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The Starlikeness of Analytic Functions of Koebe Type with Complex Order

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Abstract For $\alpha > 0$, $\lambda > 0$ and $\beta, \eta \in \mathbb{R}$, we consider the $M(\alpha,\lambda)_b$ of normalized analytic $\alpha - \lambda$ convex functions defined in the open unit disc $\mathbb{U}$. In this paper we investigate the class $M(\alpha,\lambda,\beta,\eta)_b$, with $f_b := \frac{z}{(1-z^n)^b}$ being Koebe type. By making use of Jack's Lemma as well as several differential and other inequalities, the authors derive sufficient conditions for starlikeness of the class $M(\alpha,\lambda,\beta,\eta)_b$ of n-fold symmetric analytic functions of Koebe type. Relevant connections of the results presented here with those given in earlier works are also indicated.

Keywords analytic function; subordination; superordination; Koebe type

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1. Introduction

Let $\mathcal{A}$ denote the class of normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Also, as usual, let

$$S^* = \{f : f \in \mathcal{A} \text{ and } \mathrm{Re}(\frac{zf'(z)}{f(z)}) > 0, \quad z \in \mathbb{U}\} \tag{1.2}$$

and

$$K = \{f : f \in \mathcal{A} \text{ and } \mathrm{Re}(1 + \frac{zf''(z)}{f'(z)}) > 0, \quad z \in \mathbb{U}\} \tag{1.3}$$

be the familiar classes of starlike functions in $\mathbb{U}$ and convex functions in $\mathbb{U}$, respectively.

The expressions $\frac{zf'(z)}{f(z)}$ and $(1+\frac{zf''(z)}{f'(z)})$ play an important role in the theory of univalent functions. Several new classes have been introduced and studied by various researchers by combining these expressions in different manners. As example, one can refer to the work done in [1–4].

The class $M(\alpha)$ was first introduced by Mocanu [1] who called it the class of α convex (or α-starlike) functions.

$$M(\alpha) = \{f(z) \in A : \ \mathrm{Re}\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)})\} > 0\}.$$

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Miller et al. in [5] showed that $M(\alpha)$ is a subclass of S^* for any real number α and also that $M(\alpha)$ is a subclass of K for $\alpha \geq 1$. We note that $M(0) = S^*$ and $M(1) = K$.

Motivated essentially by the aforementioned earlier works, we aim here at deriving sufficient conditions for starlikeness of n-fold symmetric function f_b of Koebe type, defined by

$$f_b(z) := \frac{z}{(1-z^n)^b}, \quad b \geq 0;\ n \in \mathbb{N} := \{1, 2, 3, \ldots\} \tag{1.4}$$

which obviously corresponds to the familiar Koebe function when $n = 1$ and $b = 2$.

In this paper we consider and denote by $M(\alpha, \lambda, \beta, \eta)_b$ the class of functions $f \in A$, for $\alpha \geq 0$, $\lambda > 0, \beta, \eta \in \mathbb{R}$ and $z \in \mathbb{U}$ that is

$$(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)}+\alpha\lambda(1+\frac{zf_b''(z)}{f_b'(z)}))]. \tag{1.5}$$

We have the following inclusion relationships:

(i) $M(0, \lambda, 0, 0) \subset S^*(0)$;

(ii) $M(\alpha, 1, 0, 0) \subset \mathcal{H}(\alpha) \subset S^*$ with $\mathcal{H}(\alpha)$, studied by Fukui et al. [6];

(iii) $M(1, 0, 0, 0) = \mathcal{H}(1) \subset S^*(1/2)$, investigated by Ramesha et al. [7];

(iv) $M(1, 0, 0, 0) = \mathcal{H}(1) \subset S^*(\gamma)$ where $\gamma < 1/2$, observed by Nunokawa et al. [8];

(v) $M(\alpha, 0, 0, 0) = \mathcal{H}(\alpha) \subset S^*$, discussed by Kamali and Srivastava [9];

(vi) $M(\alpha, \lambda, 0, 0) = \mathcal{H}(\alpha) \subset S^*$, discussed by Siregar and Darus [10];

(vii) $M(\alpha, 0, \beta, 0) = \mathcal{H}(\alpha, \beta) \subset S^*$, studied by Siregar [3];

(viii) $M(\alpha, 1, \beta, \eta) = \mathcal{H}(\alpha, \beta) \subset S^*$, investigated by Pauzi and Darus [11].

In this paper we investigate the subordination, superordination, best dominant, best subordinant, sandwich theorem and sufficient conditions for starlikeness of n-fold symmetric function of Koebe type and its applications in the class denoted by $M_b(\alpha, \lambda, \beta, \eta)_b$.

