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An Extension to the Said-Type Generalized Ball Basis Functions

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Abstract In this paper, we present a general approach to the construction of the Said-type generalized Ball basis functions. The advantage of our approach lies in the fact that it can be used to derive the expressions for polynomials of not only odd degrees but also even degrees in terms of the Said-type generalized Ball basis functions. We then define dual functionals for the Said-type generalized Ball basis in a very natural manner and bring to light the integral property of the Said-type generalized Ball basis functions. Last but not least, new polynomial basis functions are defined which include the Said-type generalized Ball basis functions as their special case, and the corresponding dual functionals and Marsden-like identity are obtained.

Keywords generalized Ball basis; dual functionals; Marsden-like identity; integral property

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1. Introduction

Since Ball developed a cubic basis in his lofting surface program Consurf [1,2] in 1974, the Ball curves and surfaces, as a new method for computer aided geometric design, have aroused more and more global attention. Said [3] introduced a kind of generalized Ball basis by extending the Ball's cubic basis to the general polynomials of arbitrary odd degrees, defined the so-called generalized Ball curves of higher orders and obtained a recursive algorithm for efficient computation of the generalized Ball curves. Wang [4] presented another kind of generalized Ball basis by extending the Ball's basis to the general polynomials of arbitrary degrees and defined the generalized Ball curves of higher orders. The generalized Ball basis defined by Said [3] and Wang [4] are usually called the Said-type generalized Ball basis and the Wang-type generalized Ball basis, respectively. Goodman and Said [5, 6] investigated some properties of the Said-type generalized Ball curves and surfaces, including degree raising and lowering properties, approximate degree lowering properties and shape preserving properties. Hu et al. [7] further extended the Said-type generalized Ball basis functions to the arbitrary polynomial space by introducing the Said-type generalized Ball basis in even cases and studied the degree elevation and reduction, recursive

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algorithm, enveloping theorem and Bézier representation of both the Said-type and the Wang-type generalized Ball curves. Xi [8], and Othman and Goldman [9] independently launched the study of the dual basis functions for the Said-type generalized Ball basis of odd degrees, and Ding et al. [10] further worked out the dual basis for the Said-type generalized Ball basis of even degrees. As we know, the Hermite two-point Taylor interpolation method, as suggested by Said in [3], can be used to construct the Said-type generalized Ball basis polynomials of odd degrees, but fails for one to obtain the generalized Ball basis polynomials of even degrees. Our purpose in this paper is to present a new approach to the unified construction of the Said-type generalized Ball basis of both odd degrees and even degrees, which will be helpful and of enlightenment to study the generalized Ball surfaces on a triangular domain. As a by-product, we define the dual functionals in a very natural and easy manner, which seems to be a hard job in Xi's paper [8] and in the paper by Ding et al. [10]. We work out the integral property of the Said-type generalized Ball basis functions and point out the difference between Bézier basis and the Said-type generalized Ball basis in this respect. Last but not least, a new kind of polynomial basis functions are defined by extending the two-point based Said-type generalized Ball basis to multiple-point based ones which include the Said-type generalized Ball basis functions as their special case, and obtain the corresponding dual functionals and Marsden-like identity.

2. Said-type generalized Ball basis

Othman and Goldman [9] gave an expression for polynomial $p(t)$ of degree $2m+1$ in terms of the Said-type generalized Ball basis, which is based on the Hermite two-point Taylor interpolation [3]. In this section we derive slightly different expressions for polynomial $p(t)$ of degree $2m+1$ and degree $2m$, respectively by a new approach.

Proposition 2.1 *For any polynomial $p(t)$ of degree $2m+1$, there holds*

$$p(t)=\sum_{i=0}^{m}t^i(1-t)^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}p^{(k)}(0)\Big]+\sum_{i=0}^{m}(1-t)^it^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big]. \tag{2.1}$$

Proof It is easy to show that the $2m+2$ polynomials $\{t^i(1-t)^{m+1}\}_{i=0}^m$ and $\{(1-t)^it^{m+1}\}_{i=0}^m$ are linearly independent and therefore form a basis for the space P_{2m+1} of all polynomials of degree $2m+1$. For $p(t)\in P_{2m+1}$, there exist $a_i,b_i,i=0,1,\dots,m$ such that

$$p(t)=\sum_{i=0}^{m}a_it^i(1-t)^{m+1}+\sum_{i=0}^{m}b_i(1-t)^it^{m+1}. \tag{2.2}$$

In order to find out the coefficients $a_i, i=0,1,\dots,m$, we rewrite the above expression as follows

$$(1-t)^{-(m+1)}p(t)=\sum_{i=0}^{m}a_it^i+(1-t)^{-(m+1)}\sum_{i=0}^{m}b_i(1-t)^it^{m+1}$$

from which we see for $k = 0, 1, \ldots, m$.

$$a_k = \frac{1}{k!}\frac{d^k}{dt^k}[(1-t)^{-(m+1)}p(t)]|_{t=0} = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\frac{(m+k-l)!}{m!}p^{(l)}(0)$$
$$= \sum_{l=0}^{k}\binom{m+k-l}{m}\frac{1}{l!}p^{(l)}(0).$$

