Relative Property A for Discrete Metric Space

Yufang LI\textsuperscript{1, 2}, Zhe DONG\textsuperscript{1}, Yuanyi WANG\textsuperscript{3,*}

1. Department of Mathematics, Zhejiang University, Zhejiang 310058, P. R. China;
2. Department of Mathematics and Statistics, Guizhou University, Guizhou 550025, P. R. China;
3. School of Information Engineering, College of Science and Technology, Ningbo University,
   Zhejiang 315211, P. R. China

Abstract Yu introduced Property A on discrete metric spaces. In this paper, a relative Property A for a discrete metric space $X$ with respect to a set $Y$ and a map $\phi_{X,Y}$ is defined. Some characterizations for relative Property A are given. In particular, a discrete metric space with relative Property A can be coarse embedding into a Hilbert space under certain condition.

Keywords relative Property A; coarse embedding; mazur map

MR(2010) Subject Classification 46B85; 46C05

1. Introduction

Coarse embeddings were introduced by Gromov in [1]. A function $f : X \to Y$ between metric spaces is a coarse embedding if there exist two non-decreasing maps $\theta_1, \theta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta_1(t) \to +\infty$ as $t \to +\infty$ and $\theta_1(d_X(x,y)) \leq d_Y(f(x),f(y)) \leq \theta_2(d_X(x,y))$ for all $x, y \in X$. The readers can refer to the book [2] for a self-contained introduction to coarse geometry.

Property A is a weak version of amenability for discrete metric space which was introduced by Yu [3], who claimed that a metric space satisfies this property can be coarse embedding into a Hilbert space. For metric spaces with bounded geometry it implies the coarse Baum Connes conjecture, and for a finitely generated group with word-length metric it implies the strong Novikov conjecture. Kasparov and Yu treated the case when the Hilbert space is replaced with a uniformly convex Banach space [4]. In [5], Nowak constructed some metric spaces which do not satisfy Property A but embed coarsely into a Hilbert space. So coarse embedding into a Hilbert space and Property A are not equivalent. In [6], Higson and Roe gave a useful equivalent definition of Property A. They claimed that a discrete metric space with bounded geometry with Property A if and only if for every $R > 0$ and $\varepsilon > 0$, there exists a map $\xi : X \to \ell^1(X)^+$ and an $S \in \mathbb{R}^+$ such that $\|\xi_x\|_{\ell^1} = 1$ and $\text{supp} \xi_x \subseteq B(x,S)$ for every $x \in X$ and $\|\xi_x - \xi_y\|_{\ell^1} < \varepsilon$ whenever $d(x,y) < R$. In [7], Ji, Ogle and Ramsey defined relative Property A for a discrete group $G$ relative to a finite family of subgroups $\mathscr{H}$, and they showed that if $G$ has Property A

Received May 1, 2018; Accepted August 12, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11701301).

* Corresponding author
E-mail address: Liyufangmail@163.com (Yufang LI); dongzhe@zju.edu.cn (Zhe DONG); wangyuanyi@nbu.edu.cn (Yuanyi WANG)
relative to a family of subgroups $\mathcal{H}$ and if each $H \in \mathcal{H}$ has Property A then $G$ has Property A. Readers can refer to [8] for more details.

In this paper, we define a relative Property A for metric space with respect to a set $Y$ and a map $\rho_{X,Y}$, which is a generalization of Yu’s Property A. Several examples and equivalent characterizations of relative Property A are given. In particular, we show that relative Property A implies coarse embedding into a Hilbert space if $d(x_1,x_2) \leq \rho_{X,Y}(x_1,y) + \rho_{X,Y}(x_2,y)$ for all $x_1,x_2 \in X$ and $y \in Y$.

2. Relative Property A

Recall that a discrete metric space $(X,d)$ has Property A if for all $R, \varepsilon > 0$, there exists a family $\{A_x\}_{x \in X}$ of finite non-empty subsets of $X \times \mathbb{N}$ such that (1) for all $x,y \in X$ with $d(x,y) \leq R$, we have $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} < \varepsilon$; (2) there exists an $S$ such that for each $x \in X$ if $(y,n) \in A_x$, then $d(x,y) \leq S$.

**Definition 2.1** Let $X$ be a discrete metric space $(X,d)$ and $Y$ be a set with $\rho_{X,Y} : X \times Y \to \mathbb{R}^+$. We say $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$ if the following are satisfied: for any $R > 0$ and $\varepsilon > 0$, there exists an $S > 0$ and a collection $\{A_x\}_{x \in X}$ of finite nonempty subsets of $Y \times \mathbb{N}$ such that:

1. For each $x \in X$ if $(y,n) \in A_x$, then $y \in B_{\rho_{X,Y}}(x,S)$, where $B_{\rho_{X,Y}}(x,S) = \{y \in Y | \rho_{X,Y}(x,y) \leq S\}$;
2. If $d(x_1,x_2) < R$, then $\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} < \varepsilon$.