For two functions f and g analytic in the open unit disk $\mathbb{U}$, we say that f is subordinate to g in $\mathbb{U}$ and write as $f \prec g$, if there exists a Schwarz function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $w(z) < 1, z \in \mathbb{U}$ such that $f(z) = g(w), z \in \mathbb{U}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\Phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{U}$ such that $(p(z), zp'(z), z^2p''(z); z) \in \mathbb{C}^3 \times \mathbb{U}$ for all $z \in \mathbb{U}$ and h be univalent in $\mathbb{U}$. Then the function p is said to satisfy a first order subordination if

$$\Phi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \text{ and } \Phi(p(0), 0; 0) = h(0). \tag{1.6}$$

A univalent function q is called a dominant of the differential subordination (1.6) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.6). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.6), is said to be best dominant of (1.6).

Let $\Psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ be an analytic function in domain $\mathbb{C}^3 \times \mathbb{U}$, h be an analytic function in $\mathbb{U}$, p be analytic and univalent in $\mathbb{U}$, with $(p(z), zp'(z), z^2p''(z); z) \in \mathbb{C}^3 \times \mathbb{U}$ for all $z \in \mathbb{U}$. Then the function p is called a solution of the first order subordination if

$$h(z) \prec \Psi(p(z), zp'(z), z^2p''(z); z) \text{ and } h(0) = \Psi(p(0), 0; 0). \tag{1.7}$$

An analytic function q is called a subordinant of the differential superordination (1.7) if $q \prec p$ for all p satisfying (1.7). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.7), is said to be best subordinant of (1.7).

The work of Siregar [3], Siregar and Darus [10] and Bansal and Raina [12] have motivated us to come to these problem. See also Frasin and Darus [10] for different studies.

2. Preliminaries

In order to prove our subordination and superodination results, we make use of the following known results.

Lemma 2.1 ([13]) *Let the function $q(z)$ be univalent in the open unit disc $\mathbb{U}$ and let the function θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) = \gamma z q'(z)\phi(q(z)), \gamma > 0 \text{ and } h(z) = \theta(q(z)) + Q(z). \tag{2.1}$$

Suppose that

(i) $Q(z)$ is starlike univalent $\mathbb{U}$ and

(ii) $\mathrm{Re}\,\frac{zh'(z)}{Q(z)} = \mathrm{Re}\,\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} > 0$ for $z \in \mathbb{U}$.

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0) = 1$, $p(\mathbb{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then

$$p(z) \prec q(z) \tag{2.3}$$

and q is the best dominant of the subordination.

Lemma 2.2 ([14]) *Let $q(z)$ be univalent in the unit disk $\mathbb{U}$ and let ϑ and φ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*

(i) $zq'(z)\varphi(q(z))$ is univalent and starlike in $\mathbb{U}$;

(ii) $\mathrm{Re}\frac{\vartheta'(q(z))}{\varphi(q(z))} > 0$ for $z \in \mathbb{U}$.

If $p(z) \in H[q(0), 1] \cap Q$ with $p(\mathbb{U}) \subseteq D$ and $\vartheta p(z) + zp'(z)\varphi(p(z))$ is univalent in $\mathbb{U}$ and $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z))$, then $q(z) \prec p(z)$ is the best subordinant.

Lemma 2.3 ([15]) *Let the (nonconstant) function $w(z)$ be analytic in $\mathbb{U}$ such that $w(0) = 0$. If $|w(z)|$ attains its maximum value on circle $|z| = r < 1$ at a point $z_o \in \mathbb{U}$, we have $z_o w'(z) = kw(z_o)$, where $k \geq 1$ is a real number.*

Lemma 2.4 ([6]) *The function defined by (1.5) is univalent if and only if*

$$0 \leq nb \leq 2. \tag{2.4}$$

Futhermore, the condition in (2.4) is necessary and sufficient for the function to be a starlike function.

Lemma 2.5 ([16]) *Let $\Theta(u, v)$ be a complex-valued function such that $\Theta : D \to \mathbb{C}(D \subset \mathbb{C} \times \mathbb{C})$.*

Here $\mathbb{C}$ is (as usual) the complex plane. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the functions $\Theta(u, v)$ satisfy each of the following conditions:

(i) $\Theta(u, v)$ *is continuous in D;*

(ii) $(1, 0) \in D$ *and* $\mathrm{Re}(\Theta(1, 0)) > 0$*;*

(iii) $\mathrm{Re}(\Theta(iu_2, v_1)) \le 0$ *for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{1}{2}(1 + u_2^2)$.*

Let $p(z) = 1 + p_1 z + p_2 z + \cdots$ be analytic (regular) in $\mathbb{U}$ such that $(p(z), zp'(z)) \in D$, $z \in \mathbb{U}$. If $\mathrm{Re}\,(\Theta(p(z), zp'(z))) \in D$, *then* $\mathrm{Re}\,(p(z)) > 0, z \in \mathbb{U}$.

