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Coefficient Estimates for Several Classes of Meromorphically Bi-Univalent Functions

Dong GUO[1,*], **Zongtao LI**[2], **Liangpeng XIONG**[3]

1. *Foundation Department, Chuzhou Vocational and Technical College, Anhui* 239000*, P. R. China;*
2. *Foundation Department, Guangzhou Civil Aviation College, Guangdong* 510403*, P. R. China;*
3. *School of Mathematics and Statistics, Wuhan University, Hubei* 430072*, P. R. China*

Abstract In this paper, we investigate the bounds of the coefficients of several classes of meromorphic bi-univalent functions. The results presented in this paper improve or generalize the recent works of other authors.

Keywords analytic functions; univalent functions; meromorphic functions; coefficient estimates; bi-univalent function; meromorphically bi-univalent functions

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z:|z|<1\}$. Further, by S we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. A function f in $\mathcal{S}$ is said to be starlike of order α $(0\le\alpha<1)$ in $\mathbb{U}$ if and only if $\mathscr{R}\{zf'(z)/f(z)\}>\alpha$ $(z\in\mathbb{U};0\le\alpha<1)$ and convex of order α $(0\le\alpha<1)$ in $\mathbb{U}$ if and only if $\mathscr{R}\{1+zf''(z)/f'(z)\}>\alpha$ $(z\in\mathbb{U};0\le\alpha<1)$. As usual, we denote these subclasses of $\mathcal{S}$ by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively.

Mocanu [1] studied linear combinations of the representations of convex and starlike functions and defined the class of α-convex functions. In [2], it was shown that if

$$\mathscr{R}\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha(1+\frac{zf''(z)}{f'(z)})\}>0,\quad z\in\mathbb{U},$$

then f is in the class of starlike functions $\mathcal{S}^*(0)$ for α being a real number and is in the class of convex functions $\mathcal{K}(0)$ for $\alpha\ge1$.

In [3], it was shown that if

$$\mathscr{R}(\frac{\alpha z^2f''(z)}{f(z)}+\frac{zf'(z)}{f(z)})>-\frac{\alpha}{2},\quad \alpha\ge0;\ z\in\mathbb{U},$$

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* Corresponding author
E-mail address: gd791217@163.com (Dong GUO); lizt2046@163.com (Zongtao LI); xlpwxf@163.com (Liangpeng XIONG)

then $f \in \mathcal{S}^*(0)$.

Babalola [4] defined the class $\mathscr{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as follows:

Definition 1.1 ([4]) *Let $f \in \mathcal{A}$ and suppose that $0 \le \beta < 1$ and $\lambda \ge 1$. Then $f(z) \in \mathscr{L}_\lambda(\beta)$, consisting of λ-pseudo-starlike functions of order β in $\mathbb{U}$ if and only if*

$$\mathscr{R}(\frac{z[f'(z)]^\lambda}{f(z)}) > \beta, \quad 0 \le \beta < 1;\ \lambda \ge 1;\ z \in \mathbb{U}.$$

In particular, Babalola [4] proved that all λ-pseudo-starlike functions are Bazilevič of type $1-\frac{1}{\lambda}$ and order $\beta^{\frac{1}{\lambda}}$ and are univalent in open unit disk $\mathbb{U}$.

It is well known that every function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{U},$$

$$f(f^{-1}(\omega)) = \omega, \quad |\omega| < r_0(f); r_0(f) \ge \frac{1}{4}.$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of all bi-univalent functions in $\mathbb{U}$ given by (1.1).

Definition 1.2 ([5]) *A function $f \in \Sigma$ is said to be in the class $S_\Sigma^\lambda(k,\beta)$ if the following conditions are satisfied:*

$$R(\frac{z[(D^kf(z))']^\lambda}{D^kf(z)}) > \beta, \quad 0 \le \beta < 1;\ \lambda \ge 1;\ z \in \mathbb{U}$$

and

$$R(\frac{z[(D^kg(\omega))']^\lambda}{D^kg(\omega)}) > \beta, \quad 0 \le \beta < 1;\ \lambda \ge 1;\ \omega \in \mathbb{U}$$

where the function $g = f^{-1}, D^kf(z) = z + \sum_{n=2}^{+\infty} n^k a_n z^n$.

The class of bi-univalent functions was introduced by Lewin in 1967 in [6] and was showed that $|a_2| < 1.51$. Brannan and Clunie [7] conjectured that $|a_2| < \sqrt{2}$ for $f \in \Sigma$. Netanyahu [8] showed that max $|a_2| = 4/3$ if $f \in \Sigma$. Recently, many authors investigated bounds for various subclasses of analytic bi-univalent functions [9–11].

