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On p-Central Automorphisms of Primitive Groups and Applications

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Abstract Let G be a primitive group. It is proved that there exits some prime p such that every p-central automorphism of G is inner. As an application, it is proved that every Coleman automorphism of the holomorph of G is inner. In particular, the normalizer property holds for such groups in question. Additionally, other related results are obtained as well.

Keywords p-central automorphism; Coleman automorphism; the normalizer conjecture

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1. Introduction

Groups considered in this paper are assumed to be finite. Let σ be an automorphism of a group G. Recall that σ is said to be a p-central automorphism provided that there exists some Sylow p-subgroup P of G such that σ centralizes P, i.e., σ fixes P elementwise. Glauberman [1] and Gross [2] investigated the structure of the group formed by all p-central automorphisms of groups with some restricted conditions. Hertweck and Kimmerle [3] proved that for any simple group G there is a prime p dividing the order of G such that every p-central automorphism of G is inner.

Another related automorphisms are Coleman automorphisms. Recall that σ is said to be a Coleman automorphism provided the restriction of σ to any Sylow subgroup of G equals that of some inner automorphism of G. Write $\mathrm{Aut}_{\mathrm{Col}}(G)$ for the group formed by all Coleman automorphisms of G and set $\mathrm{Out}_{\mathrm{Col}}(G) := \mathrm{Aut}_{\mathrm{Col}}(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the inner automorphism of G. Dade [4] proved that $\mathrm{Out}_{\mathrm{Col}}(G)$ is nilpotent and even abelian when G is solvable. Later, Hertweck and Kimmerle [3] improved this result and proved that $\mathrm{Out}_{\mathrm{Col}}(G)$ is always abelian for any group G.

Both p-central automorphisms and Coleman automorphisms of G occur naturally in the study of the normalizer conjecture of integral group ring $\mathbb{Z}G$. The normalizer conjecture [5, Problem

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43] is stated as follows: $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G) = G \cdot \mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$ where $\mathrm{U}(\mathbb{Z}G)$ denotes the unit group of $\mathbb{Z}G$. We will say that G has the normalizer property if the normalizer conjecture holds for $\mathbb{Z}G$. It is known that if $\mathrm{Out}_{\mathrm{Col}}(G) = 1$ then the normalizer property holds for G (for details, the reader may refer to the Introduction of [3]). We should mention that Hertweck [6] constructed a metabelian group of order $2^{25} \cdot 97^2$ for which the normalizer conjecture fails to hold. Nevertheless, it is still of interest to study for which groups the normalizer property holds. In this direction, many positive results have been obtained, see [7–10, 12–14] for instance.

The aim of this paper is to investigate p-central automorphisms of primitive groups, among others. Recall that a transitive permutation group G on a finite set Ω is said to be primitive provided that G possesses no domain of imprimitivity (i.e., a proper subset X of Ω with at least two elements satisfies either $Xg = X$ or $X \cap Xg = \varnothing$ for any $g \in G$). Cayley's Theorem tells us that any group may be regarded as a transitive permutation group on its underlying set. So one cannot expect to say anything on the structure of p-central automorphisms of transitive permutation groups. However, the case is different when it comes to primitive groups. One of main results of this paper is as follows.

Theorem 1.1 *Let G be a primitive group. Then there is some $p \in \pi(G)$ such that every p-central automorphism of G is inner.*

Recall that the holomorph of a group G, denoted by $\mathrm{Hol}(G)$, is defined to be the semidirect product of G by its automorphism group $\mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ acts naturally on G. As an application of Theorem A, we have the following result.

Theorem 1.2 *Let H be the holomorph of a primitive group G. Then every Coleman automorphism of H is inner, i.e.,* $\mathrm{Out}_{\mathrm{Col}}(H) = 1$.