To find out the coefficients $b_i, i = 0, 1, \ldots, m$, we rewrite (2.2) as follows

$$t^{-(m+1)}p(t) = t^{-(m+1)}\sum_{i=0}^{m} a_i t^i (1-t)^{m+1} + \sum_{i=0}^{m} b_i (1-t)^i.$$

Differentiating the above expression k times at $t = 1$, one gets for $k = 0, 1, \ldots, m$

$$\begin{aligned} b_k &= \frac{(-1)^k}{k!}\frac{d^k}{dt^k}[t^{-(m+1)}p(t)]|_{t=1} \\ &= \frac{(-1)^k}{k!}\sum_{l=0}^{k}\binom{k}{l}\frac{(m+k-l)!}{m!}p^{(l)}(1)(-1)^{k-l} \\ &= \sum_{l=0}^{k}\binom{m+k-l}{m}\frac{(-1)^l}{l!}p^{(l)}(1). \end{aligned}$$

Consequently

$$\begin{aligned} p(t) = &\sum_{i=0}^{m} t^i(1-t)^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}p^{(k)}(0)\Big] + \\ &\sum_{i=0}^{m}(1-t)^i t^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big]. \end{aligned}$$

Proposition 2.1 is proved. □

In particular, if one sets $p(t) = 1$ in (2.1), then one obtains

$$\sum_{i=0}^{m}\binom{m+i}{m}t^i(1-t)^{m+1} + \sum_{i=0}^{m}\binom{m+i}{m}(1-t)^i t^{m+1} = 1. \tag{2.3}$$

Now let

$$\begin{cases} \beta_i^{2m+1}(t) = \binom{m+i}{m}t^i(1-t)^{m+1}, & 0 \le i \le m; \\ \beta_{2m+1-i}^{2m+1}(t) = \binom{m+i}{m}(1-t)^i t^{m+1}, & 0 \le i \le m, \end{cases} \tag{2.4}$$

which is the very Said-type generalized Ball basis defined in [5, 6], then the identical relation (2.3) now turns out to be nothing but the property of partition of unity for Said-type generalized Ball basis $\{\beta_i^{2m+1}(t)\}_{i=0}^{2m+1}$ and expression (2.1) can now be rewritten as

$$\begin{aligned} p(t) = &\sum_{i=0}^{m}\beta_i^{2m+1}(t)\binom{m+i}{m}^{-1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}p^{(k)}(0)\Big] + \\ &\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t)\binom{m+i}{m}^{-1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big]. \end{aligned} \tag{2.5}$$

In particular, if one sets $p(t)=(t-x)^r$, then

$$\frac{p^{(k)}(t)}{k!}=\binom{r}{k}(t-x)^{r-k}$$

and by (2.5) we have for $0\le r\le 2m+1$

$$\begin{aligned}(t-x)^r=&\sum_{i=0}^{m}\beta_i^{2m+1}(t)\Big[\sum_{k=0}^{i}(-1)^{r-k}\frac{\binom{m+i-k}{m}\binom{r}{k}}{\binom{m+i}{m}}x^{r-k}\Big]+\\ &\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t)\Big[\sum_{k=0}^{i}(-1)^{k}\frac{\binom{m+i-k}{m}\binom{r}{k}}{\binom{m+i}{m}}(1-x)^{r-k}\Big].\end{aligned}\tag{2.6}$$

This is the Marsden's identity given in [9]. If one sets $x=0$ in (2.6), then we obtain for $0\le r\le m$

$$\begin{aligned}t^r=&\sum_{i=r}^{m}\beta_i^{2m+1}(t)\binom{m+i}{m}^{-1}\binom{m+i-r}{m}+\\ &\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t)\binom{m+i}{m}^{-1}\sum_{k=0}^{i}\binom{m+i-k}{m}(-1)^k\binom{r}{k}\\ =&\sum_{i=r}^{m}\beta_i^{2m+1}(t)\binom{m+i}{m}^{-1}\binom{m+i-r}{m}+\\ &\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t)\binom{m+i}{m}^{-1}\binom{m+i-r}{i}\end{aligned}\tag{2.7}$$

where the combinatorial identical relation

$$\sum_{k=0}^{i}\binom{m+i-k}{m}(-1)^k\binom{r}{k}=\sum_{k=0}^{i}\binom{m+i-k}{i-k}\binom{k-r-1}{k}=\binom{m-r+i}{i}$$

is used. If $m<r\le 2m+1$, then we have

$$\begin{aligned}t^r=&\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t)\binom{m+i}{m}^{-1}\sum_{k=0}^{i}\binom{m+i-k}{m}(-1)^k\binom{r}{k}\\ =&\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t)\binom{m+i}{m}^{-1}\binom{m+i-r}{i}.\end{aligned}\tag{2.8}$$

(2.7) and (2.8) are the Marsden's identities developed by Xi in [8].