Let Σ' denote the class of meromorphic univalent functions g of the form:

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \tag{1.2}$$

which are defined on the domain $\mathbb{U}^*$ given by $\mathbb{U}^* = \{z : 1 < |z| < +\infty\}$. Since g is univalent, it has inverse $g^{-1} = h$ that satisfies the following conditions:

$$g^{-1}(g(z)) = z, \quad z \in \mathbb{U}^*,$$

$$g(g^{-1}(\omega)) = \omega, \quad 0 < M < |\omega| < +\infty,$$

where

$$g^{-1}(\omega) = h(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0b_1}{w^2} - \cdots, \quad 0 < M < |\omega| < +\infty. \tag{1.3}$$

Analogous to the bi-univalent analytic functions, a function $g \in \Sigma'$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma'$. We denote the class of all meromorphic bi-univalent functions by $\mathcal{M}_{\Sigma'}$. Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature, for example, Schiffer [12] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma'$ with $|b_0| = 0$ and Duren gave an elementary proof of the inequality $|b_n| \leq \frac{2}{n+1}$ on the coefficient of meromorphic univalent functions $g \in \Sigma'$ with $|b_k| = 0$ for $1 \leq k < \frac{n}{2}$.

Motivated by the earlier works [13–19], in the present investigation, several subclasses of meromorphic bi-univalent functions are introduced and estimates for the coefficients $|b_0|$ and $|b_1|$ of functions in the newly introduced subclasses are obtained.

In order to derive our main results, we shall need the following lemma.

Lemma 1.3 ([20]) *If $p(z) \in \mathscr{P}$, then $|c_n| \leq 2$ for each n, where $\mathscr{P}$ is the family of all functions p, analytic in $\mathbb{U}$ for which $\mathscr{R}\{p(z)\} > 0$, where*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad z \in \mathbb{U}. \tag{1.4}$$

2. Coefficient estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 > 0. \tag{2.1}$$

Define the functions p and q in $\mathscr{P}$ given by

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots,$$

$$q(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + \frac{q_1}{z} + \frac{q_2}{z^2} + \cdots.$$

It follows that

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}[\frac{p_1}{z} + (p_2 - \frac{p_1^2}{2})\frac{1}{z^2} + \cdots],$$

$$v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2}[\frac{q_1}{z} + (q_2 - \frac{q_1^2}{2})\frac{1}{z^2} + \cdots].$$

Note that for the functions $p(z), q(z) \in \mathscr{P}$, we have $|p_i| \leq 2$ and $|q_i| \leq 2$ for each i. By a simple calculation, we have

$$\varphi(u(z)) = 1 + \frac{B_1 p_1}{2z} + (\frac{B_1 p_2}{2} + \frac{B_2 - B_1}{4} p_1^2)\frac{1}{z^2} + \cdots, \quad 1 < |z| < +\infty, \tag{2.2}$$

$$\varphi(v(\omega)) = 1 + \frac{B_1 q_1}{2w} + (\frac{B_1 q_2}{2} + \frac{B_2 - B_1}{4} q_1^2)\frac{1}{w^2} + \cdots, \quad 1 < |z| < +\infty. \tag{2.3}$$

Definition 2.1 *A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $M_{\Sigma'}(\lambda, \varphi)$ if the following conditions are satisfied:*

$$\frac{z[g'(z)]^\lambda}{g(z)} \prec \varphi(z), \quad \lambda \geq 1; \ z \in \mathbb{U}^*, \tag{2.4}$$

$$\frac{w[h'(w)]^\lambda}{h(w)} \prec \varphi(w), \quad \lambda \geq 1;\ w \in \mathbb{U}^*, \tag{2.5}$$

where the function h is given by (1.3).

The class $M_{\Sigma'}(\lambda, \varphi)$ includes many earlier class, which are mentioned below:

(1) $M_{\Sigma'}(\lambda, (\frac{1+z}{1-z})^\alpha) = \Sigma_{B,\lambda^*}(\alpha)$ $(\lambda \geq 1; 0 < \alpha \leq 1)$ (see [13]);

(2) $M_{\Sigma'}(1, (\frac{1+z}{1-z})^\alpha) = \tilde{\Sigma}^*_{\mathscr{B}}(\alpha)$ $(0 < \alpha \leq 1)$ (see [14]);

(3) $M_{\Sigma'}(\lambda, (\frac{1+(1-2\beta)z}{1-z})) = \Sigma_{B^*}(\lambda, \beta)$ $(\lambda \geq 1; 0 \leq \beta < 1)$ (see [13]);

(4) $M_{\Sigma'}(1, (\frac{1+(1-2\beta)z}{1-z})) = \Sigma^*_{\mathscr{B}}(\beta)$ $(0 \leq \beta < 1)$ (see [14]).