Corollary 1.3 *Let G be either a primitive group or its holomorph. Then the normalizer property holds for G.*

2. Preliminaries

In this section, we list some lemmas which will be needed in the sequel. First, we fix some notation. Write $\mathrm{F}(G)$ and $\mathrm{F}^*(G)$ for the Fitting and generalized Fitting subgroups of a group G, respectively. Set $\pi(G)$ to be the set of all primes dividing the order $|G|$ of G. Let H be a subgroup of G. Denote by $\mathrm{N}_G(H)$ and $\mathrm{C}_G(H)$ the normalizer and centralizer of H in G, respectively. Let σ be an automorphism of G. The notation $\sigma|_H$ stands for the restriction of σ to H. If H is normal in G and fixed by σ, then we write $\sigma|_{G/H}$ for the automorphism of G/H induced by σ in the natural way. For a fixed element $g \in G$, $\mathrm{conj}(g)$ is the inner automorphism of G induced by g via conjugation. Unless stated otherwise, other notations follow those in [15].

Lemma 2.1 ([3]) *Let G be a simple group. Then there exists a prime p dividing the order of G such that every p-central automorphism of G is an inner automorphism.*

Lemma 2.2 ([16]) *Let G be a primitive group with a minimal normal subgroup N which is* *abelian. Then N is the unique minimal normal subgroup of G and it is self-centralizing in G.*

Lemma 2.3 ([17]) *Let $N \trianglelefteq G$ and $\sigma \in \mathrm{Aut}(G)$. Suppose that $\sigma|_N = \mathrm{id}|_N$ and $\sigma|_{G/N} = \mathrm{id}|_{G/N}$. Then the order of σ divides the order of N.*

Lemma 2.4 ([18]) *Let G be a group and N a normal subgroup of G. Let σ be an automorphism of p-power order of G, where p is a prime. If $\sigma|_N = \mathrm{id}|_N$ and $\sigma|_{G/N} = \mathrm{id}|_{G/N}$, then $\sigma|_{G/\mathrm{O}_p(\mathrm{Z}(N))} = \mathrm{id}|_{G/\mathrm{O}_p(\mathrm{Z}(N))}$. If further σ fixes a Sylow p-subgroup of G, then σ is an inner automorphism of G induced by an element of $O_p(\mathrm{Z}(N))$.*

Lemma 2.5 *Let G be a primitive group. Then either G is an abelian simple group or $\mathrm{Z}(G) = 1$.*

Proof Suppose that $\mathrm{Z}(G) \neq 1$. Then G must have an abelian minimal normal subgroup, say N, contained in $\mathrm{Z}(G)$. It follows that $\mathrm{C}_G(N) = G$. On the other hand, by Lemma 2.2, $\mathrm{C}_G(N) = N$. Consequently, $G = N$. From this we deduce that G must be an abelian simple group. We are done. □

Lemma 2.6 ([19]) *Let H be the Holomorph of an abelian group A. Then every Coleman automorphism of H is inner, i.e., $\mathrm{Out}_{\mathrm{Col}}(H) = 1$.*

3. Proofs of main results

In this section, we will present the proofs of the main results.

Theorem 3.1 *Let G be a primitive group. Then there is some $p \in \pi(G)$ such that every p-central automorphism of G is inner.*

Proof We divide the proof into three cases:

Case 1 G is simple.

In this case, the assertion follows from Lemma 2.1.

Case 2 G has an abelian minimal normal subgroup N.

In this case, by Lemma 2.2, N is the unique minimal normal subgroup of G and $\mathrm{C}_G(N) = N$. Suppose that N is of p-power order, where p is a prime. Next we will show that every p-central automorphism of G is inner. Let σ be a such automorphism of G. Then by the definition there is a Sylow p-subgroup P of G such that

$$\sigma|_P = \mathrm{id}|_P. \tag{3.1}$$

Note that N is contained in $\mathrm{O}_p(G)$ and the latter is contained in P. So N is a subgroup of P and thus by (3.1) we have

$$\sigma|_N = \mathrm{id}|_N. \tag{3.2}$$

For any $g \in G$ and $n \in N$ we have $n^{g^\sigma} = (n^g)^\sigma = n^g$. It follows that $g^\sigma g^{-1} \in \mathrm{C}_G(N) = N$ and hence

$$\sigma|_{G/N} = \mathrm{id}|_{G/N}. \tag{3.3}$$

Now by Lemma 2.3, the equalities (3.2) and (3.3) imply that σ is of p-power order. Applying Lemma 2.4, we have $\sigma \in \mathrm{Inn}(G)$.