Similarly we can get

Proposition 2.2 *For any polynomial $p(t)$ of degree $2m$, it can be expressed as*

$$\begin{aligned}p(t)=&\sum_{i=0}^{m-1}t^i(1-t)^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}p^{(k)}(0)\Big]+\\ &\sum_{i=0}^{m-1}(1-t)^i t^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big]+\\ &t^m(1-t)^m\Big[\sum_{k=0}^{m}\binom{2m-k}{m}\frac{1}{k!}p^{(k)}(0)\Big]\end{aligned}\tag{2.9}$$

or

$$p(t) = \sum_{i=0}^{m-1} t^i(1-t)^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}p^{(k)}(0)\Big]+$$
$$\sum_{i=0}^{m-1}(1-t)^i t^{m+1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big]+$$
$$t^m(1-t)^m\Big[\sum_{k=0}^{m}\binom{2m-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big]. \tag{2.10}$$

This is the Said-type generalized Ball basis in even cases extended by Hu et al in [7]. Now the identical relation (2.9) turns out to be the property of partition of unity for even Said-type generalized Ball basis $\{\beta_i^{2m}(t)\}_{i=0}^{2m}$ and expressions (2.9) and (2.10) can now be rewritten as

$$p(t) = \sum_{i=0}^{m}\beta_i^{2m}(t)\binom{m+i}{m}^{-1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}p^{(k)}(0)\Big]+$$
$$\sum_{i=0}^{m-1}\beta_{2m-i}^{2m}(t)\binom{m+i}{m}^{-1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big] \tag{2.11}$$

or

$$p(t) = \sum_{i=0}^{m-1}\beta_i^{2m}(t)\binom{m+i}{m}^{-1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}p^{(k)}(0)\Big]+$$
$$\sum_{i=0}^{m}\beta_{2m-i}^{2m}(t)\binom{m+i}{m}^{-1}\Big[\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}p^{(k)}(1)\Big]. \tag{2.12}$$

In particular, if one sets $p(t) = (t-x)^r$, then by (2.11) and (2.12) for $0 \le r \le 2m$ we have the following Marsden's identity

$$(t-x)^r = \sum_{i=0}^{m}\beta_i^{2m}(t)\Big[\sum_{k=0}^{i}(-1)^{r-k}\frac{\binom{m+i-k}{m}\binom{r}{k}}{\binom{m+i}{m}}x^{r-k}\Big]+$$
$$\sum_{i=0}^{m-1}\beta_{2m-i}^{2m}(t)\Big[\sum_{k=0}^{i}(-1)^{k}\frac{\binom{m+i-k}{m}\binom{r}{k}}{\binom{m+i}{m}}(1-x)^{r-k}\Big] \tag{2.13}$$

or

$$(t-x)^r = \sum_{i=0}^{m-1}\beta_i^{2m}(t)\Big[\sum_{k=0}^{i}(-1)^{r-k}\frac{\binom{m+i-k}{m}\binom{r}{k}}{\binom{m+i}{m}}x^{r-k}\Big]+$$
$$\sum_{i=0}^{m}\beta_{2m-i}^{2m}(t)\Big[\sum_{k=0}^{i}(-1)^{k}\frac{\binom{m+i-k}{m}\binom{r}{k}}{\binom{m+i}{m}}(1-x)^{r-k}\Big]. \tag{2.14}$$

3. Dual functionals of Said-type generalized Ball basis

In this section, we focus on how to construct the dual functionals for Said-type generalized Ball basis functions. Although the dual basis was discussed in [8–10], our approach is constructive and natural.

Proposition 3.1 *The dual functionals for the Said-type generalized Ball basis functions $\beta_i^{2m+1}(t)$ are given by*

$$\begin{aligned}\lambda_i f &= \tbinom{m+i}{m}^{-1}\textstyle\sum_{k=0}^{i}\tbinom{m+i-k}{m}\frac{1}{k!}f^{(k)}(0), && i=0,1,\ldots,m;\\ \lambda_{2m+1-i} f &= \tbinom{m+i}{m}^{-1}\textstyle\sum_{k=0}^{i}\tbinom{m+i-k}{m}\frac{(-1)^k}{k!}f^{(k)}(1), && i=0,1,\ldots,m.\end{aligned} \tag{3.1}$$

Proof By (2.5) we have

$$\beta_j^{2m+1}(t)=\sum_{i=0}^{m}\beta_i^{2m+1}(t)\lambda_i\beta_j^{2m+1}(t)+\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t)\lambda_{2m+1-i}\beta_j^{2m+1}(t). \tag{3.2}$$

From the independence of $\{\beta_i^{2m+1}(t)\}_{i=0}^{2m+1}$ it follows

$$\lambda_i\beta_j^{2m+1}=\delta_{ij}=\begin{cases}1, & \text{for } i=j;\\ 0, & \text{for } i\neq j.\end{cases}$$

This completes the proof of Proposition 3.1. □

Similarly by using (2.14) and (3.1) one can work out the following proposition concerning the even case.