Theorem 2.2 *Let $g(z)$ given by (1.2) be in the class $M_{\Sigma'}(\lambda, \varphi)$. Then*

$$|b_0| \leq \min\{B_1, \sqrt{B_1 + |B_2 - B_1|}, \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}}\}, \tag{2.6}$$

$$|b_1| \leq \frac{B_1}{1+\lambda}. \tag{2.7}$$

Proof Let $g(z) \in M_{\Sigma'}(\lambda, \varphi)$. Then, by Definition 2.1 of meromorphically bi-univalent function class $M_{\Sigma'}(\lambda, \varphi)$, the conditions (2.4) and (2.5) can be rewritten as follows:

$$\frac{z[g'(z)]^\lambda}{g(z)} = \varphi(u(z)) \tag{2.8}$$

and

$$\frac{w[h'(w)]^\lambda}{h(w)} = \varphi(v(\omega)). \tag{2.9}$$

In light of (1.2), (1.3) and (2.2)–(2.5), we have

$$\begin{aligned}\frac{z[g'(z)]^\lambda}{g(z)} &= 1 - \frac{b_0}{z} + \frac{b_0^2 - (1+\lambda)b_1}{z^2} + \frac{b_0^3 - (2+\lambda)b_0b_1 + (1+2\lambda)b_2}{z^3} + \cdots \\ &= 1 + \frac{B_1p_1}{2z} + (\frac{B_1p_2}{2} + \frac{B_2 - B_1}{4}p_1^2)\frac{1}{z^2} + \cdots \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\frac{w[h'(w)]^\lambda}{h(w)} &= 1 + \frac{b_0}{w} + \frac{b_0^2 + (1+\lambda)b_1}{w^2} + \frac{b_0^3 + 3(1+\lambda)b_0b_1 + (1+2\lambda)b_2}{w^3} + \cdots \\ &= 1 + \frac{B_1q_1}{2w} + (\frac{B_1q_2}{2} + \frac{B_2 - B_1}{4}q_1^2)\frac{1}{w^2} + \cdots . \end{aligned} \tag{2.11}$$

Now, equating the coefficients in (2.10) and (2.11), we get

$$-b_0 = \frac{B_1}{2}p_1, \tag{2.12}$$

$$b_0^2 - (1+\lambda)b_1 = \frac{B_1p_2}{2} + \frac{B_2 - B_1}{4}p_1^2, \tag{2.13}$$

$$b_0 = \frac{B_1}{2}q_1, \tag{2.14}$$

$$b_0^2 + (1+\lambda)b_1 = \frac{B_1q_2}{2} + \frac{B_2 - B_1}{4}q_1^2. \tag{2.15}$$

From (2.12) and (2.14), we get

$$p_1 = -q_1, \tag{2.16}$$

$$b_0^2 = \frac{B_1^2(p_1^2 + q_1^2)}{8}. \tag{2.17}$$

Applying Lemma 1.3 for the coefficients p_1 and q_1, we have

$$|b_0| \leq B_1. \tag{2.18}$$

Adding (2.13) and (2.15), we have

$$2b_0^2 = \frac{B_1(p_2 + q_2)}{2} + \frac{B_2 - B_1}{4}(p_1^2 + q_1^2). \tag{2.19}$$

Applying Lemma 1.3 for the coefficients p_1 and q_1, we have

$$|b_0| \leq \sqrt{B_1 + |B_2 - B_1|}. \tag{2.20}$$

Substituting (2.16) and (2.17) into (2.19), we get

$$p_1^2 = \frac{B_1(p_2 + q_2)}{B_1^2 + B_1 - B_2}. \tag{2.21}$$

From (2.16), (2.21) and (2.17), we get

$$b_0^2 = \frac{B_1^3(p_2 + q_2)}{4(B_1^2 + B_1 - B_2)}. \tag{2.22}$$

Then, in view of Lemma 1.3, we have

$$|b_0| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}}. \tag{2.23}$$

Now, from (2.18), (2.20) and (2.23), we get

$$|b_0| \leq \min\{B_1, \sqrt{B_1 + |B_2 - B_1|}, \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}}\}.$$

Substituting (2.12) into (2.13), we get

$$-(1+\lambda)b_1 = \frac{B_1 p_2}{2} + \frac{B_2 - B_1 - B_1^2}{4}p_1^2.$$

Applying Lemma 1.3 for the coefficients p_1 and p_2, we have

$$|b_1| \leq \frac{B_1 + |B_2 - B_1 - B_1^2|}{1 + \lambda}. \tag{2.24}$$

By subtracting (2.15) from (2.13), in view of (2.16), we have

$$-2(1+\lambda)b_1 = \frac{B_1(p_2 - q_2)}{2}.$$

Applying Lemma 1.3 for the coefficients p_2 and q_2, we have

$$|b_1| \leq \frac{B_1}{1 + \lambda}. \tag{2.25}$$

By using the Eqs. (2.13) and (2.15), we get

$$2(1+\lambda)^2 b_1^2 = \frac{B_1^2(p_2^2 + q_2^2)}{4} + \frac{B_1(B_2 - B_1)}{4}(p_2 p_1^2 + q_2 q_1^2) + \frac{(B_2 - B_1)^2}{16}(p_1^4 + q_1^4) - 2b_0^4. \tag{2.26}$$

Using Lemma 1.3, we have

$$(1+\lambda)^2|b_1^2| \leq B_1^2 + 2|B_1(B_2 - B_1)| + |(B_2 - B_1)^2| + |b_0^4|.$$

Substituting (2.12) into (2.26) and using Lemma 1.3, we get

$$|b_1| \le \frac{1}{1+\lambda}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+B_1^4}. \tag{2.27}$$