3. The subordination and superordination results

Theorem 3.1 *Let $f(z) \in A$ satisfy $f(z) \neq 0$, $z \in \mathbb{U}$. Also let the function $q(z)$ be univalent in $\mathbb{U}$, with $q(0) = 1$ and $q(z) \neq 0$, such that*

$$\mathrm{Re}\{1 + (\beta + \eta i)\frac{zq'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\} > 0, \quad z \in \mathbb{U} \tag{3.1}$$

and

$$\mathrm{Re}\{1 + \frac{1}{\lambda}[(\beta + \eta i + 2)q(z) + \frac{1}{\alpha}(\beta + \eta i + 1) - (\beta + \eta i + 1)] + (\beta + \eta i)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} > 0\}, \quad z \in \mathbb{U} \tag{3.2}$$

for $|\beta + \eta i| \le 1$ and $\alpha > 0$. If

$$(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1 - \alpha + \alpha(1 - \lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda(1 + \frac{zf_b''(z)}{f_b'(z)}))] \prec h(z), \tag{3.3}$$

where

$$h(z) = \alpha[q(z)]^{\beta+\eta i+2} + (1 - \alpha)[q(z)]^{\beta+\eta i+1} + \alpha\lambda zq'(z)[q(z)]^{\beta+\eta i}, \tag{3.4}$$

then $\frac{zf_b'(z)}{f_b(z)} \prec q(z), z \in \mathbb{U}$ and $q(z)$ is the best dominant of (3.3).

Proof Firstly, choose

$$p(z) = \frac{zf_b'(z)}{f_b(z)},\ \theta(w) = w^{\beta+\eta i}[(1 - \alpha)w + aw^2] \text{ and } \phi(w) = w^{\beta+\eta i},$$

then $\theta(w)$ and $\phi(w)$ exist and are analytic inside the domain $D^* = \mathbb{C} \setminus \{0\}$ which contains $q(\mathbb{U}), q(0) = 1$ and $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$.

Now, if we define the functions $Q(z)$ and $h(z)$ by

$$Q(z) = \alpha\lambda zq'(z)\phi[q(z)] = \alpha\lambda zq'(z)[q(z)]^{\beta+\eta i}$$

and

$$h(z) = \theta[q(z)] + Q(z) = \alpha[q(z)]^{\beta+\eta i+2} + (1 - \alpha)[q(z)]^{\beta+\eta i+1} + \alpha\lambda zq'(z)[q(z)]^{\beta+\eta i},$$

then it follows from (3.1) and (3.2) that $Q(z)$ is starlike in $\mathbb{U}$ and

$$\mathrm{Re}(\frac{zh'(z)}{Q(z)}) = 1 + \frac{1}{\lambda}[(\beta + \eta i + 2)q(z) + \frac{1}{\alpha}(\beta + \eta i + 1) - (\beta + \eta i + 1)] + (\beta + \eta i)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} > 0, \quad z \in \mathbb{U}.$$

We also note that the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0) = 1$. Since $0 \notin p(\mathbb{U})$, therefore $p(\mathbb{U}) \subset D^*$ and $\alpha\lambda > 0$. Hence, the hypotheses of Lemma 2.1 are satisfied.

Since $p(z) = \frac{zf_b'(z)}{f_b(z)}$, we have

$$\begin{aligned} p'(z) =& \frac{f_b(z)[zf_b''(z) + f_b'(z)] - z[f_b'(z)]^2}{[f_b(z)]^2} \\ =& \frac{[zf_b(z)f_b''(z) + f_b(z)f_b'(z)] - z[f_b'(z)]^2}{[f_b(z)]^2} \\ =& z\frac{f_b''(z)}{f_b(z)} + \frac{f_b'(z)}{f_b(z)} - z[\frac{f_b'(z)}{f_b(z)}]^2. \end{aligned}$$

Multiplying $p'(z)$ with z, we have

$$zp'(z) = z^2\frac{f_b''(z)}{f_b(z)} + z\frac{f_b'(z)}{f_b(z)} - [z\frac{f_b'(z)}{f_b(z)}]^2.$$