Case 3 G has a nonabelian minimal normal subgroup N.

In this case, by O'nan-Scott Theorem [15, 6.6.12], the generalized Fitting subgroup $\mathrm{F}^*(G)$ of G must be a characteristically simple group. So we may assume that $\mathrm{F}^*(G)$ is the direct product of copies of a nonabelian simple group, say S. By Lemma 2.1, there is a prime $p \in \pi(S)$ such that every p-central automorphism of S is inner. It follows that every p-central automorphism of $\mathrm{F}^*(G)$ is inner. We claim that every p-central automorphisms of G is also inner. For this, let σ be a such automorphism of G. Then by the definition there is some Sylow p-subgroup P of G such that

$$\sigma|_P = \mathrm{id}|_P. \tag{3.4}$$

Let Q be the intersection of P with $\mathrm{F}^*(G)$. Then Q is a Sylow p-subgroup of $\mathrm{F}^*(G)$ since $\mathrm{F}^*(G)$ is a normal subgroup of G. Then, by (3.4), we have

$$\sigma|_Q = \mathrm{id}|_Q. \tag{3.5}$$

This shows that $\sigma|_{\mathrm{F}^*(G)}$ is also a p-central automorphism of $\mathrm{F}^*(G)$. It follows that there is some $x \in \mathrm{F}^*(G)$ such that

$$\sigma|_{\mathrm{F}^*(G)} = \mathrm{conj}(x)|_{\mathrm{F}^*(G)}. \tag{3.6}$$

Let g be an arbitrary element in G and y an arbitrary element in $\mathrm{F}^*(G)$. Then we have

$$y^{xg^{\sigma}} = (y^x)^{g^{\sigma}} = (y^{\sigma})^{g^{\sigma}} = (y^g)^{\sigma} = y^{gx}. \tag{3.7}$$

As y is arbitrary, from (3.7) we obtain that

$$xg^{\sigma}x^{-1}g^{-1} \in \mathrm{C}_G(\mathrm{F}^*(\mathrm{G})). \tag{3.8}$$

But note that

$$\mathrm{C}_G(\mathrm{F}^*(\mathrm{G})) = \mathrm{Z}(\mathrm{F}^*(\mathrm{G})) = 1. \tag{3.9}$$

So (3.8) yields that

$$g^{\sigma} = g^x = g^{\mathrm{conj}(x)}. \tag{3.10}$$

As g is arbitrary, we have $\sigma = \mathrm{conj}(x)$. This completes the proof of Theorem 3.1. $\square$

It is known that a group G is a primitive group if and only if G has a primitive maximal subgroup. Recall that a maximal subgroup M of G is said to be primitive provided that $\mathrm{C}_G(N) \leq M$ for any nontrivial normal subgroup N of M. So Theorem 3.1 may be restated as follows.

Theorem 3.2 *Let G be a group with a primitive maximal subgroup. Then there exists some prime $p \in \pi(G)$ such that every p-central automorphism of G is inner.*

Similar to the proof of Theorem 3.1, the following result can be proved as well. The proof is left to the readers.

