such that $(\Sigma^m C)_l = C_{l-m}$, and whose boundary operators are $(-1)^m \delta_{l-m}$. The lth homology module of C is the module $\mathrm{H}_l(C) = \mathrm{Z}_l(C)/\mathrm{B}_l(C)$ where $\mathrm{Z}_l(C) = \mathrm{Ker}(\delta_l^C)$ and $\mathrm{B}_l(C) = \mathrm{Im}(\delta_{l+1}^C)$.

Let C and D be complexes of left R-modules. We will denote by $\mathrm{Hom}_R(C, D)$ the complex of abelian groups with $\mathrm{Hom}_R(C, D)_n = \prod_{t\in\mathbb{Z}} \mathrm{Hom}_R(C_t, D_{n+t})$ such that if $f \in \mathrm{Hom}_R(C, D)_n$, then $(\delta_n(f))_m = \delta^D_{m+n} f_m - (-1)^n f_{m+1}\delta^C_m$. f is called a chain map of degree n if $\delta_n(f) = 0$. A chain map of degree 0 is called a morphism. We use $\mathrm{Hom}(C, D)$ to denote the abelian group of morphisms from C to D and Ext^i for $i \geq 0$ will denote the groups we get from the right derived functor of Hom. We let $\mathrm{Ext}^1_{dw}(C, D)$ be the subgroup of $\mathrm{Ext}^1(C, D)$ consisting of those short exact sequences which are split in each degree.

Let X be a complex of right R-modules and Y a complex of left R-modules, $X \otimes_R Y$ denotes the usual tensor product of X and Y, where $(X \otimes_R Y)_n = \bigoplus_{t\in\mathbb{Z}} X_t \otimes_R Y_{n-t}$ and $\delta(x \otimes y) = \delta^X_t(x) \otimes y + (-1)^t x \otimes \delta^Y_{n-t}(y)$ for $x \in X_t, y \in Y_{n-t}$.

General background materials about complexes of R-modules can be found in [5].

We first recall some notions and results needed in the paper.

A left R-module M is called Gorenstein injective if there exists an exact sequence of injective left R-modules $\cdots \to E_{-1} \to E_0 \to E_1 \to \cdots$ with $M \cong \mathrm{Ker}(E_0 \to E_1)$ which remains exact after applying $\mathrm{Hom}_R(I, -)$ for any injective left R-module I (see [1, Definition 2.1]).

Recall from [11, p.227] that a complex I is said to be CE injective if $I, \mathrm{Z}(I), \mathrm{B}(I)$ and $\mathrm{H}(I)$ are complexes consisting of injective modules. A complex P is said to be CE projective if P, $\mathrm{Z}(P)$, $\mathrm{B}(P)$ and $\mathrm{H}(P)$ are complexes consisting of projective modules.

Let $\mathcal{W}$ be a class of R-modules. $\mathcal{W}$ is called self-orthogonal if it satisfies the following condition:

$$\mathrm{Ext}^i_R(W, W') = 0 \text{ for all } W, W' \in \mathcal{W} \text{ and all } i \geq 1.$$

In the following, $\mathcal{W}$ always denotes a self-orthogonal class of R-modules which is closed under extensions, finite direct sums and direct summands. Geng and Ding in [12, Remark 2.3] enumerated a variety of interesting examples of self-orthogonal classes.

Let R be a commutative ring. Following [13, Definition 2.1], an R-module C is semidualizing if:

(1) C_R admits a degreewise finite R-projective resolution.

(2) The homothety map ${}_R R_R \to \mathrm{Hom}_S(C, C)$ is an isomorphism.

(3) $\mathrm{Ext}^{\geq 1}_R(C, C) = 0$.

Examples include the rank 1 free module and a dualizing (canonical) module, when one exists.

An R-module is C-projective if it has the form $C \otimes_R P$ for some projective R-module P. An R-module is C-injective if it has the form $\mathrm{Hom}_R(C, E)$ for some injective R-module E. Let $\mathcal{P}_C = \{C \otimes_R P \mid P \text{ is a projective } R\text{-module}\}$ and $\mathcal{I}_C = \{\mathrm{Hom}_R(C, E) \mid E \text{ is an injective } R\text{-module}\}$ denote the class of C-projective and C-injective modules, respectively. Then $\mathcal{P}_C$ and $\mathcal{I}_C$ are self-orthogonal [12, Corollary 3.2].