Proposition 3.2 *The dual functionals for the Said-type generalized Ball basis functions $\beta_i^{2m}(t)$ are given by*

$$\begin{aligned}\mu_i f &= \binom{m+i}{m}^{-1}\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}f^{(k)}(0), && i=0,1,\ldots,m;\\ \mu_{2m-i} f &= \binom{m+i}{m}^{-1}\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}f^{(k)}(1), && i=0,1,\ldots,m-1\end{aligned} \tag{3.3}$$

or

$$\begin{aligned}\mu_i f &= \binom{m+i}{m}^{-1}\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{1}{k!}f^{(k)}(0), && i=0,1,\ldots,m-1;\\ \mu_{2m-i} f &= \binom{m+i}{m}^{-1}\sum_{k=0}^{i}\binom{m+i-k}{m}\frac{(-1)^k}{k!}f^{(k)}(1), && i=0,1,\ldots,m.\end{aligned} \tag{3.4}$$

4. Integral property of Said-type generalized Ball basis

For the Bézier basis functions

$$B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i},$$

there is a well known integral formula

$$\int_0^1 B_i^n(t)\mathrm{d}t=\frac{1}{n+1},\quad \forall i=0,1,\ldots,n.$$

This formula shows that the Bézier basis functions are uniformly distributed on the interval $[0,1]$, or in other words, each of the $n+1$ basis functions has the same moment. However, this is not the case for the Said-type generalized Ball basis functions.

Proposition 4.1 *For the Said-type generalized Ball basis functions $\beta_i^{2m+1}(t)$ defined in (2.4)* *there holds*

$$\int_0^1 \beta_i^{2m+1}(t)\mathrm{d}t = \int_0^1 \beta_{2m+1-i}^{2m+1}(t)\mathrm{d}t = \frac{m+1}{(m+i+1)(m+i+2)}, \quad i = 0, 1, \ldots, m, \tag{4.1}$$

and for the Said-type generalized Ball basis functions $\beta_i^{2m}(t)$ defined in (2.13) there holds

$$\int_0^1 \beta_i^{2m}(t)\mathrm{d}t = \int_0^1 \beta_{2m-i}^{2m}(t)\mathrm{d}t = \frac{m+1}{(m+i+1)(m+i+2)}, \quad i = 0, 1, \ldots, m-1, \tag{4.2}$$

and

$$\int_0^1 \beta_m^{2m}(t)\mathrm{d}t = \frac{1}{2m+1}. \tag{4.3}$$

Proof Making use of the definition of the beta functions and the relation between beta functions and gamma functions, we have

$$\begin{aligned}
&\int_0^1 \beta_i^{2m+1}(t)\mathrm{d}t = \int_0^1 \beta_{2m+1-i}^{2m+1}(t)\mathrm{d}t = \binom{m+i}{m} \int_0^1 t^i (1-t)^{m+1}\mathrm{d}t \\
&= \binom{m+i}{m} \int_0^1 (1-t)^i t^{m+1}\mathrm{d}t = \binom{m+i}{m} B(i+1, m+2) \\
&= \frac{(m+i)!}{m!i!} \frac{\Gamma(i+1)\Gamma(m+2)}{\Gamma(m+i+3)} = \frac{(m+i)!}{m!i!} \frac{i!(m+1)!}{(m+i+2)!} \\
&= \frac{m+1}{(m+i+1)(m+i+2)}, \quad i = 0, 1, \ldots, m.
\end{aligned}$$

Similarly one can prove (4.2) and (4.3). Proposition 4.1 is proved. □

Clearly the Said-type generalized Ball basis functions neither in odd cases nor in even cases are uniformly distributed on the interval $[0, 1]$. Their moments are symmetric and decrease with increasing index i. $\beta_m^{2m+1}(t)$ and $\beta_{m+1}^{2m+1}(t)$ have the minimum moment $1/(4m+2)$ while $\beta_{m-1}^{2m}(t)$ and $\beta_{m+1}^{2m}(t)$ share the minimum moment $(m+1)/[m(4m+2)]$. It is easy to get

$$\begin{aligned}
\sum_{i=0}^{m} \int_0^1 \beta_i^{2m+1}(t)\mathrm{d}t &= \sum_{i=0}^{m} \int_0^1 \beta_{2m+1-i}^{2m+1}(t)\mathrm{d}t = \frac{1}{2}, \\
\sum_{i=0}^{m-1} \int_0^1 \beta_i^{2m}(t)\mathrm{d}t &= \sum_{i=0}^{m-1} \int_0^1 \beta_{2m-i}^{2m}(t)\mathrm{d}t = \frac{m}{2m+1},
\end{aligned}$$

which implies that the total moments of the Said-type generalized Ball basis functions in both odd and even cases sum up to the unit one.

5. New polynomial basis functions

In this section, we define a new kind of polynomial basis functions based on divided d-ifferences, which include the Said-type generalized Ball basis functions as their special case. As we know, the divided differences [11] of a bivariate function $f(x, y)$ at the grid of points

$\{x_1, x_2, \dots, x_r\} \times \{y_1, y_2, \dots, y_s\}$ are defined as follows

$$\begin{aligned}
&f[x;y] = f(x,y),\\
&f[x;y_1,y_2] = \frac{f[x;y_2]-f[x;y_1]}{y_2-y_1},\\
&f[x;y_1,y_2,\dots,y_s] = \frac{f[x;y_1,\dots,y_{s-2,}y_s]-f[x;y_1,\dots,y_{s-1}]}{y_s-y_{s-1}},\\
&f[x_1,x_2;y] = \frac{f[x_2;y]-f[x_1;y]}{x_2-x_1},\\
&f[x_1,x_2,\dots,x_r;y] = \frac{f[x_1,\dots,x_{r-2},x_r;y]-f[x_1,\dots,x_{r-1};y]}{x_r-x_{r-1}},\\
&f[x_1,\dots,x_r;y_1,\dots,y_s] = \frac{f[x_1,\dots,x_r;y_1,\dots,y_{s-2,}y_s]-f[x_1,\dots,x_r;y_1,\dots,y_{s-1}]}{y_s-y_{s-1}}\\
&\qquad\qquad\qquad\qquad\quad = \frac{f[x_1,\dots,x_{r-2},x_r;y_1,\dots,y_s]-f[x_1,\dots,x_{r-1};y_1,\dots,y_s]}{x_r-x_{r-1}}.
\end{aligned}$$