Theorem 3.3 *Suppose that the generalized Fitting subgroup* $\mathrm{F}^*(G)$ *of a group* G *is characteristically simple. Then there exists some prime* $p \in \pi(G)$ *such that every* p*-central automorphism of* G *is inner.*

Let G be a permutation group on a finite set Ω. Recall that G is said to be k-transitive on Ω provided that the natural action of G on the set Ω^k is transitive, where Ω^k consists of all ordered k-tuples $(\omega_1, \omega_2, \ldots, \omega_k)$ with ω_i's being distinct elements in Ω and $1 \leq k \leq |\Omega|$. It is known that every 2-transitive permutation group is primitive [16, 7.2.4] and that $(k+1)$-transitivity implies that k-transitivity [16, Exercise 7.1.1]. With these facts in hand, we can announce the following corollary of Theorem 3.1.

Corollary 3.4 *Let* G *be a* k*-transitive permutation group with* $k \geq 2$*. Then there is a prime* $p \in \pi(G)$ *such that every* p*-central automorphism of* G *is inner.*

In addition, it is known that any transitive permutation group of prime degree is primitive [16, 7.2.2]. As a direct consequence of Theorem 3.1, we have

Corollary 3.5 *Let* G *be a transitive permutation group of prime degree. Then there is a prime* $p \in \pi(G)$ *such that every* p*-central automorphism of* G *is inner.*

Corollary 3.6 *Let* G *be* k*-transitive permutation group with* $k \geq 4$*. Then for any normal subgroup* N *of* G *there is a prime* $p \in \pi(N)$ *such that every* p*-central automorphism of* N *is inner.*

Proof Suppose that G is of degree n. We divide the proof into two subcases.

Case 1 G is similar to S_n, the symmetric group of degree n.

In this case, we may assume that $G = S_n$. Since $k \geq 4$, it follows that $n \geq 4$.

Subcase 1.1 $n \neq 4$.

Note that in this case the only normal subgroups N of G are 1, A_n and S_n.

If $N = 1$, there is nothing to prove.

If $N = A_n$, then N is nonabelian simple and thus the assertion follows from Lemma 2.1.

If $N = S_n$, then N is primitive and thus the assertion follows from Theorem 3.1.

Subcase 1.2 $n = 4$.

In this case, the only normal subgroups of G are 1, $C_2 \times C_2$, A_4 and S_4.

If either $N = 1$ or $N = C_2 \times C_2$, then the assertion is trivial.

If $N = A_4$, then it is easy to see that $\mathrm{F}^*(N) = C_2 \times C_2$ and thus the assertion follows from Theorem 3.3.

If $N = S_4$, the assertion follows from the fact that S_4 is primitive.

Case 2 G is not similar to S_n.

In this case, by [16, Exercise 7.2.15], every nontrivial normal subgroup N of G is $(k-1)$-transitive. Remember that $k \geq 4$. So $k - 1 \geq 3$ and thus by Corollary 3.4 there is a prime

$p \in \pi(N)$ such that every p-central automorphism of N is inner. We are done. □

Corollary 3.7 *Let G be a group as in either Theorems 3.1 or 3.3. Then the normalizer property holds for G.*

4. Applications of main results

In this section, we apply the results obtained in Section 3 to the study of Coleman automorphisms of Holomorphs of some classes of groups. As an application of Theorem 3.1, we can state the following result.

Theorem 4.1 *Let H be the holomorph of a primitive group G. Then every Coleman automorphism of H is inner, i.e.,* $\mathrm{Out}_{\mathrm{Col}}(H) = 1$.

Proof Let $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(H)$. We have to show that σ is an inner. The proof splits into two cases according to Lemma 2.5.

Case 1 G is an abelian simple group.

In this case, the assertion follows from Lemma 2.6.

Case 2 G is not an abelian simple group.