Substituting (2.19) into (2.26) and using Lemma 1.3, we get

$$|b_1| \le \frac{1}{1+\lambda}\sqrt{2B_1^2+4|B_1(B_2-B_1)|+2|(B_2-B_1)^2|}. \tag{2.28}$$

Substituting (2.22) into (2.26) and using Lemma 1.3, we get

$$|b_1| \le \frac{1}{1+\lambda}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+\frac{B_1^6}{|(B_1^2+B_1-B_2)^2|}}. \tag{2.29}$$

Then, from (2.24), (2.25) and (2.27)–(2.29), we have

$$\begin{aligned}|b_1| \le & \min\{\frac{B_1}{1+\lambda}, \frac{B_1+|B_2-B_1-B_1^2|}{1+\lambda}, \frac{1}{1+\lambda}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+B_1^4},\\ & \frac{1}{1+\lambda}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+\frac{B_1^6}{|(B_1^2+B_1-B_2)^2|}},\\ & \frac{1}{1+\lambda}\sqrt{2B_1^2+4|B_1(B_2-B_1)|+2|(B_2-B_1)^2|}\} = \frac{B_1}{1+\lambda}.\end{aligned}$$

This completes the proof of Theorem 2.2. □

Corollary 2.3 *Let $g(z)$ given by (1.2) be in the class $M_{\Sigma'}(\lambda, (\frac{1+z}{1-z})^{\alpha}) = \Sigma_{B,\lambda^*}(\alpha)$. Then*

$$|b_0| \le \min\{2\alpha, \sqrt{4\alpha-2\alpha^2}, \frac{2\alpha}{\sqrt{1+\alpha}}\} = \frac{2\alpha}{\sqrt{1+\alpha}}, \tag{2.30}$$

$$|b_1| \le \frac{2\alpha}{1+\lambda}. \tag{2.31}$$

Remark 2.4 Recall Srivastava et al. [13, Theorem 2.1] coefficient estimate, $|b_0| \le 2\alpha, |b_1| \le \frac{2\sqrt{5}\alpha^2}{1+\lambda}$ for functions $g(z) \in \Sigma_{B,\lambda^*}(\alpha)$, where the coefficient of $|b_1|$ should be $|b_1| \le \frac{2\alpha\sqrt{5\alpha^2-4\alpha+4}}{1+\lambda}$. The bounds on $|b_0|$ and $|b_1|$ given in (2.30) and (2.31) are more accurate than that given by [13, Theorem 2.1].

Corollary 2.5 *Let $g(z)$ given by (1.2) be in the class $M_{\Sigma'}(1, (\frac{1+z}{1-z})^{\alpha}) = \tilde{\Sigma}^*_{\mathscr{B}}(\alpha)$. Then*

$$|b_0| \le \min\{2\alpha, \sqrt{4\alpha-2\alpha^2}, \frac{2\alpha}{\sqrt{1+\alpha}}\} = \frac{2\alpha}{\sqrt{1+\alpha}}, \tag{2.32}$$

$$|b_1| \le \alpha. \tag{2.33}$$

Remark 2.6 Recall Halim et al. [14, Theorem 2] coefficient estimate, $|b_0| \le 2\alpha, |b_1| \le \sqrt{5}\alpha^2$ for functions $g(z) \in M_{\Sigma'}(1, (\frac{1+z}{1-z})^{\alpha}) = \Sigma^*_{\mathscr{B}}(\alpha)$, where the coefficient of $|b_1|$ should be $|b_1| \le \alpha\sqrt{5\alpha^2-4\alpha+4}$. Obviously, the bounds on $|b_0|$ and $|b_1|$ given in (2.32) and (2.33) are more accurate than that given by [14, Theorem 2].

Corollary 2.7 *Let $g(z)$ given by (1.2) be in the class $M_{\Sigma'}(\lambda, \frac{1+(1-2\beta)z}{1-z}) = \Sigma_{B^*}(\lambda,\beta)$. Then*

$$|b_0| \le \min\{2(1-\beta), \sqrt{2(1-\beta)}\} = \begin{cases} \sqrt{2(1-\beta)}, & \text{if } 0 \le \beta < \frac{1}{2},\\ 2(1-\beta), & \text{if } \frac{1}{2} \le \beta < 1\end{cases} \tag{2.34}$$

and

$$|b_1| \leq \frac{2(1-\beta)}{1+\lambda}. \tag{2.35}$$

Remark 2.8 Recall Srivastava et al. [13, Theorem 3.1] coefficient estimate, $|b_0| \leq 2(1-\beta), |b_1| \leq \frac{2(1-\beta)\sqrt{4\beta^2-8\beta+5}}{1+\lambda}$ for functions $g(z) \in \Sigma_{B^*}(\lambda, \beta)$, the bounds on $|b_0|$ and $|b_1|$ given in (2.34) and (2.35) are more accurate than that given by [13, Theorem 3.1].