The following proposition gives the relationships between Property A and relative Property A.

**Proposition 2.2** Suppose $(Z,d)$ is a discrete metric space, $X$ is a subspace of $Z$, $Y$ is a subset of $Z$ and the map $\rho_{X,Y}(x,y) = d(x,y)$ for all $x \in X, y \in Y$, then

1. If $X \subseteq Y$ and $X$ has Property A, then $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$;
2. If $Y \subseteq X$ and $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$, then $X$ has Property A;
3. If $X = Y$, $X$ has Property A if and only if $X$ has relative property A with respect to $Y$ and $\rho_{X,Y}$.

**Proof** (1) Suppose $X$ has Property A. Then for any $R > 0$ and $\varepsilon > 0$, there exists a collection $\{A_x\}_{x \in X}$ and an $S$ satisfying the definition of Property A. Since $X \subseteq Y$, we can see $A_x$ as subset of $Y \times \mathbb{N}$. Then if $(y,n) \in A_x$, we have $d(x,y) \leq S$. So $y \in B_{\rho_{X,Y}}(x,S)$. For any $x_1,x_2 \in X$ with $d(x_1,x_2) < R$, we have

$$\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} < \varepsilon.$$ 

So $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$.

(2) The proof is similar to that in (1).
(3) It is clear from (1) and (2). □

Now we give some examples about relative Property A.

**Example 2.3** Let $X$ be a finite metric space with a set $Y$ and a map $\rho_{X,Y} : X \times Y \to \mathbb{R}^+$. For any $R > 0$ and $\varepsilon > 0$, fix $y_0 \in Y$, let $S = \max_{x \in X} \rho_{X,Y}(x, y_0)$ and $A_x = (y_0, 1) \subseteq Y \times \mathbb{N}$ for any $x \in X$. If $(y, n) \in A_x$, then $y = y_0$, $n = 1$. So $y \in B_{\rho_{X,Y}}(x, S)$. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have $\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = 0 < \varepsilon$. So $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$.

**Example 2.4** Let $X$ be a discrete metric space with a set $Y$ and a map $\rho_{X,Y} : X \times Y \to \mathbb{R}^+$. If $\rho_{X,Y}$ is uniformly bounded, then there exists an $S$ such that $\rho_{X,Y}(x, y) \leq S$ for all $x \in X$ and $y \in Y$. For any $R > 0$ and $\varepsilon > 0$, we fix a $y_0 \in Y$ and let $A_x = (y_0, 1) \subseteq Y \times \mathbb{N}$ for all $x \in X$. It is clear that if $(y, n) \in A_x$, we have $\rho_{X,Y}(x, y) \leq S$. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have $\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = 0 < \varepsilon$. So $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$.

**Example 2.5** If $X = \mathbb{Z}$ and $Y$ is a countable set. Let $f$ be a bijection from $Y$ to $\mathbb{Z}$. We define $\rho_{X,Y}(x, y) = |x - f(y)|$. Fix $R > 0$ and $\varepsilon > 0$ where $\varepsilon < 1$ and choose an $S \in \mathbb{N}$ such that $S > 2R\varepsilon^{-1}$. Define $A_x = \{(y, 1) \in Y \times \mathbb{N} | \rho_{X,Y}(x, y) \leq S\}$. It is clear that each $A_x$ is a finite subset in $Y \times \mathbb{N}$. By the definition of $A_x$, $\rho_{X,Y}(x, y) \leq S$ if $(y, n) \in A_x$. For any $x_1, x_2 \in X$ with $|x_1 - x_2| < R$, we have $|A_{x_1} \cup A_{x_2}| = 2S + |x_1 - x_2| + 1$, $|A_{x_1} \triangle A_{x_2}| = 2|x_1 - x_2|$ and $|A_{x_1} \cap A_{x_2}| = 2S - |x_1 - x_2| + 1$. So

$$\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{2|x_1 - x_2|}{2S - |x_1 - x_2| + 1} < \frac{2R}{2S - R + 1} < \frac{2R}{2R\varepsilon^{-1} - R} = \frac{2\varepsilon}{1 - \varepsilon}.$$

So $Z$ has relative Property A with respect to a countable set $Y$ and $\rho_{X,Y}$. Let $Y = \mathbb{Z}$ and $\rho_{X,Y}(x, y) = |x - y|$. From Proposition 2.2, we conclude that $\mathbb{Z}$ has Property A.

Next we give an example of discrete space with Property A but without relative Property A with a map $\rho_{X,Y}$ and some set $Y$.

**Example 2.6** If $X = \mathbb{Z}$ and $Y = \{0\} \subseteq \mathbb{Z}$ is a single point set. We define $\rho_{X,Y}(x, y) = |x - y|$ for all $x \in X$ and $y \in Y$. For any $S > 0$, there exists an $x_0 \in \mathbb{Z}$ such that $|x_0 - 0| = |x_0| > S$. Since $A_{x_0}$ is nonempty, there exists $n \in \mathbb{N}$ such that $(0, n) \in A_{x_0}$ and $\rho_{X,Y}(x_0, 0) > S$. So $\mathbb{Z}$ does not have relative Property A with respect to $Y = \{0\}$ and $\rho_{X,Y}(x, y) = |x - y|$.