Applying Lemma 2.1, we find that

$$\begin{aligned} &(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda(1 + \frac{zf_b''(z)}{f_b'(z)}))] \\ &= (\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(1 - \alpha + \alpha\lambda)(\frac{zf_b'(z)}{f_b(z)}) + \alpha(1-\lambda)(\frac{zf_b'(z)}{f_b(z)})^2 + \alpha\lambda(\frac{z^2f_b''(z)}{f_b(z)})] \\ &= (p(z))^{\beta+\eta i}[(1 - \alpha + \alpha\lambda)p(z) + \alpha(1-\lambda)(p(z))^2 + \alpha\lambda(zp'(z) - p(z) + [p(z)]^2)] \\ &= (p(z))^{\beta+\eta i}[(1-\alpha)p(z) + \alpha(p(z))^2 + \alpha\lambda zp'(z)] \\ &= \alpha(p(z))^2(p(z))^{\beta+\eta i} + (1-\alpha)p(z)(p(z))^{\beta+\eta i} + \alpha\lambda zp'(z)(p(z))^{\beta+\eta i} \\ &= (p(z))^{\beta+\eta i}[(1-\alpha)p(z) + \alpha(p(z))^2] + \alpha\lambda zp'(z)(p(z))^{\beta+\eta i} \\ &= \theta(p(z)) + \alpha\lambda zp'(z)(p(z))^{\beta+\eta i} \\ &\prec h(z) = \alpha(q(z))^2(q(z))^{\beta+\eta i} + (1-\alpha)q(z)(q(z))^{\beta+\eta i} + \alpha\lambda zq'(z)(q(z))^{\beta+\eta i} \\ &= \theta(q(z)) + \alpha\lambda zq'(z)(q(z))^{\beta+\eta i} \end{aligned}$$

which implies

$$(\frac{zf_b'(z)}{f_b(z)}) \prec q(z), \quad z \in \mathbb{U}$$

and it is proved that $q(z)$ is the best dominant of (3.3). □

Theorem 3.2 *Let f be analytic in $\mathbb{U}$ such that $f(0) = 0$, h be convex univalent in $\mathbb{U}$ and $h \in H[0,1] \cap Q$. Assume that*

$$(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda(1 + \frac{zf_b''(z)}{f_b'(z)}))]$$

is a univalent function in $\mathbb{U}$, where $|\beta + \eta i| \leq 1$ and $\alpha > 0$.

If $h \in A$ and the subordination

$$\begin{aligned} h(z) &= \theta(q(z)) + \alpha\lambda zq'(z)(q(z))^{\beta+\eta i} \\ &\prec (\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda(1 + \frac{zf_b''(z)}{f_b'(z)}))] \end{aligned}$$

holds, then $q(z) \prec (\frac{zf_b'(z)}{f_b(z)})$ *implies that* $q(z) \prec p(z)$, *where* $p(z) = (\frac{zf_b'(z)}{f_b(z)})$ *and q(z) is the best subordinant.*

Proof Our aim is to apply Lemma 2.2. By setting

$$p(z) = \frac{zf_b'(z)}{f_b(z)},\ \vartheta(w) = w^{\beta+\eta i}[(1-\alpha)w + aw^2],\ \varphi(w) = w^{\beta+\eta i},$$

then $\vartheta(w)$ and $\varphi(w)$ exist and are analytic inside the domain $D^* = \mathbb{C} \setminus \{0\}$ which contains $p(\mathbb{U})$, $p(0) = 1$ and $\varphi(w) \neq 0$ when $w \in p(\mathbb{U})$.

It can be observed that $\vartheta(w)$ and $\varphi(w)$ are analytic in $\mathbb{C}$. Thus

$$\text{Re}\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0.$$

Now, we must show that

$$h(z) = \vartheta(q(z)) + \alpha\lambda zq'(z)\varphi(q(z)) \prec \varphi(p(z)) + \alpha\lambda zp'(z)\varphi(p(z)).$$

By the assumption of the theorem

$$\begin{aligned} h(z) &= \vartheta(q(z)) + \alpha\lambda zq'(z)\varphi(q(z)) \\ &= \alpha(q(z))^2(q(z))^{\beta+\eta i} + (1-\alpha)q(z)(q(z))^{\beta+\eta i} + \alpha\lambda zq'(z)(q(z))^{\beta+\eta i} \\ &\prec \alpha(q(z))^2(q(z))^{\beta+\eta i} + (1-\alpha)q(z)(q(z))^{\beta+\eta i} + \alpha\lambda zq'(z)(q(z))^{\beta+\eta i} \\ &= \vartheta(p(z)) + \alpha\lambda zp'(z)\varphi(p(z)) \\ &= (\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda(1+\frac{zf_b''(z)}{f_b'(z)}))]. \end{aligned}$$

Thus in view of Lemma 2.2, $q(z) \prec p(z)$ which implies $q(z) \prec (\frac{zf_b'(z)}{f_b(z)})$ and $q(z)$ is the best subordinant. □

If we combine Theorems 3.1 and 3.2, then we obtain the differential Sandwich-Type theorem.