Corollary 2.9 *Let $g(z)$ given by (1.2) be in the class $M_{\Sigma'}(1, (\frac{1+(1-2\beta)z}{1-z})) = \Sigma^*_{\mathscr{B}}(\beta)$. Then*

$$|b_0| \leq \min\{2(1-\beta), \sqrt{2(1-\beta)}\} = \begin{cases} \sqrt{2(1-\beta)}, & \text{if } 0 \leq \beta < \frac{1}{2}, \\ 2(1-\beta), & \text{if } \frac{1}{2} \leq \beta < 1 \end{cases} \tag{2.36}$$

and

$$|b_1| \leq 1-\beta. \tag{2.37}$$

Remark 2.10 Recall Halim et al. [14, Theorem 1] coefficient estimate, $|b_0| \leq 2(1-\beta), |b_1| \leq (1-\beta)\sqrt{4\beta^2-8\beta+5}$ for functions $g(z) \in M_{\sum'}(1, (\frac{1+(1-2\beta)z}{1-z})) = \Sigma^*_{\mathscr{B}}(\beta)$. Obviously, the bounds on $|b_0|$ and $|b_1|$ given in (2.36) and (2.37) are more accurate than that given by [14, Theorem 1].

Definition 2.11 *A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $B(\varphi, \mu)$ if the following conditions are satisfied:*

$$\frac{zg'(z)}{g(z)}(\frac{g(z)}{z})^{\mu} \prec \varphi(z), \quad 0 \leq \mu < 1;\ z \in \mathbb{U}^*,$$

$$\frac{wh'(w)}{h(w)}(\frac{h(w)}{w})^{\mu} \prec \varphi(w), \quad 0 \leq \mu < 1;\ w \in \mathbb{U}^*,$$

where the function h is given by (1.3).

The class $B(\varphi, \mu)$ includes many earlier class, which are mentioned below:

(1) $B((\frac{1+z}{1-z})^{\alpha}, \beta) = \Sigma^B_{\mathscr{B}}(\beta, \alpha)$ $(0 < \alpha \leq 1; \beta > 0)$ (see [14]);

(2) $B(\frac{1+(1-2\alpha)z}{1-z}, \beta) = B[\alpha, \beta]$ $(0 \leq \alpha < 1; 0 \leq \beta < 1)$ (see [15]).

By applying the method of the proof of Theorem 2.2, we can prove the following result.

Theorem 2.12 *Let $g(z)$ given by (1.2) be in the class $B(\varphi, \mu)$. Then*

$$|b_0| \leq \min\{\frac{B_1}{1-\mu}, \sqrt{\frac{2B_1+2|B_2-B_1|}{(2-\mu)(1-\mu)}}, \frac{B_1\sqrt{2B_1}}{\sqrt{|(\mu-2)(\mu-1)B_1^2+2(B_1-B_2)(\mu-1)^2|}}\},$$

$$\begin{aligned} |b_1| \leq \min\{&\frac{B_1}{2-\mu}, \frac{B_1+|B_2-B_1|+\frac{(2-\mu)B_1^2}{2(1-\mu)}}{2-\mu}, \frac{1}{2-\mu}\sqrt{2B_1^2+4|B_1(B_2-B_1)|+2|(B_2-B_1)^2|}, \\ &\frac{1}{2-\mu}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+\frac{(2-\mu)^2B_1^6}{|[(\mu-2)B_1^2+2(B_1-B_2)(\mu-1)]^2|}}, \\ &\frac{1}{2-\mu}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+\frac{(2-\mu)^2B_1^4}{4(1-\mu)^2}}\} \\ =&\frac{B_1}{2-\mu}. \end{aligned}$$

Corollary 2.13 *Let $g(z)$ given by (1.2) be in the class $B((\frac{1+z}{1-z})^{\alpha},\beta)=\Sigma_{\mathscr{B}}^{B}(\beta,\alpha)$. Then*

$$|b_0|\le \min\{\frac{2\alpha}{1-\beta},\frac{2\alpha}{\sqrt{|(\beta-2)(\beta-1)\alpha+(1-\alpha)(\beta-1)^2|}},\sqrt{\frac{8\alpha-4\alpha^2}{(2-\beta)(1-\beta)}}\}, \tag{2.38}$$

$$|b_1|\le \frac{2\alpha}{2-\beta}. \tag{2.39}$$

Remark 2.14 The bounds on $|b_0|$ and $|b_1|$ given in (2.38) and (2.39) are more accurate than that given by [14, Theorem 3].

Corollary 2.15 *Let $g(z)$ given by (1.2) be in the class $B(\frac{1+(1-2\alpha)z}{1-z},\beta)=B[\alpha,\beta]$. Then*

$$|b_0|\le \min\{\frac{2(1-\alpha)}{1-\beta},\frac{2\sqrt{1-\alpha}}{\sqrt{(1-\beta)(2-\beta)}}\}=\begin{cases}\frac{2\sqrt{1-\alpha}}{\sqrt{(1-\beta)(2-\beta)}}, & \text{if } 0\le\alpha<\frac{1}{2-\beta},\\ \frac{2(1-\alpha)}{1-\beta}, & \text{if } \frac{1}{2-\beta}\le\beta<1\end{cases} \tag{2.40}$$

and

$$|b_1|\le \frac{2(1-\alpha)}{2-\beta}. \tag{2.41}$$

Remark 2.16 The bounds on $|b_0|$ and $|b_1|$ given in (2.40) and (2.41) are more accurate than that given by [15, Theorem 2].