**Proposition 2.7** If a discrete metric space $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$, then any subspace of $X$ also has relative Property A with respect to $Y$ and $\rho_{X,Y}$.

**Proof** Suppose $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$ and $X'$ is a subspace of $X$. Let $\{A_x\}_{x \in X}$ be the sets satisfying the definition of relative Property A with respect to $Y$ and $\rho_{X,Y}$. We can choose the subfamilies $\{A_x\}_{x \in X'}$. It is easy to check these sets satisfying the definition of $X'$ having relative Property A with respect to $Y$ and $\rho_{X,Y}$. □

**Proposition 2.8** Let $X$ be a discrete metric space $(X, d)$ and $Y_1$, $Y_2$ be subsets of $Y$. If $X$ has relative Property A with respect to both $Y_1$ and $Y_2$ and $\rho_{X,Y}$, then $X$ has relative Property A with respect to both $Y_1 \times Y_2$ and $\rho_{X,Y}$. □
with respect to $Y_1 \cup Y_2$ and $\rho_{X,Y}$.

**Proof** Suppose that $X$ has relative Property A with respect to both $Y_1$ and $Y_2$ and $\rho_{X,Y}$. For any $R > 0$ and $\varepsilon > 0$, let $\{A_i^x\}_{x \in X}$ and $\{A_i^y\}_{y \in X}$ be the sets satisfying the definition of relative Property A with respect to $Y_1$ and $Y_2$ and $\rho_{X,Y}$, and $S', S''$ be the relevant constants. Let $A_x = A_x' \cup A_x''$. If $(y, n) \in A_x$, then $(y, n) \in A_x'$ or $(y, n) \in A_x''$. In the first case, $\rho_{X,Y}(x, y) \leq S'$; in the second case, $\rho_{X,Y}(x, y) \leq S''$. So $\rho_{X,Y}(x, y) \leq S$ where $S = \max\{S', S''\}$.

For each $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, from the condition (2) of relative Property A, we get

$$\frac{|A_{x_1} \Delta A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} < \varepsilon$$

and

$$\frac{|A_{y_1} \Delta A_{y_2}|}{|A_{y_1} \cap A_{y_2}|} < \varepsilon.$$ Then

$$\frac{|A_{x_1} \Delta A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{(|A_{x_1}' \cup A_{x_2}' \Delta A_{x_1}' \cup A_{x_2}') + (|A_{x_1}'' \cup A_{x_2}'' \Delta A_{x_1}'' \cup A_{x_2}'|)}{|A_{x_1}' \cup A_{x_2}'\cap (A_{x_1}'' \cup A_{x_2}'')|}$$

$$\leq \frac{|A_{x_1}' \Delta A_{x_2}'|}{|A_{x_1}' \cap A_{x_2}'|} + \frac{|A_{x_1}'' \Delta A_{x_2}'|}{|A_{x_1}'' \cap A_{x_2}'|} + \frac{|A_{x_1}' \Delta A_{x_2}''|}{|A_{x_1}' \cap A_{x_2}''|} + \frac{|A_{x_1}'' \Delta A_{x_2}''|}{|A_{x_1}'' \cap A_{x_2}''|}$$

$$< 2\varepsilon.$$

So the proposition is proved. □

**Proposition 2.9** If $X_1$ is a discrete metric space with relative Property A with respect to $Y$ and $\rho_{X_1,Y}$, and $X_2$ is a discrete metric space with relative Property A with respect to $Y$ and $\rho_{X_2,Y}$. Then $X_1 \times X_2$ is a discrete metric space with relative Property A with respect to $Y$ and $\rho_{X_1 \times X_2,Y}$, where $\rho_{X_1 \times X_2,Y}((x_1, x_2), y) = \min\{\rho_{X_1,Y}(x_1, y), \rho_{X_2,Y}(x_2, y)\}$ for all $x_1 \in X_1, x_2 \in X_2$ and $y \in Y$, and $d_{X_1 \times X_2}((x_1', x_2'), (x_1'', x_2'')) = d_{X_1}(x_1', x_1'') + d_{X_2}(x_2', x_2'')$ for all $x_1', x_1'', x_2', x_2'' \in X_2$.