Definition 2.1 ([8, Definition 2.3]) *A complex X is said to be a CE $\mathcal{W}$ complex if X, $\mathrm{Z}(X)$, $\mathrm{B}(X)$ and $\mathrm{H}(X)$ are complexes each of whose terms belongs to $\mathcal{W}$.*

A complex X is called a $\mathcal{W}$ complex if X is acyclic and $\mathrm{Z}_n(X) \in \mathcal{W}$ for any $n \in \mathbb{Z}$. We will denote the class of $\mathcal{W}$ complexes by $\widetilde{\mathcal{W}}$.

Remark 2.2 (1) For any module $M \in \mathcal{W}$ and any $n \in \mathbb{Z}$, $D^n(M)$ and $S^n(M)$ are CE $\mathcal{W}$ complexes.

(2) In particular, if $\mathcal{W}$ denotes the class of all projective (respectively, injective) modules, then CE $\mathcal{W}$ complexes are precisely CE projective (respectively, CE injective) complexes.

(3) If we put $\mathcal{W} = \mathcal{P}_C$ (respectively, $\mathcal{W} = \mathcal{I}_C$), then a CE $\mathcal{W}$ complex above is particularly called CE C-projective (respectively, CE C-injective) complexes.

(4) It is obviously that every complex $X \in \widetilde{\mathcal{W}}$ is a CE $\mathcal{W}$ complex.

3. Strongly $\mathcal{W}$-Gorenstein complex

This section is devoted to define and study strongly $\mathcal{W}$-Gorenstein complexes.

Definition 3.1 *A complex G is said to be strongly $\mathcal{W}$-Gorenstein, if there exists an exact sequence of $\mathcal{W}$ complexes*

$$\mathbb{W} = \cdots \longrightarrow W^{-1} \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

such that

(1) $G = \mathrm{Ker}(W^0 \to W^1)$;

(2) The sequence remains exact when $\mathrm{Hom}(V, -)$ *and* $\mathrm{Hom}(-, V)$ *are applied to it for any CE $\mathcal{W}$ complex V.*

And in this case, $\mathbb{W}$ is called a complete strongly $\mathcal{W}$-resolution of G.

Remark 3.2 (1) If $\mathcal{W}$ is the class of injective (respectively, projective) R-modules, then strongly $\mathcal{W}$-Gorenstein complexes are said to be strongly Gorenstein injective (respectively, strongly Gorenstein projective).

(2) Let R be a commutative ring and $\mathcal{W}$ be the C-injective (respectively, C-projective) R-modules. Then a strongly $\mathcal{W}$-Gorenstein complex above is particularly called strongly C-Gorenstein injective (respectively, strongly C-Gorenstein projective). That is, a complex X is said to be strongly C-Gorenstein injective (respectively, strongly C-Gorenstein projective) if there exists an exact sequence of C-injective (respectively, C-projective) complexes $\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X_{-1} \longrightarrow \cdots$ with $X \cong \mathrm{Ker}(X_0 \to X_{-1})$, such that $\mathrm{Hom}(V, -)$ (respectively, $\mathrm{Hom}(-, V)$) exacts the above sequence for any CE C-injective (respectively, CE C-projective) complex V.

To prove our results in Theorem 1.1, we establish several facts.

Lemma 3.3 ([14, Lemma 2.1]) *For chain complexes X, Y and $n \in \mathbb{Z}$, we have*

$$\mathrm{Ext}^1_{dw}(\Sigma^{n+1}X, Y) = \mathrm{H}_n\mathrm{Hom}(X, Y) = \mathrm{Hom}_{K(R)}(\Sigma^n X, Y).$$

Lemma 3.4 *For any CE $\mathcal{W}$ complex X and $\mathcal{W}$ complex Y,* $\mathrm{Ext}^1(X, Y) = 0$, $\mathrm{Ext}^1(Y, X) = 0$, *and then* $\mathrm{Hom}_R(X, Y)$ *and* $\mathrm{Hom}_R(Y, X)$ *are exact.*

Proof Note that $({}^{\perp}\mathcal{W}, \mathcal{W})$ is a cotorsion pair in R-Mod and $\mathcal{W}$ is a self-orthogonal class of