Remark 3.3 From Theorems 3.1 and 3.2 it follows

(i) Setting $\beta = \eta = 0$, we obtain the result as asserted by Siregar and Darus [10];

(ii) Putting $\beta = \eta = 0$ and $\lambda = 1$, we get the result obtained by Siregar [3];

(iii) Substituting $\lambda = 1$, we have the result as shown by Pauzi and Darus [14].

4. The properties of the class $M_b(\alpha, \lambda, \beta, \eta)$

The method of proving the next theorem is similar to Kamali and Srivastava [9].

Theorem 4.1 *Let the n-fold symmetric function* $f_b(z)$, *defined by (1.4), be analytic in* $\mathbb{U}$, *with* $\frac{f_b(z)}{z} \neq 0$, $z \in \mathbb{U}$. *If* $f_b(z)$ *satisfies the inequality*

$$\begin{aligned} &(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda(1+\frac{zf_b''(z)}{f_b'(z)}))] \\ &\quad > (1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) - \frac{\alpha\lambda nb}{4}], \end{aligned} \tag{4.1}$$

then $f_b(z)$ *is starlike in* $\mathbb{U}$ *for* $\alpha > 0$ *and* $|\beta + \eta i| \le 1$ *and*

$$(0 \le nb < 2; \frac{\alpha\lambda + 2\alpha + 2 - \sqrt{\Delta}}{2\alpha} \le nb \le \frac{\alpha\lambda + 2\alpha + 2 + \sqrt{\Delta}}{2\alpha}),$$

where $\Delta := \alpha^2(\lambda + 2)^2 + 4\alpha(\lambda - 2) + 4$.

Proof Let $\alpha > 0$, $\lambda > 0$ and $f_b(z)$ satisfy the hypothesis of Theorem 4.1. We put

$$\frac{zf_b'(z)}{f_b(z)} = \frac{1 + (nb-1)w(z)}{1 - w(z)}, \tag{4.2}$$

where $w(\mathbb{U})$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $w(z) \ne 1$.

Then by differentiating (4.2) respect to z, we have

$$\frac{[f_b'(z) + zf_b''(z)]f_b(z) - z[f_b'(z)]^2}{[f_b(z)]^2} = \frac{(nb-1)w'(z)(1 - w(z)) + w'(z)[1 + (nb-1)w(z)]}{(1 - w(z))^2}$$

which implies that

$$\frac{zf_b''(z)}{f_b(z)} + \frac{f_b'(z)}{f_b(z)} - z[\frac{f_b'(z)}{f_b(z)}]^2 = \frac{nbw'(z)}{[1 - w(z)]^2}$$

and when multiplied by $\frac{f_b(z)}{f_b'(z)}$ gives

$$1 + \frac{zf_b''(z)}{f_b'(z)} - \frac{zf_b'(z)}{f_b(z)} = \frac{nbzw'(z)}{[1 - w(z)][1 + (nb-1)w(z)]}.$$

We can write

$$1 + \frac{zf_b''(z)}{f_b'(z)} = \frac{nbzw'(z)}{[1 - w(z)][1 + (nb-1)w(z)]} + \frac{1 + (nb-1)w(z)}{1 - w(z)} \tag{4.3}$$

which in turn from (4.2) and (4.3) implies (1.5). Therefore

$$\begin{aligned}
&(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda(1 + \frac{zf_b''(z)}{f_b'(z)}))] \\
&= (\frac{1 + (nb-1)w(z)}{1 - w(z)})^{\beta+\eta i}[(\frac{1 + (nb-1)w(z)}{1 - w(z)})(1 - \alpha + \alpha(1-\lambda)(\frac{1 + (nb-1)w(z)}{1 - w(z)}) + \\
&\quad \alpha\lambda(\frac{nbzw'(z)}{(1 - w(z))[1 + (nb-1)w(z)]} + \frac{1 + (nb-1)w(z)}{1 - w(z)}))] \\
&= (\frac{1 + (nb-1)w(z_0)}{1 - w(z_0)})^{\beta+\eta i}[(1-\alpha)(\frac{1 + (nb-1)w(z)}{1 - w(z)}) + \\
&\quad \alpha(\frac{[1 + (nb-1)w(z)]^2 + \lambda nbzw'(z)}{(1 - w(z))^2})].
\end{aligned}$$