In this case, by Lemma 2.5, we must have $\mathrm{Z}(G) = 1$. Let N be the product of G and its centralizer $\mathrm{C}_H(G)$. Then N is normal in H since G is normal in H. Note further that σ is a Coleman automorphism of H. So $\sigma|_N$ is an automorphism of N. Similarly, the restrictions $\sigma|_G$ and $\sigma|_{\mathrm{C}_H(G)}$ are automorphisms of G and $\mathrm{C}_H(G)$, respectively. By Theorem 3.1, there is a prime p such that every p-central automorphism of G is inner. The same is valid for $\mathrm{C}_H(G)$ since $\mathrm{C}_H(G)$ is isomorphic to G. Let P be a Sylow p-subgroup of H. Then $\sigma|_P = \mathrm{conj}(x)|_P$. Replacing σ with $\sigma\mathrm{conj}(x^{-1})$, we may assume that

$$\sigma|_P = \mathrm{id}|_P. \tag{4.1}$$

In particular, $\sigma|_G$ is a p-central automorphism of G. Thus there exists some $y \in G$ such that

$$\sigma|_G = \mathrm{conj}(y)|_G. \tag{4.2}$$

Similarly, there exists some $z \in \mathrm{C}_H(G)$ such that

$$\sigma|_{\mathrm{C}_H(G)} = \mathrm{conj}(z)|_{\mathrm{C}_H(G)}. \tag{4.3}$$

Combining (4.2) with (4.3), we obtain that

$$\sigma|_N = \mathrm{conj}(yz)|_N. \tag{4.4}$$

Note that

$$\mathrm{C}_H(N) \leq \mathrm{C}_H(G) \cap \mathrm{C}_H(\mathrm{C}_H(G)) = \mathrm{Z}(\mathrm{C}_H(G)) = 1.$$

So we have $\mathrm{C}_H(N) = 1$. This, together with (4.4), implies that $\sigma = \mathrm{conj}(yz)$, as desired. □

As a direct consequence of Theorem 4.1, we have the following result.

Corollary 4.2 *Let H be the holomorph of a k-transitive permutation group G with $k \geq 2$. Then every Coleman automorphism of H is inner.*

As an application of Theorem 3.3, we have the following result.

Theorem 4.3 *Let G be a group with $\mathrm{F}^*(G)$ being a centerless characteristically simple group. Then $\mathrm{Out}_{\mathrm{Col}}(H) = 1$, where H is the holomorph of G.*

Proof The proof is similar to that of Case 2 of Theorem 4.1, so we omit it. □

Recall that a group G is said to be monolith if G has only a unique minimal normal subgroup. As far as Coleman automorphisms of holomorphs of monolith groups are concerned, we have the following result.

Corollary 4.4 *Let G be a monolith group with $\mathrm{F}^*(G)$ being the unique minimal normal subgroup. Then $\mathrm{Out}_{\mathrm{Col}}(H) = 1$, where H is the holomorph of G.*

Proof The proof splits into two cases according to whether $\mathrm{F}^*(G)$ is abelian or not.

Case 1 $\mathrm{F}^*(G)$ is nonabelian.

In this case, $\mathrm{F}^*(G)$ is centerless characteristically simple and thus the assertion follows from Theorem 4.3.

Case 2 $\mathrm{F}^*(G)$ is abelian.

In this case, we claim that either $\mathrm{Z}(G) = 1$ or G is an abelian simple group. In fact, if $\mathrm{Z}(G) \neq 1$, then $\mathrm{Z}(G) = \mathrm{F}^*(G)$. But note that $\mathrm{F}^*(G)$ is self-centralizing in G. So we have $G = \mathrm{Z}(G) = \mathrm{F}^*(G)$. As $\mathrm{F}^*(G)$ is the unique minimal normal subgroup, G must be an abelian simple group, as claimed. To this point, the subsequent argument is similar to that of Theorem 4.1, so we omit it. □

Remark 4.5 The restriction that $\mathrm{F}^*(G)$ is the unique minimal normal subgroup in Corollary 4.4 is needless when the minimal normal subgroup of G is nonabelian. In fact, the Fitting subgroup $\mathrm{F}^*(G)$ coincides with the unique minimal normal subgroup since in the current situation $\mathrm{F}(G)$ is trivial.

We conclude this paper by recording the following corollary.

Corollary 4.6 *Let H be as in either Theorems 4.1 or 4.3. Then the normalizer property holds for H.*

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