R-modules. Then X is also a CE ${}^{\perp}\mathcal{W}$ complex, and so $\mathrm{Ext}^1(X,Y)=0$ by [15, Remark 2.11]. Dually, we can show that $\mathrm{Ext}^1(Y,X)=0$. Thus, $\mathrm{Hom}_R(X,Y)=0$ and $\mathrm{Hom}_R(Y,X)=0$ by Lemma 3.3. $\square$

Lemma 3.5 *If G is strongly $\mathcal{W}$-Gorenstein complexes, then G_n is $\mathcal{W}$-Gorenstein in R-*Mod *for $n\in\mathbb{Z}$, $\mathrm{Hom}_R(G,V)$ and $\mathrm{Hom}_R(V,G)$ are exact for any CE $\mathcal{W}$ complex V.*

Proof It follows from [6, Corollary 4.8] that G_n is $\mathcal{W}$-Gorenstein.

Since G is strongly $\mathcal{W}$-Gorenstein, there is an exact sequence

$$0\to G\longrightarrow W\longrightarrow L\to 0$$

such that W is a $\mathcal{W}$ complex and L is a strongly $\mathcal{W}$-Gorenstein complex. This induces an exact sequence

$$0\to \mathrm{Hom}(V,G)\longrightarrow \mathrm{Hom}(V,W)\longrightarrow \mathrm{Hom}(V,L)\to \mathrm{Ext}^1(V,G)\to \mathrm{Ext}^1(V,W)=0$$

by Lemma 3.4. Then $\mathrm{Ext}^1(V,G)=0$ using the definition of strongly $\mathcal{W}$-Gorenstein complexes. This implies that $\mathrm{Hom}_R(V,G)$ is exact using Lemma 3.3. Also $\mathrm{Hom}_R(G,V)$ is exact using a similar method. $\square$

Theorem 3.6 *Let G be a complex. Then the following statements are equivalent:*

(1) G is strongly $\mathcal{W}$-Gorenstein.

(2) G is a complex of $\mathcal{W}$-Gorenstein R-modules and any morphism $f: V\longrightarrow G$ and $g: G\longrightarrow V$ are null homotopic whenever V is a CE $\mathcal{W}$ complex.

Proof (1)$\Longrightarrow$(2). It follows from Lemmas 3.3 and 3.5.

(2)$\Longrightarrow$(1). Now suppose G is a complex of $\mathcal{W}$-Gorenstein R-modules and any morphism $f: V\longrightarrow G$ and $g: G\longrightarrow V$ are null homotopic whenever V is a CE $\mathcal{W}$ complex. We wish to show that G is strongly $\mathcal{W}$-Gorenstein.

Using Lemma 3.3, we get $\mathrm{Hom}_R(V,G)$ and $\mathrm{Hom}_R(V,G)$ are exact for any CE $\mathcal{W}$ complex V.

Note that there is a short exact sequence

$$0\to \Sigma^{-1}G\xrightarrow{(1,0)}\mathrm{Cone}(1^G)\xrightarrow{(0,1)} G\to 0.$$

Since each G_n is $\mathcal{W}$-Gorenstein, we can find a short exact sequence $0\to Y_n\xrightarrow{\alpha_n} J_n\xrightarrow{\beta_n} G_n\to 0$ with $J_n\in\mathcal{W}$ and Y_n $\mathcal{W}$-Gorenstein. This gives us another short exact sequence

$$0\to \bigoplus_{n\in\mathbb{Z}} D^n(Y_n)\xrightarrow{\bigoplus_{n\in\mathbb{Z}} D^n(\alpha_n)}\bigoplus_{n\in\mathbb{Z}} D^n(J_n)\xrightarrow{\bigoplus_{n\in\mathbb{Z}} D^n(\beta_n)}\bigoplus_{n\in\mathbb{Z}} D^n(G_n)\to 0.$$

Put $W^{-1}=\bigoplus_{n\in\mathbb{Z}} D^n(J_n)$. Then W^{-1} is a $\mathcal{W}$ complex. Notice that $\mathrm{Cone}(1^G)=\bigoplus_{n\in\mathbb{Z}} D^n(G_n)$, let β be the composite

$$\bigoplus_{n\in\mathbb{Z}} D^n(J_n)\xrightarrow{\bigoplus_{n\in\mathbb{Z}} D^n(\beta_n)}\bigoplus_{n\in\mathbb{Z}} D^n(G_n)\xrightarrow{(0,1)} G$$

i.e., $\beta = (0,1)\bigoplus_{n\in\mathbb{Z}} D^n(\beta_n)$, which implies that β is an epimorphism. Setting $K^{-1} = \ker\beta$, we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K^{-1} \arrow[r] \arrow[d] & \bigoplus\limits_{n\in\mathbb{Z}} D^n(J_n) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma^{-1}G \arrow[r] \arrow[d] & \bigoplus\limits_{n\in\mathbb{Z}} D^n(G_n) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram 2 Exact commutative diagram