Now, we claim that $w(z) < 1$ $(z \in \mathbb{U})$. If there exists a z_o in $\mathbb{U}$ such that $|w(z_o)| = 1$, then by using Lemma 2.3, $z_o w'(z) = kw(z_o)$, where $k \ge 1$ is a real number. Setting $w(z_o) = e^{i\theta}$ $(0 \le \theta \le 2\pi)$

gives

$$\begin{aligned}
&\mathrm{Re}\{(\frac{z_of_b'(z_o)}{f_b(z_o)})^{\beta+\eta i}[(\frac{z_of_b'(z_o)}{f_b(z_o)})(1-\alpha+\alpha(1-\lambda)\frac{z_of_b'(z_o)}{f_b(z_o)}+\alpha\lambda(1+\frac{z_of_b''(z_o)}{f_b'(z_o)}))]\}\\
&=\mathrm{Re}\{(\frac{1+(nb-1)w(z_0)}{1-w(z_0)})^{\beta+\eta i}[(1-\alpha)(\frac{1+(nb-1)w(z_0)}{1-w(z_0)})+\\
&\quad\alpha(\frac{[1+(nb-1)w(z_0)]^2+\lambda nbz_0w'(z_0)}{(1-w(z_0))^2})]\}\\
&=\mathrm{Re}\{(\frac{1+(nb-1)w(z_0)}{1-w(z_0)})^{\beta+\eta i}[(1-\alpha)(\frac{1+(nb-1)w(z_0)}{1-w(z_0)})+\\
&\quad\alpha(\frac{[1+(nb-1)w(z_0)]^2+\lambda nbkw(z_0)}{(1-w(z_0))^2})]\}\\
&=\mathrm{Re}\{(\frac{1+(nb-1)\mathrm{e}^{i\theta}}{1-\mathrm{e}^{i\theta}})^{\beta+\eta i}[(1-\alpha)(\frac{1+(nb-1)\mathrm{e}^{i\theta}}{1-\mathrm{e}^{i\theta}})+\\
&\quad\alpha(\frac{\lambda nbk\mathrm{e}^{i\theta}+[1+(nb-1)\mathrm{e}^{i\theta}]^2}{(1-\mathrm{e}^{i\theta})^2})]\}\\
&=\mathrm{Re}\{(\frac{1+(nb-1)\mathrm{e}^{i\theta}}{1-\mathrm{e}^{i\theta}})^{\beta+\eta i}[(1-\alpha)(\frac{1+(nb-1)\mathrm{e}^{i\theta}}{1-\mathrm{e}^{i\theta}})+\\
&\quad\alpha(\frac{\lambda nbk\mathrm{e}^{i\theta}}{(1-\mathrm{e}^{i\theta})^2}+[\frac{1+(nb-1)\mathrm{e}^{i\theta}}{(1-\mathrm{e}^{i\theta})}]^2)]\}\\
&=\mathrm{Re}\{(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)+\sin\ln(2-nb)i}{2}][(1-\alpha)(1-\frac{nb}{2})+\\
&\quad\alpha(\frac{-\lambda nbk}{4\sin^2(\frac{\theta}{2})}+(1-\frac{nb}{2})^2+(\frac{nb}{2})^2(\frac{1+\cos\theta}{1-\cos\theta}))]\}\\
&=(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\alpha)(1-\frac{nb}{2})+\\
&\quad\alpha(\frac{-\lambda nbk}{4\sin^2(\frac{\theta}{2})}+(1-\frac{nb}{2})^2+(\frac{nb}{2})^2(\frac{1+\cos\theta}{1-\cos\theta}))]\\
&=(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{\alpha nb}{2})(1-\frac{nb}{2})-\frac{\alpha nb}{4}(\frac{k+\lambda nbk\cos^2(\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})})]\\
&\le(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2})-\frac{\alpha\lambda nb}{4}],\quad z\in\mathbb{U},
\end{aligned}$$

since $k\ge 1$.