Proposition 5.1 *Suppose that $p(t)$ is a polynomial of degree $2m+1$. Then $p(t)$ can be expressed as*

$$\begin{aligned}
p(t) =& \sum_{i=0}^{m}\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)\sum_{k=1}^{i+1}p[u_1,\dots,u_k]q[u_k,\dots,u_{i+1};v_1,\dots,v_{m+1}]+\\
&\sum_{i=0}^{m}\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j)\sum_{k=1}^{i+1}p[v_1,\dots,v_k]q[v_k,\dots,v_{i+1};u_1,\dots,u_{m+1}],
\end{aligned}$$

where $u_i \neq v_j$ for $i = 0,1,\dots,m$; $j = 0,1,\dots,m$, $q(x,y) = 1/(x-y)$, and $p[u_1,\dots,u_k]$ denotes the divided difference of $p(t)$ at points $u_1,\dots,u_k$ while $q[u_k,\dots,u_{i+1};v_1,\dots,v_{m+1}]$ stands for the divided difference of bivariate function $q(x,y)$ at grid of points $\{u_k,\dots,u_{i+1}\}\times\{v_1,\dots,v_{m+1}\}$.

Proof It is easy to verify that $\{\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)\}_{i=0}^{m}, \{\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j)\}_{i=0}^{m}$ are the basis functions of the polynomial space P_{2m+1} of degree $2m+1$, which means that there exist $a_i, b_i, i = 0,1,\dots,m$ such that

$$p(t) = \sum_{i=0}^{m}a_i\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j) + \sum_{i=0}^{m}b_i\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j).$$

Therefore

$$\begin{aligned}
a_i &= (\frac{p(t)}{\prod_{j=1}^{m+1}(t-v_j)})[u_1,\dots,u_{i+1}]\\
&= \sum_{k=1}^{i+1}p[u_1,\dots,u_k](\frac{1}{\prod_{j=1}^{m+1}(t-v_j)})[u_k,\dots,u_{i+1}]\\
&= \sum_{k=1}^{i+1}p[u_1,\dots,u_k]q[u_k,\dots,u_{i+1};v_1,\dots,v_{m+1}]
\end{aligned}$$

and

$$b_i = \Big(\frac{p(t)}{\prod_{j=1}^{m+1}(t-u_j)}\Big)[v_1,\ldots,v_{i+1}]$$
$$= \sum_{k=1}^{i+1} p[v_1,\ldots,v_k]\Big(\frac{1}{\prod_{j=1}^{m+1}(t-u_j)}\Big)[v_k,\ldots,v_{i+1}]$$
$$= \sum_{k=1}^{i+1} p[v_1,\ldots,v_k]q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}].$$

Substituting the above a_i, b_i into the expression of $p(t)$, one gets

$$p(t) = \sum_{i=0}^{m}\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)\sum_{k=1}^{i+1}p[u_1,\ldots,u_k]q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]+$$
$$\sum_{i=0}^{m}\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j)\sum_{k=1}^{i+1}p[v_1,\ldots,v_k]q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]$$

as asserted. □

Let $p(t) = 1$. Then we have by Proposition 5.1

$$\sum_{i=0}^{m} q[u_1,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)+$$
$$\sum_{i=0}^{m} q[v_1,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j) = 1.$$

Set for $i = 0, 1, \ldots, m$

$$\beta_i^{2m+1}(t;u,v) = q[u_1,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)$$

and

$$\beta_{2m+1-i}^{2m+1}(t;u,v) = q[v_1,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j).$$

Then every $p(t)$ in P_{2m+1} can be expressed in terms of the basis functions $\{\beta_i^{2m+1}(t;u,v)\}_{i=0}^{2m+1}$

$$p(t) = \sum_{i=0}^{m}\beta_i^{2m+1}(t;u,v)\sum_{k=1}^{i+1}\frac{q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]}{q[u_1,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]}p[u_1,\ldots,u_k]+$$
$$\sum_{i=0}^{m}\beta_{2m+1-i}^{2m+1}(t;u,v)\sum_{k=1}^{i+1}\frac{q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]}{q[v_1,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]}p[v_1,\ldots,v_k].$$

Now we define the following linear functionals

$$\lambda_i f = \sum_{k=1}^{i+1}\frac{q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]}{q[u_1,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]}f[u_1,\ldots,u_k], \quad i = 0,1,\ldots,m,$$
$$\lambda_{2m+1-i} f = \sum_{k=1}^{i+1}\frac{q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]}{q[v_1,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]}f[v_1,\ldots,v_k], \quad i = 0,1,\ldots,m.$$

Then

$$p(t) = \sum_{i=0}^{m} (\lambda_i p)\beta_i^{2m+1}(t;u,v) + \sum_{i=0}^{m} (\lambda_{2m+1-i}p)\beta_{2m+1-i}^{2m+1}(t;u,v)$$

and the independence of $\{\beta_i^{2m+1}(t;u,v)\}_{i=0}^{2m+1}$ implies

$$\lambda_i \beta_j^{2m+1}(t;u,v) = \delta_{i,j}, \quad i,j = 0,1,\ldots,2m+1.$$

Therefore $\lambda_i f$, $i = 0,1,\ldots,2m+1$, are the dual functionals with respect to the basis functions $\{\beta_i^{2m+1}(t;u,v)\}_{i=0}^{2m+1}$. In order to derive the dual functionals in even cases, we need the following lemma.