**Proof** Suppose $X_1$ and $X_2$ have relative Property A with respect to $Y$ and $\rho_{X_1,Y}$ and $\rho_{X_2,Y}$. For any $R > 0$ and $\varepsilon > 0$, let $\{A_i^{x_1}\}_{x_1 \in X_1}$, $\{A_i^{x_2}\}_{x_2 \in X_2}$ be the sets satisfying the definition of relative Property A and $S_1, S_2$ be the relevant constants. Set $A_{(x_1, x_2)} = A_{x_1} \cup A_{x_2}$. Then if $(y, n) \in A_{(x_1, x_2)}$, we have $(y, n) \in A_{x_1}$ or $(y, n) \in A_{x_2}$. In the first case, $\rho_{X_1,Y}(x_1, y) \leq S_1$; in the second case, $\rho_{X_2,Y}(x_2, y) \leq S_2$. Since $\rho_{X_1 \times X_2,Y}((x_1, x_2), y) = \min\{\rho_{X_1,Y}(x_1, y), \rho_{X_2,Y}(x_2, y)\}$, so $\rho_{X_1 \times X_2,Y}((x_1, x_2), y) \leq S$ where $S = \max\{S_1, S_2\}$.

Suppose $x_1', x_1'' \in X_1$ and $x_2', x_2'' \in X_2$ with $d_{X_1 \times X_2}((x_1', x_2'), (x_1'', x_2'')) < R$. We have $d_{X_1}(x_1', x_1'') < R$ and $d_{X_2}(x_2', x_2'') < R$, so

$$\frac{|A_{x_1'} \Delta A_{x_1''}|}{|A_{x_1'} \cap A_{x_1''}|} < \varepsilon$$

and

$$\frac{|A_{x_2'} \Delta A_{x_2''}|}{|A_{x_2'} \cap A_{x_2''}|} < \varepsilon.$$ As the proof of above proposition, we can see that

$$\frac{|A_{(x_1', x_2')} \Delta A_{(x_1'', x_2'')}|}{|A_{x_1'} \cap A_{x_1''}| \cap A_{(x_1', x_2')} \cap A_{x_1''} \cap A_{x_2'} \cap A_{x_2''}|} < 2\varepsilon.$$

So $X_1 \times X_2$ is a metric space with relative Property A with respect to $Y$ and $\rho_{X_1 \times X_2,Y}$. □
One of Yu’s main motivations for introducing Property A was that it implies coarse embedding into Hilbert space. Now we prove that a discrete metric space with relative Property A with respect to $Y$ and $\rho_{X,Y}$ can be coarse embedding into a Hilbert space under the condition of $d(x_1, x_2) \leq \rho_{X,Y}(x_1, y) + \rho_{X,Y}(x_2, y)$ for all $x_1, x_2 \in X$ and $y \in Y$. The main idea of the following theorem comes from [8].

**Theorem 2.10** Suppose $(X, d)$ is a discrete metric space with relative Property A with respect to $Y$ and $\rho_{X,Y}$. If $d(x_1, x_2) \leq \rho_{X,Y}(x_1, y) + \rho_{X,Y}(x_2, y)$ for all $x_1, x_2 \in X$ and $y \in Y$, then $X$ can be coarse embedding into a Hilbert space.

**Proof** First we define a Hilbert space

$$ \mathcal{H} = \bigoplus_{k=1}^{\infty} \ell^2(Y \times \mathbb{N}). $$

Since $X$ is a discrete metric space with relative Property A with respect to $Y$ and $\rho_{X,Y}$, then for $R = k \in \mathbb{N}$ and $\varepsilon = \frac{1}{2k+1} > 0$, we can define a sequence of sets $\{A^k_n\}_{n \in X} \subseteq Y \times \mathbb{N}$ satisfying:

1. There exists an $S_k > k$, such that if $(y, n) \in A^k_n$, then $y \in B_{\rho_{X,Y}}(x, \frac{1}{2k}S_k)$;
2. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < k$, we have $\frac{|A^k_{x_1} \triangle A^k_{x_2}|}{|A^k_{x_1}| \cap |A^k_{x_2}|} < \frac{1}{2^{k+1}}$.

Let $\chi_{A_k}$ be the characteristic function of $A_k$. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < k$, we have

$$ \frac{\|\chi_{A^k_{x_1}} - \chi_{A^k_{x_2}}\|_{\ell^2(Y \times \mathbb{N})}}{|A^k_{x_1}|^{\frac{1}{2}} - |A^k_{x_2}|^{\frac{1}{2}}} = \frac{|A^k_{x_1} \setminus A^k_{x_2}|}{|A^k_{x_1}|} + \frac{|A^k_{x_2} \setminus A^k_{x_1}|}{|A^k_{x_2}|} + \frac{|A^k_{x_1} \cap A^k_{x_2}|}{|A^k_{x_1}|} \left( \frac{1}{|A^k_{x_1}|^{\frac{1}{2}}} - \frac{1}{|A^k_{x_2}|^{\frac{1}{2}}} \right)^2
$$

$$ = \frac{|A^k_{x_1} \setminus A^k_{x_2}|}{|A^k_{x_1}|} + \frac{|A^k_{x_2} \setminus A^k_{x_1}|}{|A^k_{x_2}|} + \frac{|A^k_{x_1} \cap A^k_{x_2}|}{|A^k_{x_1}|} \left( \frac{1}{|A^k_{x_1}|^{\frac{1}{2}}} - \frac{1}{|A^k_{x_2}|^{\frac{1}{2}}} \right)^2
$$