Definition 2.17 *A function $g(z)\in\Sigma'$ given by (1.2) is said to be in the class $\mathcal{M}_{\Sigma'}^{\mu}(\varphi)$ if the following conditions are satisfied:*

$$\frac{zg'(z)}{(1-\mu)g(z)+\mu zg'(z)}\prec\varphi(z),\quad 0\le\mu<1;\ z\in\mathbb{U}^*,$$
$$\frac{wh'(w)}{(1-\mu)h(w)+\mu wh'(w)}\prec\varphi(w),\quad 0\le\mu<1;\ w\in\mathbb{U}^*,$$

where the function h is given by (1.3).

The class $\mathcal{M}_{\Sigma'}^{\mu}(\varphi)$ includes many earlier class, which are mentioned below:

(1) $\mathcal{M}_{\Sigma'}^{\lambda}((\frac{1+z}{1-z})^{\alpha})=\mathscr{M}_{\sigma}(\alpha,\lambda)$ $(0<\alpha\le1;0\le\lambda<1)$ (see [16]);

(2) $\mathcal{M}_{\Sigma'}^{\lambda}(\frac{1+(1-2\beta)z}{1-z})=\mathscr{M}_{\sigma}(\beta,\lambda)$ $(0\le\beta<1;0\le\lambda<1)$ (see [16]).

By applying the method of the proof of Theorem 2.2, we can prove the following result.

Theorem 2.18 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}_{\Sigma'}^{\mu}(\varphi)$. Then*

$$|b_0|\le\min\{\frac{B_1}{1-\mu},\frac{\sqrt{B_1+|B_2-B_1|}}{1-\mu},\frac{B_1\sqrt{B_1}}{(1-\mu)\sqrt{|B_1^2-B_2+B_1|}}\},$$

$$\begin{aligned}|b_1|\le&\min\{\frac{B_1}{2(1-\mu)},\frac{B_1+|B_2-B_1-B_1^2|}{2(1-\mu)},\frac{1}{2(1-\mu)}\sqrt{2B_1^2+4|B_1(B_2-B_1)|+2|(B_2-B_1)^2|},\\ &\frac{1}{2(1-\mu)}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+\frac{B_1^6}{|(B_1^2+B_1-B_2)^2|}},\\ &\frac{1}{2(1-\mu)}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+B_1^4}\}\\ =&\frac{B_1}{2(1-\mu)}.\end{aligned}$$

Corollary 2.19 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}_{\Sigma'}^{\lambda}((\frac{1+z}{1-z})^{\alpha}) = \mathscr{M}_{\sigma}(\alpha, \lambda)$. Then*

$$|b_0| \leq \min\{\frac{2\alpha}{1-\lambda}, \frac{\sqrt{4\alpha - 2\alpha^2}}{1-\lambda}, \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}}\} = \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}}, \tag{2.42}$$

$$|b_1| \leq \frac{\alpha}{1-\lambda}. \tag{2.43}$$

Remark 2.20 The bounds on $|b_0|$ and $|b_1|$ given in (2.42) and (2.43) are more accurate than that given by [16, Theorem 5].

Corollary 2.21 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}_{\Sigma'}^{\lambda}(\frac{1+(1-2\beta)z}{1-z}) = \mathscr{M}_{\sigma}(\beta, \lambda)$. Then*

$$|b_0| \leq \min\{\frac{2(1-\beta)}{1-\lambda}, \frac{\sqrt{2(1-\beta)}}{1-\lambda}\} = \begin{cases} \frac{\sqrt{2(1-\beta)}}{1-\lambda}, & \text{if } 0 \leq \beta < \frac{1}{2}, \\ \frac{2(1-\beta)}{1-\lambda}, & \text{if } \frac{1}{2} \leq \beta < 1, \end{cases} \tag{2.44}$$

and

$$|b_1| \leq \frac{1-\beta}{1-\lambda}. \tag{2.45}$$

Remark 2.22 The bounds on $|b_0|$ given in (2.44) are more accurate than that given by [16, Theorem 10].

Definition 2.23 ([17]) *A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $\mathcal{M}_{\Sigma'}^{\gamma}(\lambda, \varphi)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}[\lambda \frac{zg'(z)}{g(z)} + (1-\lambda)(1 + \frac{zg''(z)}{g'(z)}) - 1] \prec \varphi(z), \quad \gamma \in \mathbb{C} - \{0\};\ 0 < \lambda \leq 1;\ z \in \mathbb{U}^*,$$

$$1 + \frac{1}{\gamma}[\lambda \frac{wh'(w)}{h(w)} + (1-\lambda)(1 + \frac{wh''(w)}{h'(w)}) - 1] \prec \varphi(w), \quad \gamma \in \mathbb{C} - \{0\};\ 0 < \lambda \leq 1;\ z \in \mathbb{U}^*,$$

where the function h is given by (1.3).