Lemma 5.2 *For $p(t) \in P_{2m}$ and $q(x,y) = 1/(x-y)$, there holds*

$$\begin{aligned}&\sum_{k=1}^{m+1} p[u_1,\ldots,u_k]q[u_k,\ldots,u_{m+1};v_1,\ldots,v_{m+1}]\\ &\quad = -\sum_{k=1}^{m+1} p[v_1,\ldots,v_k]q[v_k,\ldots,v_{m+1};u_1,\ldots,u_{m+1}].\end{aligned}$$

Proof It suffices to prove that Lemma 5.2 is valid for $p(t) = \prod_{j=1}^{n}(t-u_j)$ for $0 \le n \le m$ and $p(t) = \prod_{j=1}^{m+1}(t-u_j)\prod_{j=1}^{n}(t-v_j)$ for $0 \le n \le m-1$. From

$$\begin{aligned}&\sum_{k=1}^{m+1} p[v_1,\ldots,v_k]q[v_k,\ldots,v_{m+1};u_1,\ldots,u_{m+1}]\\ &\quad = \Big(\frac{p(t)}{\prod_{j=1}^{m+1}(t-u_j)}\Big)[v_1,\ldots,v_{m+1}],\end{aligned}$$

it follows for $p(t) = \prod_{j=1}^{n}(t-u_j)$ for $0 \le n \le m$

$$\begin{aligned}&\sum_{k=1}^{m+1} p[v_1,\ldots,v_k]q[v_k,\ldots,v_{m+1};u_1,\ldots,u_{m+1}]\\ &\quad = \Big(\frac{1}{\prod_{j=n+1}^{m+1}(t-u_j)}\Big)[v_1,\ldots,v_{m+1}]\\ &\quad = q[v_1,\ldots,v_{m+1};u_{n+1},\ldots,u_{m+1}]\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=1}^{m+1} p[u_1,\ldots,u_k]q[u_k,\ldots,u_{m+1};v_1,\ldots,v_{m+1}]\\ &\quad = p[u_1,\ldots,u_{n+1}]q[u_{n+1},\ldots,u_{m+1};v_1,\ldots,v_{m+1}]\\ &\quad = -q[v_1,\ldots,v_{m+1};u_{n+1},\ldots,u_{m+1}],\end{aligned}$$

since $q(x,y) = -q(y,x)$.

Therefore Lemma 5.2 holds true for $p(t) = \prod_{j=1}^{n}(t-u_j)$, $0 \le n \le m$. Since $p(t) = \prod_{j=1}^{m+1}(t-u_j)\prod_{j=1}^{n}(t-v_j)$ for $0 \le n \le m-1$, we have $p[u_1,\ldots,u_k] = 0$, $k = 1,2,\ldots,m+1$, which leads to

$$\sum_{k=1}^{m+1} p[u_1,\ldots,u_k]q[u_k,\ldots,u_{m+1};v_1,\ldots,v_{m+1}] = 0.$$

In this case, we also have

$$\begin{aligned}
&\sum_{k=1}^{m+1} p[v_1,\ldots,v_k]q[v_k,\ldots,v_{m+1};u_1,\ldots,u_{m+1}]\\
&=\Big(\frac{p(t)}{\prod_{j=1}^{m+1}(t-u_j)}\Big)[v_1,\ldots,v_{m+1}]\\
&=\Big(\prod_{j=1}^{n}(t-v_j)\Big)[v_1,\ldots,v_{m+1}]=0.
\end{aligned}$$

As a result, Lemma 5.2 also holds true for $p(t)=\prod_{j=1}^{m+1}(t-u_j)\prod_{j=1}^{n}(t-v_j)$ for $0\le n\le m-1$. This completes the proof of Lemma 5.2. □

Proposition 5.3 *For $p(t)\in P_{2m}$ and $q(x,y)=1/(x-y)$ there holds*

$$\begin{aligned}
p(t)=&\sum_{i=0}^{m-1}\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)\sum_{k=1}^{i+1}p[u_1,\ldots,u_k]q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]+\\
&\sum_{i=0}^{m-1}\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j)\sum_{k=1}^{i+1}p[v_1,\ldots,v_k]q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]+\\
&\prod_{j=1}^{m}(t-u_j)(t-v_j)(u_{m+1}-v_{m+1})\sum_{k=1}^{m+1}p[u_1,\ldots,u_k]q[u_k,\ldots,u_{m+1};v_1,\ldots,v_{m+1}].
\end{aligned}$$