If we let

$$\begin{aligned}
&\mathrm{Re}\{(\frac{z_of_b'(z_o)}{f_b(z_o)})^{\beta+\eta i}[(\frac{z_of_b'(z_o)}{f_b(z_o)})(1-\alpha+\alpha(1-\lambda)\frac{z_of_b'(z_o)}{f_b(z_o)}+\alpha\lambda(1+\frac{z_of_b''(z_o)}{f_b'(z_o)}))]\}\\
&\le(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2})-\frac{\alpha\lambda nb}{4}]=\tau(nb),
\end{aligned}\tag{4.4}$$

then $\tau(nb)\le 0$. From (4.4) we get $0\le nb<2$ and

$$\frac{\alpha\lambda+2\alpha+2\sqrt{\Delta}}{2\alpha}\le nb\le\frac{\alpha\lambda+2\alpha+2\sqrt{\Delta}}{2\alpha};\ \Delta:=\alpha^2(\lambda+2)^2+4\alpha(\lambda-2)+4.$$

Thus we have

$$\mathrm{Re}\{(\frac{z_o f_b'(z_o)}{f_b(z_o)})^{\beta+\eta i}[(\frac{z_o f_b'(z_o)}{f_b(z_o)})(1-\alpha+\alpha(1-\lambda)\frac{z_o f_b'(z_o)}{f_b(z_o)}+\alpha\lambda(1+\frac{z_o f_b''(z_o)}{f_b'(z_o)}))]\} \leq 0, \tag{4.5}$$

where $0 \leq nb < 2$ and $\frac{\alpha\lambda+2\alpha+2\sqrt{\Delta}}{2\alpha} \leq nb \leq \frac{\alpha\lambda+2\alpha+2\sqrt{\Delta}}{2\alpha}$ with $\Delta := \alpha^2(\lambda+2)^2+4\alpha(\lambda-2)+4$, which is a contradiction to the hypotheses of (4.1).

Therefore, $|w(z)| < 1$ for all z in $\mathbb{U}$. Hence f_b is starlike in $\mathbb{U}$. This completes the proof of our theorem. □

By taking $\beta = \eta = 0$ in Theorem 4.1, then we get the following corollary as asserted by Siregar and Darus [10].

Corollary 4.2 *Let the n-fold symmetric function $f_b(z)$, defined by (1.7), be analytic in $\mathbb{U}$, with $\frac{f_b(z)}{z} \neq 0$, $z \in \mathbb{U}$. If $f_b(z)$ satisfies the inequality:*

$$\mathrm{Re}\{\frac{zf_b'(z)}{f_b(z)}[1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)}+\alpha\lambda(1+\frac{zf_b''(z)}{f_b'(z)})]\} > (1-\frac{nb}{2})(1-\frac{\alpha nb}{2})-\frac{\alpha\lambda nb}{4}, \tag{4.6}$$

then $f_b(z)$ is starlike in $\mathbb{U}$ for $\alpha > 0$ and $|\beta+\eta i| \leq 1$ and

$$(\frac{\alpha\lambda+2\alpha+2-\sqrt{\Delta}}{2\alpha} \leq nb \leq \frac{\alpha\lambda+2\alpha+2+\sqrt{\Delta}}{2\alpha}),$$

where $\Delta := \alpha^2(\lambda+2)^2+4\alpha(\lambda-2)+4$.

Remark 4.3 Setting $\lambda = 1$ in Theorem 4.1, we arrive to Theorem 4.1 obtained by Pauzi and Darus [11].

5. Applications of differential inequalities

We apply the following result involving differential inequalities with a view to deriving several further sufficient conditions for starlikeness of the n-fold symmetric function f_b defined by (1.4) by using Lemma 2.5.

Let us now consider the following implication:

$$\mathrm{Re}\{(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})(1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)}+\alpha\lambda(1+\frac{zf_b''(z)}{f_b'(z)})])\} > (1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2})-\frac{\alpha\lambda nb}{4}] \tag{5.1}$$

$$\Rightarrow \mathrm{Re}\{(\frac{zf_b'(z)}{f_b(z)})^{\mu}\} > 0, \quad z \in \mathbb{U} \tag{5.2}$$

and

$$(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2})-\frac{\alpha\lambda nb}{4}] < 1, \quad \alpha \geq 0,\ \lambda > 0,\ \beta \in \mathbb{R};\ \mu \geq 1.$$

If we put $p(z) = \{\frac{zf_b'(z)}{f_b(z)}\}^{\mu}$, then (5.2) is equivalent to

$$\begin{aligned}&\text{Re}\{\frac{\alpha\lambda}{\mu}\{p(z)\}^{\frac{1-\mu}{\mu}}zp'(z) + \alpha\{p(z)\}^{2/\mu} + (1-\alpha)p(z)^{\frac{1}{\mu}} - (1-\frac{nb}{2})^{\beta}\\ &[\frac{\cos\ln(2-nb)}{2}]((1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) + \frac{\lambda\alpha nb}{4})\} > 0\\ &\Rightarrow \text{Re}(p(z)) > 0, \quad z \in \mathbb{U}.\end{aligned} \tag{5.3}$$