$$ \leq \frac{|A^k_{x_1} \triangle A^k_{x_2}|}{|A^k_{x_1}| \cap |A^k_{x_2}|} + \frac{|A^k_{x_1} \cap A^k_{x_2}|}{|A^k_{x_1}|} \left( \frac{|A^k_{x_1}| + |A^k_{x_2}| - 2|A^k_{x_1}|^{\frac{1}{2}}|A^k_{x_2}|^{\frac{1}{2}}}{|A^k_{x_1}| \cap |A^k_{x_2}|} \right)
$$

$$ \leq \frac{|A^k_{x_1} \triangle A^k_{x_2}|}{|A^k_{x_1}| \cap |A^k_{x_2}|} + \frac{|A^k_{x_1} \cap A^k_{x_2}|}{|A^k_{x_1} \cap |A^k_{x_2}|} \left( \frac{|A^k_{x_1}| + |A^k_{x_2}| - 2|A^k_{x_1}|^{\frac{1}{2}}|A^k_{x_2}|^{\frac{1}{2}}}{|A^k_{x_1}| \cap |A^k_{x_2}|} \right)
$$

$$ = \frac{2|A^k_{x_1} \triangle A^k_{x_2}|}{|A^k_{x_1} \cap |A^k_{x_2}|} < \frac{1}{2^{k+1}} $$

Hence

$$ \frac{\|\chi_{A^k_{x_1}} - \chi_{A^k_{x_2}}\|_{\ell^2(Y \times \mathbb{N})}}{|A^k_{x_1}|^{\frac{1}{2}} - |A^k_{x_2}|^{\frac{1}{2}}} < \frac{1}{2^{k+1}} $$

for any $x_1, x_2 \in X$ with $d(x_1, x_2) < k$.

Now, choose $x_0 \in X$ and define $f : X \to \mathcal{H}$ by

$$ f(x) = \bigoplus_{k=1}^{\infty} \frac{\chi_{A^k_{x}}}{|A^k_{x}|^{\frac{1}{2}}} - \frac{\chi_{A^k_{x_0}}}{|A^k_{x_0}|^{\frac{1}{2}}} $$

For any $x \in X$, there exists a $k' \in \mathbb{N} > 0$ such that $d(x, x_0) < k'$. We have

$$ \|f(x) - f(x_0)\| = \sum_{k=1}^{\infty} \frac{\|\chi_{A^k_{x}} - \chi_{A^k_{x_0}}\|_{\ell^2(Y \times \mathbb{N})}}{|A^k_{x}|^{\frac{1}{2}} - |A^k_{x_0}|^{\frac{1}{2}}} $$
\[\sum_{k=1}^{k'-1} \left\| \frac{X_k}{|A_k|^2} - \frac{X_{k+1}}{|A_{k+1}|^2} \right\|_{\ell^2(Y \times N)} + \sum_{k=k'}^{\infty} \left\| \frac{X_k}{|A_k|^2} - \frac{X_{k+1}}{|A_{k+1}|^2} \right\|_{\ell^2(Y \times N)} \]

So \(f\) is well-defined.

For any \(m \in \mathbb{N}\), if \(x_1, x_2 \in X\) with \(m \leq d(x_1, x_2) < m + 1\), we can get the estimate
\[
\|f(x_1) - f(x_2)\|^2 = \sum_{k=1}^{\infty} \left\| \frac{X_k}{|A_k|^2} - \frac{X_{k+1}}{|A_{k+1}|^2} \right\|^2_{\ell^2(Y \times N)}
\]
\[
\leq \sum_{k=1}^{m} \left\| \frac{X_k}{|A_k|^2} - \frac{X_{k+1}}{|A_{k+1}|^2} \right\|^2_{\ell^2(Y \times N)} + 1
\]
\[
\leq \sum_{k=1}^{m} \left( \left\| \frac{X_k}{|A_k|^2} \right\|^2_{\ell^2(Y \times N)} + \left\| \frac{X_{k+1}}{|A_{k+1}|^2} \right\|^2_{\ell^2(Y \times N)} \right) + 1
\]
\[
= 4m + 1.
\]

Hence \(\|f(x_1) - f(x_2)\| \leq \sqrt{4m + 1}\).