Proof By Proposition 5.1 and Lemma 5.2

$$\begin{aligned}
p(t)=&\sum_{i=0}^{m-1}\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)\sum_{k=1}^{i+1}p[u_1,\ldots,u_k]q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]+\\
&\sum_{i=0}^{m-1}\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j)\sum_{k=1}^{i+1}p[v_1,\ldots,v_k]q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]+\\
&\prod_{j=1}^{m}(t-u_j)(t-v_j)((t-v_{m+1})\sum_{k=1}^{m+1}p[u_1,\ldots,u_k]q[u_k,\ldots,u_{m+1};v_1,\ldots,v_{m+1}]+\\
&(t-u_{m+1})\sum_{k=1}^{m+1}p[v_1,\ldots,v_k]q[v_k,\ldots,v_{m+1};u_1,\ldots,u_{m+1}])\\
=&\sum_{i=0}^{m-1}\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j)\sum_{k=1}^{i+1}p[u_1,\ldots,u_k]q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]+\\
&\sum_{i=0}^{m-1}\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j)\sum_{k=1}^{i+1}p[v_1,\ldots,v_k]q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]+\\
&\prod_{j=1}^{m}(t-u_j)(t-v_j)(u_{m+1}-v_{m+1})\sum_{k=1}^{m+1}p[u_1,\ldots,u_k]q[u_k,\ldots,u_{m+1};v_1,\ldots,v_{m+1}].
\end{aligned}$$

Proposition 5.3 is proved. □

Let

$$\beta_i^{2m}(t;u,v) = q[u_1,\dots,u_{i+1};v_1,\dots,v_{m+1}]\prod_{j=1}^{i}(t-u_j)\prod_{j=1}^{m+1}(t-v_j), \quad 0\le i\le m-1,$$

$$\beta_{2m-i}^{2m}(t;u,v) = q[v_1,\dots,v_{i+1};u_1,\dots,u_{m+1}]\prod_{j=1}^{i}(t-v_j)\prod_{j=1}^{m+1}(t-u_j), \quad 0\le i\le m-1,$$

$$\beta_m^{2m}(t;u,v) = (u_{m+1}-v_{m+1})q[u_1,\dots,u_{m+1};v_1,\dots,v_{m+1}]\prod_{j=1}^{m}(t-u_j)(t-v_j).$$

Then $p(t)$ can be expressed as the linear combination of $\beta_i^{2m}(t;u,v)$, $i=0,1,\dots,2m$

$$\begin{aligned} p(t) = &\sum_{i=0}^{m-1}\beta_i^{2m}(t;u,v)\sum_{k=1}^{i+1}\frac{q[u_k,\dots,u_{i+1};v_1,\dots,v_{m+1}]}{q[u_1,\dots,u_{i+1};v_1,\dots,v_{m+1}]}p[u_1,\dots,u_k]+\\ &\sum_{i=0}^{m-1}\beta_{2m-i}^{2m}(t;u,v)\sum_{k=1}^{i+1}\frac{q[v_k,\dots,v_{i+1};u_1,\dots,u_{m+1}]}{q[v_1,\dots,v_{i+1};u_1,\dots,u_{m+1}]}p[v_1,\dots,v_k]+\\ &\beta_m^{2m}(t;u,v)\sum_{k=1}^{m+1}\frac{q[u_k,\dots,u_{m+1};v_1,\dots,v_{m+1}]}{q[u_1,\dots,u_{m+1};v_1,\dots,v_{m+1}]}p[u_1,\dots,u_k]. \end{aligned}$$

Now we define the following linear functionals

$$\lambda_i f = \sum_{k=1}^{i+1}\frac{q[u_k,\dots,u_{i+1};v_1,\dots,v_{m+1}]}{q[u_1,\dots,u_{i+1};v_1,\dots,v_{m+1}]}f[u_1,\dots,u_k], \quad i=0,1,\dots,m,$$

$$\lambda_{2m-i} f = \sum_{k=1}^{i+1}\frac{q[v_k,\dots,v_{i+1};u_1,\dots,u_{m+1}]}{q[v_1,\dots,v_{i+1};u_1,\dots,u_{m+1}]}f[v_1,\dots,v_k], \quad i=0,1,\dots,m-1.$$

Then

$$p(t) = \sum_{i=0}^{m}(\lambda_i p)\beta_i^{2m}(t;u,v) + \sum_{i=0}^{m-1}(\lambda_{2m-i}p)\beta_{2m-i}^{2m}(t;u,v).$$

From

$$\beta_j^{2m}(t;u,v) = \sum_{i=0}^{m}(\lambda_i\beta_j^{2m}(t;u,v))\beta_i^{2m}(t;u,v) + \sum_{i=0}^{m-1}(\lambda_{2m-i}\beta_j^{2m}(t;u,v))\beta_{2m-i}^{2m}(t;u,v)$$

and the independence of $\{\beta_i^{2m}(t;u,v)\}_{i=0}^{2m}$ it follows

$$\lambda_i\beta_j^{2m}(t;u,v) = \delta_{i,j}, \quad i,j=0,1,\dots,2m.$$

Therefore $\lambda_i f$, $i=0,1,\dots,2m$, are the dual functionals with respect to the basis functions $\{\beta_i^{2m}(t;u,v)\}_{i=0}^{2m}$. Notice