By setting $p(z) = u$ and $zp'(z) = v$, and letting

$$\begin{aligned}\Theta(z) =&\frac{\alpha\lambda}{\mu}u^{\frac{1-\mu}{\mu}}v + \alpha u^{2/\mu} + (1-\alpha)u^{\frac{1}{\mu}} -\\ &(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}]((1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) + \frac{\lambda\alpha nb}{4})\end{aligned}$$

for $\alpha \geq 0$ and $\mu \geq 1$, we have $\Theta(u, v)$ is continuous in $\mathbb{D} = (\mathbb{C} \setminus \{0\} \times \mathbb{C})$, $(1, 0) \in \mathbb{D}$ and

$$\text{Re}(\Theta(1,0)) = 1 - (1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}]((1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) + \frac{\lambda\alpha nb}{4}) > 0.$$

Since

$$(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}]((1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) + \frac{\lambda\alpha nb}{4}) < 1,$$

the condition (i) and (ii) of Lemma 2.5 are satisfied. Moreover, for $(iu_2, v_1) \in \mathbb{D}$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we obtain

$$\begin{aligned}\text{Re}(\Theta(iu_2, v_1)) =&\frac{\alpha\lambda}{\mu}|u_2|^{\frac{(1-\mu)}{\mu}}v_1\cos(\frac{(1-\mu)\pi}{2\mu}) + \alpha|u_2|^{\frac{2}{\mu}}\cos(\frac{\pi}{\mu}) + (1-\alpha)|u_2|^{\frac{1}{\mu}}\cos(\frac{\pi}{2\mu}) -\\ &(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) + \frac{\lambda\alpha nb}{4}]\\ \leq& -\frac{\alpha\lambda}{2\mu}(1+u_2^2)|u_2|^{\frac{(1-\mu)}{\mu}}\sin(\frac{\pi}{2\mu}) + \alpha|u_2|^{\frac{2}{\mu}}\cos(\frac{\pi}{\mu}) + (1-\alpha)|u_2|^{\frac{1}{\mu}}\cos(\frac{\pi}{2\mu}) -\\ &(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) + \frac{\lambda\alpha nb}{4}],\end{aligned}$$

which upon putting $|u_2| = \zeta$ $(\zeta > 0)$, yields

$$\text{Re}(\Theta(iu_2, v_1)) \leq \Phi(\zeta), \tag{5.4}$$

where

$$\begin{aligned}\Phi(\zeta) :=& -\frac{\alpha\lambda}{2\mu}(1+\zeta^2)\zeta^{\frac{(1-\mu)}{\mu}}\sin(\frac{\pi}{2\mu}) + \alpha\zeta^{\frac{2}{\mu}}\cos(\frac{\pi}{\mu}) + (1-\alpha)\zeta^{\frac{1}{\mu}}\cos(\frac{\pi}{2\mu}) -\\ &(1-\frac{nb}{2})^{\beta}[\frac{\cos\ln(2-nb)}{2}][(1-\frac{nb}{2})(1-\frac{\alpha nb}{2}) + \frac{\lambda\alpha nb}{4}].\end{aligned} \tag{5.5}$$

Remark 5.1 If, for some choices of the parameters α, λ, μ and nb, we find that $\Phi(\zeta) \leq 0$, $\zeta > 0$, then we can conclude from (5.4) and Lemma 2.5 that the corresponding implication (5.2) holds true.

First of all, for the choice: $\mu = 1$ and $nb = 1$, we have

Theorem 5.2 *If n-fold symmetric function f_b, defined by (1.4) and analytic in $\mathbb{U}$ with*

$$\frac{f_b(z)}{z} \neq 0, \quad z \in \mathbb{U},$$

satisfies the following inequality:

$$\mathrm{Re}\{(\frac{zf_b'(z)}{f_b(z)})^{\beta+\eta i}[(\frac{zf_b'(z)}{f_b(z)})([1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)}+\alpha\lambda(1+\frac{zf_b''(z)}{f_b'(z)}))]\}>0, \quad (5.6)$$

then $f_b \in S^*$ *for any real* $\alpha \geq 0$ *and* $\lambda > 0$.

Proof For $\mu = 1$ and $nb = 1$, from (5.5) we find that

$$\Phi(\zeta) := -\frac{\alpha}{2}(1+3\alpha s^2) - \frac{1}{8}(\alpha-2)(1-\frac{\alpha\lambda}{2})^{\beta+\eta i} \leq 0, \quad \zeta \in \mathbb{R},$$

which implies Theorem 5.2 in view of the remark. □

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