On the other hand, we define a map \(Q : \mathbb{R}^+ \to \mathbb{N}\) by \(Q(t) = |\{S_k | S_k < t\}|\) for \(t \in \mathbb{R}^+\). From the choice of the families \(\{A_k\}_{k \in X}\), \(Q(t)\) exists for all \(t \in \mathbb{R}^+\) since \(S_k \geq k\), and \(Q(t) \to \infty\) as \(t \to \infty\). Now we claim that for any \(x_1, x_2 \in X\) if \(d(x_1, x_2) > S_k\), then \(A_{x_1}^k \cap A_{x_2}^k = \emptyset\). Indeed, if there exists \((y, u) \in A_{x_1}^k \cap A_{x_2}^k\), then \(p_{X,Y}(x_1, y) \leq \frac{1}{2}S_k\) and \(p_{X,Y}(x_2, y) \leq \frac{1}{2}S_k\). Since for any \(x_1, x_2 \in X\), we have \(d(x_1, x_2) \leq p_{X,Y}(x_1, x_2)\), this is a contradiction. For any \(m \in \mathbb{N}\), if \(m \leq d(x_1, x_2) < m + 1\), we write \(Q(d(x_1, x_2)) = r\). So the number of \(S_k\) satisfies \(S_k < d(x_1, x_2)\) is \(r\). Then we can write the set \(\{S_k | S_k < d(x_1, x_2)\}\) as \(\{S_{k_1}, S_{k_2}, ..., S_{k_r}\}\). Since \(d(x_1, x_2) > S_{k_j}\), we have \(A_{x_1}^k \cap A_{x_2}^k = \emptyset\) for \(j = 1, ..., r\). We have
\[
\|f(x_1) - f(x_2)\|^2 = \sum_{k=1}^{\infty} \left\| \frac{X_k}{|A_k|^2} - \frac{X_{k+1}}{|A_{k+1}|^2} \right\|^2_{\ell^2(Y \times N)}
\]
\[
\geq \sum_{j=1}^{r} \left\| \frac{X_{k_j}}{|A_{k_j}|^2} - \frac{X_{k_{j+1}}}{|A_{k_{j+1}}|^2} \right\|^2_{\ell^2(Y \times N)} = 2r.
\]

Hence \(\|f(x_1) - f(x_2)\| \geq \sqrt{2r}\).

Define the maps \(\theta_1, \theta_2 : \mathbb{R}^+ \to \mathbb{R}^+\) by \(\theta_1(t) = \sqrt{2Q(t)}\) and \(\theta_2(t) = \sqrt{4m + 1}\) for \(t \in \mathbb{R}^+\). It is clear that both \(\theta_1(t)\) and \(\theta_2(t)\) are non-decreasing and \(\theta_1(t) \to +\infty\) as \(t \to +\infty\). We have
\[
\theta_1(d(x_1, x_2)) \leq d(f(x_1), f(x_2)) \leq \theta_2(d(x_1, x_2))
\]
for any \(x_1, x_2 \in X\).

So \(X\) can be coarse embedding into a Hilbert space. \(\square\)

In [8], the author collected many equivalent formulations of Property A under the condition of bounded geometry. Recall that a metric space \(X\) is bounded geometry if for every \(C > 0\), there is an absolute bound on the number of elements in any ball within \(X\) of radius \(C\). In this paper, we need to restrict the discrete metric space with relative bounded geometry in order to
prove the similar results.

**Definition 2.11** A discrete metric space \((X,d)\) with respect \(Y\) and \(\rho_{X,Y}\) is relative bounded geometry if for all \(L > 0\), there exists an \(N_L \in \mathbb{N}\) such that \(|\{y \in Y| y \in B_{\rho_{X,Y}}(x,L)\}| < N_L\) for all \(x \in X\).

If we let \(X = Y\) and \(\rho_{X,Y} = d\), then it is exactly the definition of bounded geometry. In the following, we will give some equivalent formulations of relative Property A under this condition.

**Theorem 2.12** Let \(X\) be a discrete metric space with relative bounded geometry with respect to \(Y\) and \(\rho_{X,Y}\). Then \(X\) has relative Property A with respect to \(Y\) and \(\rho_{X,Y}\) if and only if for every \(R > 0\) and \(\varepsilon > 0\) there exists an \(S > 0\) and \(\xi : X \rightarrow \ell^1(Y)^+\) satisfying:

1. \(\|\xi_x\|_{\ell^1} = 1\), for any \(x \in X\);
2. If \(\xi_x(y) \neq 0\), then \(y \in B_{\rho_{X,Y}}(x,S)\);
3. If \(d(x_1,x_2) < R\), then \(\|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} < \varepsilon\).

**Proof** Suppose that \(X\) has relative Property A with respect to \(Y\) and \(\rho_{X,Y}\). For any \(R > 0\) and \(\varepsilon > 0\), let \(\{A_x\}_{x \in X}\) and \(S\) satisfy the definition of relative Property A with respect to \(Y\) and \(\rho_{X,Y}\).