The class $\mathcal{M}_{\Sigma'}^{\gamma}(\lambda, \varphi)$ includes many earlier class, which are mentioned below:

(1) $\mathcal{M}_{\Sigma'}^{\gamma}(1, \varphi) = \mathcal{S}_{\Sigma'}^{\gamma}(\varphi)$ (see [17]);

(2) $\mathcal{M}_{\Sigma'}^{1}(1, \varphi) = \mathcal{S}_{\Sigma'}(\varphi)$ (see [17]).

By applying the method of the proof of Theorem 2.2, we can prove the following result.

Theorem 2.24 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}_{\Sigma'}^{\gamma}(\lambda, \varphi)$. Then*

$$|b_0| \leq \min\{\frac{|\gamma|B_1}{\lambda}, \sqrt{\frac{|\gamma|(B_1 + |B_2 - B_1|)}{\lambda}}, \frac{B_1|\gamma|\sqrt{B_1}}{\sqrt{\lambda|\gamma B_1^2 + \lambda(B_1 - B_2)|}}\}, \tag{2.46}$$

$$\begin{aligned} |b_1| \leq \min\{ & \frac{|\gamma|B_1}{2|1-2\lambda|}, \frac{|\gamma|}{2|1-2\lambda|}[B_1 + \frac{|\lambda(B_2 - B_1) - \gamma B_1^2|}{\lambda}], \\ & \frac{|\gamma|}{2|1-2\lambda|}\sqrt{2B_1^2 + 4|B_1(B_2 - B_1)| + 2|(B_2 - B_1)^2|}, \\ & \frac{|\gamma|}{2|1-2\lambda|}\sqrt{B_1^2 + 2|B_1(B_2 - B_1)| + |(B_2 - B_1)^2| + \frac{|\gamma^2|B_1^6}{|[\gamma B_1^2 + \lambda(B_1 - B_2)]^2|}}, \\ & \frac{|\gamma|}{2|1-2\lambda|}\sqrt{B_1^2 + 2|B_1(B_2 - B_1)| + |(B_2 - B_1)^2| + \frac{|\gamma^2|B_1^4}{\lambda^2}}\} = \frac{|\gamma|B_1}{2|1-2\lambda|}. \end{aligned} \tag{2.47}$$

Remark 2.25 Recall Murugusundaramoorthy et al. [17, Theorem 2.2] coefficient estimate, $|b_0| \le \frac{|\gamma|B_1}{\lambda}, |b_1| \le \frac{|\gamma|}{2(2\lambda-1)}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+\frac{|\gamma^2|B_1^4}{\lambda^2}}$ for functions $g(z) \in \mathcal{M}^{\gamma}_{\Sigma'}(\lambda,\varphi)$. We find that the bounds on $|b_0|$ and $|b_1|$ given in (2.46) and (2.47) are more accurate than that given by [17, Theorem 2.2].

Corollary 2.26 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}^{\gamma}_{\Sigma'}(1,\varphi) = \mathcal{S}^{\gamma}_{\Sigma'}(\varphi)$. Then*

$$|b_0| \le \min\{|\gamma|B_1, \sqrt{|\gamma|(B_1+|B_2-B_1|}, \frac{B_1|\gamma|\sqrt{B_1}}{\sqrt{|\gamma B_1^2+B_1-B_2|}}\}, \tag{2.48}$$

$$|b_1| \le \frac{|\gamma|B_1}{2}. \tag{2.49}$$

Remark 2.27 The bounds on $|b_0|$ and $|b_1|$ given in (2.48) and (2.49) are more accurate than that given by [17, Theorem 2.3].

Corollary 2.28 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}^{1}_{\Sigma'}(1,\varphi) = \mathcal{S}_{\Sigma'}(\varphi)$. Then*

$$|b_0| \le \min\{B_1, \sqrt{(B_1+|B_2-B_1|}, \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2+B_1-B_2|}}\}, \tag{2.50}$$

$$|b_1| \le \frac{B_1}{2}. \tag{2.51}$$

Remark 2.29 The bounds on $|b_0|$ and $|b_1|$ given in (2.50) and (2.51) are more accurate than that given by [17, Theorem 2.4].

Corollary 2.30 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}^{\gamma}_{\Sigma'}(\lambda,(\frac{1+z}{1-z})^{\alpha}) = \mathcal{M}_{\Sigma'}(\lambda,\alpha)$. Then*

$$|b_0| \le \min\Big\{\frac{2|\gamma|\alpha}{\lambda}, \sqrt{\frac{|\gamma|(4\alpha-2\alpha^2)}{\lambda}}, \frac{2\alpha|\gamma|}{\sqrt{\lambda|2\gamma\alpha+\lambda(1-\alpha)|}}\Big\}, \tag{2.52}$$

$$|b_1| \le \frac{\alpha|\gamma|}{|1-2\lambda|}. \tag{2.53}$$

Remark 2.31 The bounds on $|b_0|$ and $|b_1|$ given in (2.52) and (2.53) are more accurate than that given by [17, Corollary 3.1].