$$(\cdot - x)^r[u_1,\dots,u_k] = \begin{cases} 0, & k>r+1;\\ \sum_{j=1}^{k}\frac{(u_j-x)^r}{(u_j-u_1)\cdots(u_j-u_{j-1})(u_j-u_{j+1})\cdots(u_j-u_k)}, & k\le r+1, \end{cases}$$

we obtain the following Marsden-like identity

$$
\begin{aligned}
(t-x)^r =& \sum_{i=0}^{m} \beta_i^{2m+1}(t;u,v) \sum_{k=1}^{\min\{r+1,i+1\}} \frac{q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]}{q[u_1,\ldots,u_{i+1};v_1,\ldots,v_{m+1}]} \cdot \\
& \sum_{j=1}^{k} \frac{(u_j-x)^r}{(u_j-u_1)\cdots(u_j-u_{j-1})(u_j-u_{j+1})\cdots(u_j-u_k)} + \\
& \sum_{i=0}^{m} \beta_{2m+1-i}^{2m+1}(t;u,v) \sum_{k=1}^{\min\{r+1,i+1\}} \frac{q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]}{q[v_1,\ldots,v_{i+1};u_1,\ldots,u_{m+1}]} \cdot \\
& \sum_{j=1}^{k} \frac{(v_j-x)^r}{(v_j-v_1)\cdots(v_j-v_{j-1})(v_j-v_{j+1})\cdots(v_j-v_k)}.
\end{aligned}
$$

Denote by $I(u_1,\ldots,u_k)$ the smallest interval that contains $u_1,\ldots,u_k$. Then making use of the relation between divided differences and derivatives, one gets

$$p[u_1,\ldots,u_k] = \frac{p^{(k-1)}(\xi_k)}{(k-1)!}, \quad \xi_k \in I(u_1,\ldots,u_k),$$

$$p[v_1,\ldots,v_k] = \frac{p^{(k-1)}(\eta_k)}{(k-1)!}, \quad \eta_k \in I(v_1,\ldots,v_k),$$

$$
\begin{aligned}
q[u_k,\ldots,u_{i+1};v_1,\ldots,v_{m+1}] =& \frac{1}{(i-k+1)!m!} \frac{\partial^{i-k+m+1}}{\partial x^{i-k+1}\partial y^m} q(\xi_{k,i},\eta) \\
=& (-1)^{m+1}\binom{m+i-k+1}{m}(\eta-\xi_{k,i})^{-(m+i-k+2)}
\end{aligned}
$$

and

$$q[v_k,\ldots,v_{i+1};u_1,\ldots,u_{m+1}] = (-1)^{m+1}\binom{m+i-k+1}{m}(\xi-\eta_{k,i})^{-(m+i-k+2)},$$

where $\xi_{k,i} \in I(u_k,\ldots,u_{i+1}), \eta \in I(v_1,\ldots,v_{m+1}), \xi \in I(u_1,\ldots,u_{m+1}), \eta_{k,i} \in I(v_k,\ldots,v_{i+1})$. Therefore for $p(t) \in P_{2m+1}$, we have

$$
\begin{aligned}
p(t) =& \sum_{i=0}^{m} \prod_{j=1}^{i}(t-u_j) \prod_{j=1}^{m+1}(v_j-t) \sum_{k=0}^{i} \binom{m+i-k}{m} \frac{p^{(k)}(\xi_k)}{k!}(\eta-\xi_{k+1,i})^{-(m+i-k+1)} + \\
& \sum_{i=0}^{m} \prod_{j=1}^{i}(v_j-t) \prod_{j=1}^{m+1}(t-u_j) \sum_{k=0}^{i} \binom{m+i-k}{m} \frac{(-1)^k p^{(k)}(\eta_k)}{k!}(\eta_{k+1,i}-\xi)^{-(m+i-k+1)}.
\end{aligned}
$$

In the particular case when $u_1 = u_2 = \cdots = u_{m+1} = 0, v_1 = v_2 = \cdots = v_{m+1} = 1$, the above expression turns out to be

$$
\begin{aligned}
p(t) =& \sum_{i=0}^{m} t^i(1-t)^{m+1} \sum_{k=0}^{i} \binom{m+i-k}{m} \frac{p^{(k)}(0)}{k!} + \\
& \sum_{i=0}^{m} (1-t)^i t^{m+1} \sum_{k=0}^{i} \binom{m+i-k}{m} \frac{(-1)^k p^{(k)}(1)}{k!}
\end{aligned}
$$

which is the very expression in Proposition 2.1.

6. Conclusion

We first present a new approach to the unified construction of Said-type generalized Ball basis of both odd degrees and even degrees. The advantage of our approach is based on the fact that it can be used to derive the expressions for polynomials of not only odd degrees but also even degrees in terms of the Said-type generalized Ball basis functions. We then define the dual functionals for the Said-type generalized Ball basis in a very natural manner and bring to light the integral property of the Said-type generalized Ball basis functions. Last we extend the Said-type generalized Ball basis functions to multi-point case by defining new polynomial basis functions, which include Said-type generalized Ball basis functions as their special case, and the corresponding dual functionals and Marsden-like identity are obtained. Our future work will be focused on exploring the applications of the new polynomial basis functions in numerical approximation and geometric modeling.

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