For any \(x \in X\), define

\[
\xi_x : Y \rightarrow \mathbb{R}^+, \xi_x(y) = \frac{|A_x \cap (y \times N)|}{|A_x|}.
\]

Then

\[
\|\xi_x\|_{\ell^1} = \sum_{y \in Y} |\xi_x(y)| = \sum_{y \in Y} \frac{|A_x \cap (y \times N)|}{|A_x|} = 1.
\]

For any \(x_1, x_2 \in X\),

\[
\|\xi_{x_1}|A_{x_1}| - \xi_{x_2}|A_{x_2}|\|_{\ell^1} = \sum_{y \in Y} \left| |(y \times N) \cap A_{x_1}| - |(y \times N) \cap A_{x_2}| \right|
\]

\[
= |A_{x_1} \triangle A_{x_2}|.
\]

Since for any \(x_1, x_2 \in X\) with \(d(x_1,x_2) < R\), we have

\[
\varepsilon > \frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{|A_{x_1}| + |A_{x_2}| - 2|A_{x_1} \cap A_{x_2}|}{|A_{x_1} \cap A_{x_2}|}.
\]

Whence

\[
2 + \varepsilon > \frac{|A_{x_1}| + |A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{|A_{x_1}| + |A_{x_2}|}{|A_{x_1}|} = 1 + \frac{|A_{x_2}|}{|A_{x_1}|}.
\]

So by symmetry

\[
1 + \varepsilon > \frac{|A_{x_2}|}{|A_{x_1}|} > \frac{1}{1 + \varepsilon}.
\]

Combining these two comments, we conclude that

\[
\|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} \leq \|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} + \|\xi_{x_2} - \xi_{x_2}\|_{\ell^1}
\]

\[
\leq \frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1}|} + 1 - \frac{|A_{x_2}|}{|A_{x_1}|}.
\]
Finally, note that if \( \xi_x(y) \neq 0 \), then \( A_x \cap \{ y \times \mathbb{N} \} \neq \emptyset \). So there exists an \( n \in \mathbb{N} \) such that \( (y, n) \in A_x \). By the definition of \( A_x \), we have \( y \in B_{\rho_{x,y}}(x, S) \).

Conversely, suppose that for any \( \varepsilon > 0 \) and \( R > 0 \), there exists an \( S \) and \( \xi \) satisfying the conditions (1), (2) and (3). By relative bounded geometry, for all \( x \in X \) the number of elements of \( y \) within the support of \( \xi_x \) is uniformly bounded. So we can use the same approximation method as proving [6, Lemma 3.5]. We may assume that there is a natural number \( M \) such that for all \( x \in X \), the function \( \xi_x \in \ell^1(Y) \) assumes only values in the range

\[
\frac{0}{M}, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M}{M}.
\]

Define

\[
A_x = \{(y, j) \in Y \times \mathbb{N} | \xi_x(y) \geq \frac{j}{M}, \ j > 0, \ j \in \mathbb{N} \}.
\]

Set

\[
A_x^j = \{ y \in Y | \xi_x(y) \geq \frac{j}{M} \}, \ j = 0, \ldots, M.
\]

It is clear that \( A_x^j \subseteq A_x^{j-1} \) for \( j = 1, \ldots, M \). Let \( Z = \bigcup_{j=1}^{M}(A_x^{j-1}\setminus A_x^j) \cup A_x^M \). We can see that the sets \( \{A_x^{j-1}\setminus A_x^j\} \) \( (j = 1, \ldots, M) \) and \( A_x^M \) are all disjoint. Since \( \|\xi_x\| = 1 \) for all \( x \in X \), we have

\[
1 = \sum_{y \in Y} \xi_x(y) = \sum_{j=1}^{M}(|A_x^{j-1}| - |A_x^j|)j - \frac{1}{M} + |A_x^M|\frac{M}{M}
\]

\[
= \frac{1}{M}\sum_{j=1}^{M}(|A_x^{j-1}| - |A_x^j|)(j - 1) + M|A_x^M|
\]

\[
= \frac{1}{M}(|A_x^1| + |A_x^2| + \cdots + |A_x^M|).
\]

So \( \sum_{j=1}^{M}|A_x^j| = M \). Set

\[
\hat{A}_x^j = \{(y, j) \in Y \times \mathbb{N} | y \in A_x^j \}.
\]

We have \( |\hat{A}_x^j| = |A_x^j| \) and \( A_x = \bigcup_{j=1}^{M} \hat{A}_x^j \). Since \( \hat{A}_x^j \) are all disjoint for \( j = 1, \ldots, M \). So \( |A_x| = \sum_{j=1}^{M}|\hat{A}_x^j| = \sum_{j=1}^{M}|A_x^j| = M \). If \( (y, n) \in A_x \), then \( \xi_x(y) \geq \frac{n}{M} > 0 \). So \( y \in B_{\rho_{x,y}}(x, S) \).