Corollary 2.32 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}^{1}_{\Sigma'}(\lambda,(\frac{1+z}{1-z})^{\alpha}) = \mathcal{S}_{\Sigma'}(\lambda,\alpha)$. Then*

$$|b_0| \le \{\frac{2\alpha}{\lambda}, \sqrt{\frac{4\alpha-2\alpha^2}{\lambda}}, \frac{2\alpha}{\sqrt{\lambda|2\alpha+\lambda(1-\alpha)|}}\}, \tag{2.54}$$

$$|b_1| \le \frac{\alpha}{|1-2\lambda|}. \tag{2.55}$$

Remark 2.33 The bounds on $|b_0|$ and $|b_1|$ given in (2.54) and (2.55) are more accurate than that given by [17, Corollary 3.3].

Corollary 2.34 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}^{\gamma}_{\Sigma'}(\lambda,\frac{1+(1-2\beta)z}{1-z}) = \mathcal{M}^{\gamma}_{\Sigma'}(\lambda,\beta)$. Then*

$$|b_0| \le \min\{\frac{2|\gamma|(1-\beta)}{\lambda}, \sqrt{\frac{2|\gamma|(1-\beta)}{\lambda}}\} = \begin{cases} \sqrt{\frac{2|\gamma|(1-\beta)}{\lambda}}, & \text{if } 0 \le \beta < 1-\frac{\lambda}{2|\gamma|}, \\ \frac{2|\gamma|(1-\beta)}{\lambda}, & \text{if } 1-\frac{\lambda}{2|\gamma|} \le \beta < 1 \end{cases} \tag{2.56}$$

and

$$|b_1| \leq \frac{|\gamma|(1-\beta)}{|1-2\lambda|}. \tag{2.57}$$

Remark 2.35 The bounds on $|b_0|$ and $|b_1|$ given in (2.56) and (2.57) are more accurate than that given by [17, Corollary 3.2].

Definition 2.36 *A function $g(z) \in \Sigma'$ given by (1.2) is said to be in the class $\mathcal{M}_{\Sigma'}^{\lambda}(\varphi)$ if the following conditions are satisfied:*

$$\frac{zg'(z)}{g(z)} + \lambda\frac{z^2g''(z)}{g'(z)} \prec \varphi(z), \quad 0 \leq \lambda < 1;\ z \in \mathbb{U}^*,$$

$$\frac{wh'(w)}{h(w)} + \lambda\frac{w^2h''(w)}{h'(w)} \prec \varphi(w), \quad 0 \leq \lambda < 1;\ w \in \mathbb{U}^*,$$

where the function h is given by (1.3).

By applying the method of the proof of Theorem 2.2, we can prove the following result.

Theorem 2.37 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}_{\Sigma'}^{\lambda}(\varphi)$. Then*

$$|b_0| \leq \min\{B_1, \sqrt{B_1+|B_2-B_1|}, \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2+B_1-B_2|}}\},$$

$$\begin{aligned}|b_1| \leq &\min\{\frac{B_1}{2(1-\lambda)}, \frac{B_1+|B_2-B_1-B_1^2|}{2(1-\lambda)}, \frac{1}{2(1-\lambda)}\sqrt{2B_1^2+4|B_1(B_2-B_1)|+2|(B_2-B_1)^2|},\\ &\frac{1}{2(1-\lambda)}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+B_1^4},\\ &\frac{1}{2(1-\lambda)}\sqrt{B_1^2+2|B_1(B_2-B_1)|+|(B_2-B_1)^2|+\frac{B_1^6}{|(B_1^2+B_1-B_2)^2|}}\}\\ =&\frac{B_1}{2(1-\lambda)}.\end{aligned}$$

Corollary 2.38 *If $g(z)$ given by (1.2) is in the class $\mathcal{M}_{\Sigma'}^{\lambda}(\frac{1+(1-2\alpha)}{1-\alpha})$, $0 \leq \alpha < 1$, and $0 \leq \lambda < 1$, then*

$$|b_0| \leq \min\{2(1-\alpha), \sqrt{2(1-\alpha)}\} = \begin{cases} \sqrt{2(1-\alpha)}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ 2(1-\alpha), & \text{if } \frac{1}{2} \leq \alpha < 1 \end{cases}$$

and

$$|b_1| \leq \frac{1-\alpha}{1-\lambda}.$$

Corollary 2.39 *Let $g(z)$ given by (1.2) be in the class $\mathcal{M}_{\Sigma'}^{\lambda}((\frac{1+z}{1-z})^{\beta})$ $(0 < \beta \leq 1; 0 \leq \lambda < 1)$. Then*

$$|b_0| \leq \min\{2\beta, \sqrt{4\beta-2\beta^2}, \frac{2\beta}{\sqrt{1+\beta}}\} = \frac{2\beta}{\sqrt{1+\beta}},$$

$$|b_1| \leq \frac{\beta}{1-\lambda}.$$

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