It follows from the snake lemma that there is a short exact sequence

$$0 \to \bigoplus_{n\in\mathbb{Z}} D^n(Y_n) \longrightarrow K^{-1} \longrightarrow \Sigma^{-1}G \to 0.$$

So each K_n^{-1} is a $\mathcal{W}$-Gorenstein module.

For any CE $\mathcal{W}$ complex V, applying $\mathrm{Hom}_R(V,-)$ and $\mathrm{Hom}_R(-,V)$ to the short exact sequence $0 \to K^{-1} \longrightarrow W^{-1} \longrightarrow G \to 0$, respectively, it yields two exact sequences $0 \to \mathrm{Hom}_R(V,K^{-1}) \to \mathrm{Hom}_R(V,W^{-1}) \to \mathrm{Hom}_R(V,G) \to 0$ and $0 \to \mathrm{Hom}_R(G,V) \to \mathrm{Hom}_R(W^{-1},V) \to \mathrm{Hom}_R(K^{-1},V) \to 0$, where $\mathrm{Hom}_R(V,G)$, $\mathrm{Hom}_R(V,W^{-1})$, $\mathrm{Hom}_R(G,V)$ and $\mathrm{Hom}_R(W^{-1},V)$ are exact by Lemmas 3.3 and 3.4. Then $\mathrm{Hom}_R(V,K^{-1})$ and $\mathrm{Hom}_R(K^{-1},V)$ are exact, and so $\mathrm{Ext}^1(V,K^{-1}) = 0$ and $\mathrm{Ext}^1(K^{-1},V) = 0$. This means that

$$0 \to K^{-1} \longrightarrow W^{-1} \longrightarrow G \longrightarrow 0$$

remains exact after applying $\mathrm{Hom}(V,-)$ and $\mathrm{Hom}(-,V)$ for any CE $\mathcal{W}$ complex V.

It also follows that K^{-1} has the same properties as G. Thus, we may continue inductively to obtain the following resolution

$$\cdots \to W^{-2} \xrightarrow{d_{-2}} W^{-1} \xrightarrow{d_{-1}} G \xrightarrow{\beta} 0,$$

and this resolution also remains exact after applying $\mathrm{Hom}(V,-)$ and $\mathrm{Hom}(-,V)$ for any CE $\mathcal{W}$ complex V by Lemma 3.3 again.

Similarly, we can obtain a $\mathcal{W}$-coresolution

$$0 \to G \xrightarrow{\alpha} W^0 \xrightarrow{d_0} W^1 \xrightarrow{d_1} W^2 \to \cdots$$

with each $L^i = \mathrm{Coker}d_i$ strongly $\mathcal{W}$-Gorenstein and $W^i \in \widetilde{\mathcal{W}}$ for $i \geq 0$ and this coresolution also remains exact after applying $\mathrm{Hom}(V,-)$ and $\mathrm{Hom}(-,V)$ for any CE $\mathcal{W}$ complex V.

Therefore, G is strongly $\mathcal{W}$-Gorenstein. $\square$

4. Remarks

In this section, we illustrate the main results in Section 3 by some corollaries and examples.

Lemma 4.1 ([8, Proposition 3.5]) *W is a CE $\mathcal{W}$ complex if and only if W can be divided into direct sums $W = W' \oplus W''$ where W' is a $\mathcal{W}$ complex and W'' is a graded module with all items in $\mathcal{W}$, i.e., $W = \bigoplus_{i\in\mathbb{Z}}(D^i(W_i') \oplus S^i(W_i''))$.*

Take $\mathcal{W}$ to be the subcategory of injective R-modules. The following corollary which appears in [3, Proposition 3.3] can be obtained using Lemma 4.1.

Corollary 4.2 *A complex X is CE injective if and only if X can be divided into direct sums $X = X' \oplus X''$ where X' is an injective complex and X'' is a graded module with all items injective R-modules.*

If R is a commutative ring and $\mathcal{W} = \mathcal{I}_C$ is the class of C-injective R-modules, we get the following corollary.

Corollary 4.3 *A complex X is CE C-injective if and only if X can be divided into direct sums $X = X' \oplus X''$ where X' is a C-injective complex and X'' is a graded module with all items C-injective R-modules.*

Using Lemma 4.1, as an immediate consequence of Theorem 3.6, the following result is obtained.

Corollary 4.4 *Let G be a complex. Then the following statements are equivalent:*

(1) G is strongly $\mathcal{W}$-Gorenstein.

(2) G is a complex of $\mathcal{W}$-Gorenstein R-modules and any morphism $f : \underline{V} \longrightarrow G$ and $g : G \longrightarrow \underline{V}$ are null homotopic whenever $V \in \mathcal{W}$.

Lemma 4.5 ([15, Lemma 3.2 and its dual version]) *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes with A or C exact. Then $0 \to A \to B \to C \to 0$ is CE exact, the definition of CE exact sequences can be found in [3, Definition 5.3].*

As we know, a complex I is injective if and only if I is an exact complex and each $\mathrm{Z}_n(I)$ is an injective R-module. The following corollary establishes a similar result for strongly Gorenstein injective complexes.

Proposition 4.6 *Let G be a complex. Then the following statements are equivalent:*

(1) G is strongly Gorenstein injective.

*(2) G is an exact complex with $\mathrm{Z}_n(G)$ Gorenstein injective in R-*Mod *for $n \in \mathbb{Z}$.*

Proof (1)$\Longrightarrow$(2). Note that $\underline{\mathbb{Q}/\mathbb{Z}}$ is CE injective. Then G is an exact complex by Lemma 3.3 and Corollary 4.4.

It follows from Lemma 4.5 that there is a CE exact sequence of injective complexes

$$\cdots \to I^{-1} \to I^0 \to I^1 \to \cdots$$

such that $G \cong \mathrm{Ker}(I^0 \to I^1)$, and so the following sequence

$$\cdots \to \mathrm{Z}_n(I^{-1}) \to \mathrm{Z}_n(I^0) \to \mathrm{Z}_n(I^1) \to \cdots \tag{4.1}$$

is exact such that $\mathrm{Z}_n(G) \cong \mathrm{Ker}(\mathrm{Z}_n(I^0) \to \mathrm{Z}_n(I^1))$ and each $\mathrm{Z}_n(I^i)$ is injective for integers

n, i. Note that $\mathrm{S}^n(E)$ is CE injective for any injective R-module E. Using [14, Lemma 3.1], the sequence (4.1) is $\mathrm{Hom}_R(E, -)$ exact. Therefore, each $\mathrm{Z}_n(G)$ is a Gorenstein injective module.

(2)$\Longrightarrow$(1). It is obvious that G_n is Gorenstein injective R-modules. For any injective R-module I, the exactness of G leads to the exactness of $\mathrm{Hom}_R(G, I)$, on the other hand, $\mathrm{Z}_n(G)$ is Gorenstein injective R-modules for $n \in \mathbb{Z}$, then $\mathrm{Hom}_R(I, G)$ is exact by [9, Theorem 2.7]. Thus G is strongly Gorenstein injective by Lemma 3.3 and Corollary 4.4. $\square$

In the following, R will be a commutative ring.

Take $\mathcal{W}$ to be the class of C-injective R-modules. As an example of Theorem 3.6, the following observation will be established.

Example 4.7 Let G be a complex of R-modules. Then the following statements are equivalent:

(1) G is strongly C-Gorenstein injective.

(2) G is a complex of C-Gorenstein injective R-modules and any morphism $f : \underline{V} \longrightarrow G$ is null homotopic whenever V is a C-injective module and $G \otimes_R C$ is exact for any semidualizing R-module C.

Proof It follows from Corollary 4.4 and adjoint isomorphisms. $\square$

Recall from [16, the dual version of Definition 2.1] that an R-module M is G_C-injective if there exists an exact sequence of R-modules $\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X_{-1} \longrightarrow \cdots$ with each X_i injective for $i < 1$ and X_i C-injective for $i \geq 1$, such that $\mathrm{Hom}_{\mathcal{C}(R)}(X, -)$ exacts the above sequence for any C-injective R-module X and $M \cong \mathrm{Ker}(X_0 \to X_{-1})$. Set $\mathcal{GI}_C(R)$=the subcategory of G_C-injective R-modules.

Let C be a semidualizing R-module. Following [13, Definition 4.1], the Auslander class of C is the subcategory $\mathcal{A}_C(R)$ of R-modules M such that:

(1) $\mathrm{Tor}^R_{\geq 1}(C, M) = 0 = \mathrm{Ext}^{\geq 1}_R(C, C \otimes_R M)$, and

(2) the natural map $M \longrightarrow \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

Using [12, Proposition 3.6] and Example 4.7, we obtain the following example which gives the relationship between a complex G and the corresponding entries G_n.

Example 4.8 A complex G is strongly C-Gorenstein injective if and only if $G_n \in \mathcal{GI}_C(R) \bigcap \mathcal{A}_C(R)$, any morphism $f : \underline{V} \longrightarrow G$ is null homotopic whenever V is a C-injective module and $G \otimes_R C$ is exact for any semidualizing R-module C.

We round off the paper with the following specific example of strongly $\mathcal{W}$-Gorenstein complexes.

Example 4.9 Let E be an injective module and C a semidualizing module. Then

$$X =: \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_R(C, E) \xrightarrow{1} \mathrm{Hom}_R(C, E) \longrightarrow 0 \longrightarrow \cdots$$

is a strongly $\mathcal{W}$-Gorenstein complex for the self-orthogonal class of C-injective modules.

Proof Since any C-injective module is C-Gorenstein injective, X is a complex of C-Gorenstein injective R-modules. Note that X is a contractible complex. Then any morphism $f : \underline{V} \longrightarrow X$ is null homotopic whenever V is a C-injective module and $X \otimes_R C$ is exact for any semidualizing R-module C. Thus X is strongly C-Gorenstein injective by Example 4.7, i.e., X is a strongly $\mathcal{W}$-Gorenstein complex for the self-orthogonal class of C-injective modules. $\square$

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References

[1] E. E. ENOCHS, O. M. G. JENDA. *Gorenstein injective and projective modules*. Math. Z., 1995, **220**(4): 611–633.

[2] M. AUSLANDER, M. BRIDGER. *Stable Module Theory*. American Mathematical Society, Providence, R.I. 1969.

[3] E. E. ENOCHS. *Cartan-Eilenberg complexes and resolutions*. J. Algebra, 2011, **342**: 16–39.

[4] E. E. ENOCHS, J. R. GARCÍA ROZAS. *Gorenstein injective and projective complexes*. Comm. Algebra, 1998, **26**(5): 1657–1674.

[5] J. R. GARCÍA ROZAS. *Covers and Envelopes in the Category of Complexes*. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[6] Li LIANG, Nanqing DING, Gang YANG. *Some remarks on projective generators and injective cogenerators*. Acta Math. Sin. (Engl. Ser.), 2014, **30**(12): 2063–2078.

[7] Zhongkui LIU, Chunxia ZHANG. *Gorenstein injective complexes of modules over Noetherian rings*. J. Algebra, 2009, **342**(5): 1546–1554.

[8] Bo LU, Zhongkui LIU. *A note on Cartan-Eilenberg Gorenstein categories*. Kodai Math. J., 2015, **38**(1): 209–227.

[9] Aimin XU, Nanqing DING. *On stability of Gorenstein categories*. Comm. Algebra, 2013, **41**(10): 3793–3804.

[10] Xiaoyan YANG, Zhongkui LIU. *Gorenstein projective, injective, and flat complexes*. Comm. Algebra, 2011, **39**(5): 1705–1721.

[11] J. L. VERDER. *Catégories dérivées, état 0*. Springer, Berlin, 1977.

[12] Yuxian GENG, Nanqing DING. *$\mathcal{W}$-Gorenstein modules*. J. Algebra, 2011, **325**: 132–146.

[13] H. HOLM, D. WHITE. *Foxby equivalence over associative rings*. J. Math. Kyoto Univ., 2007, **47**(4): 781–808.

[14] J. GILLESPIE. *The flat model structure on Ch(R)*. Trans. Amer. Math. Soc., 2004, **356**(8): 3369–3390.

[15] Bo LU, Zhongkui LIU. *Cartan-Eilenberg complexes with respect to cotorsion pairs*. Arch. Math.(Basel), 2014, **102**(1): 35–48.

[16] D. WHITE. *Gorenstein projective dimension with respect to a semidualizing module*. J. Commut. Algebra, 2010, **2**(1): 111–137.