In addition, for any \( x_1, x_2 \in X \), we have

\[
|A_{x_1} \triangle A_{x_2}| = M\|\xi_{x_1} - \xi_{x_2}\|_1 = |A_{x_1}|\|\xi_{x_1} - \xi_{x_2}\|_1.
\]

So we obtain

\[
\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1}|} < \varepsilon
\]

when \( d(x_1, x_2) < R \). Since

\[
\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1}|} = \frac{|A_{x_1}| + |A_{x_2}| - 2|A_{x_1} \cap A_{x_2}|}{|A_{x_1}|},
\]

we have

\[
\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1}|} < \varepsilon
\]

when \( d(x_1, x_2) < R \).
we have
\[ |A_{x_1} \cap A_{x_2}| > \frac{(2-\varepsilon)M}{2}. \]

Then
\[ \frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{2M - 2|A_{x_1} \cap A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} < 2M - \frac{(2-\varepsilon)M}{\varepsilon} = \frac{2\varepsilon}{2-\varepsilon}. \]

So X has relative Property A with respect to Y and \( \rho_{X,Y} \). \( \square \)

The result above can be generalized to the \( \ell^p(Y) \), where \( 1 \leq p < \infty \) and Y is a countable set. In order to obtain the next corollary, we introduce the Mazur map [9] first. The Mazur map \( M_{p,q} : S(\ell^p) \rightarrow S(\ell^q) \) is defined by the formula
\[ M_{p,q}(x) = \text{sign}(x) |x|^{\frac{p}{q}} \]
where \( S(\ell^p) \) is the unit sphere of \( \ell^p \) and \( x \in S(\ell^p) \). It is a uniform homeomorphism between unit spheres of \( \ell^p \) and \( \ell^q \). More precisely, it satisfies the following inequalities:
\[ \frac{p}{q} \|x - y\|_p \leq \|M_{p,q}(x) - M_{p,q}(y)\|_q \leq C \|x - y\|_p \]
for all \( x, y \in S(\ell^p) \) and \( p < q \), where the constant \( C \) depends only on \( \frac{p}{q} \). We have the opposite inequalities if \( p > q \).

**Corollary 2.13** Let Y be a countable set and X be a discrete metric space with relative bounded geometry with respect to Y and \( \rho_{X,Y} \). Then X has relative Property A with respect to Y and \( \rho_{X,Y} \) if and only if the following hold for any \( 1 \leq p < \infty \): for every \( R > 0 \) and \( \varepsilon > 0 \) there exists an \( S > 0 \) and \( \eta : X \rightarrow \ell^p(Y)^+ \) satisfying:

1. \( \|\eta_x\|_{\ell^p} = 1 \) for any \( x \in X \);
2. If \( \eta_x(y) \neq 0 \), then \( y \in B_{\rho_{X,Y}}(x,S) \);
3. If \( d(x_1,x_2) < R \), then \( \|\eta_{x_1} - \eta_{x_2}\|_{\ell^p} \leq \varepsilon \).

**Proof** We prove the sufficiency first. Suppose X has relative Property A with respect to Y and \( \rho_{X,Y} \). For any \( R > 0 \) and \( \varepsilon > 0 \), there exists \( \xi : X \rightarrow \ell^1(Y)^+ \) and an S satisfying (1), (2) and (3) of the Theorem 2.12. Take any \( 1 \leq p < \infty \) and define a function \( \eta : X \rightarrow \ell^p(Y)^+ \) satisfying
\[ \eta_x(y) = \xi_x(y)^{\frac{p}{1}}. \]

Then
\[ \|\eta_x\|_{\ell^p} = \sum_{y \in Y} |\eta_x(y)|^p = \sum_{y \in Y} \xi_x(y)^p = \sum_{y \in Y} |\xi_x(y)| = \|\xi_x\|_{\ell^1} = 1. \]

So \( \|\eta_x\|_{\ell^p} = 1 \). If \( \eta_x(y) \neq 0 \), then \( \xi_x(y) \neq 0 \). So \( y \in B_{\rho_{X,Y}}(x,S) \).

For any \( x_1, x_2 \in X \) with \( d(x_1, x_2) < R \), we have
\[ \|\eta_{x_1} - \eta_{x_2}\|_{\ell^p} = \sum_{y \in Y} |\eta_{x_1}(y) - \eta_{x_2}(y)|^p \leq \sum_{y \in Y} |\eta_{x_1}(y)|^p - |\eta_{x_2}(y)|^p \]
\[ = \sum_{y \in Y} |\xi_{x_1}(y) - \xi_{x_2}(y)| = \|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} < \varepsilon. \]

So \( \|\eta_{x_1} - \eta_{x_2}\|_{\ell^p} < \varepsilon^{\frac{1}{p}} \).
Conversely, for any $R > 0$ and $\varepsilon > 0$ give a map $\eta : X \to \ell^p(Y)^+$ and an $S$ which satisfy (1), (2), (3) of the corollary. Define $\xi : X \to \ell^1(Y)^+$ by the formula

$$\xi_x = M_{p,1}(\eta_x),$$

where $M_{p,1}$ is the Mazur map. Then for any $x \in X$, $\|\xi_x\|_{\ell^1} = 1$ and $\eta_x(y) = 0$ if and only if $\xi_x(y) = 0$ for all $y \in Y$. So if $\xi_x(y) \neq 0$, we have $y \in B_{\rho_X,Y}(x,S)$.

For any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have

$$\|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} = \|M_{p,1}(\eta_{x_1}) - M_{p,1}(\eta_{x_2})\|_{\ell^p} \leq p\|\eta_{x_1} - \eta_{x_2}\|_{\ell^p} < p\varepsilon.$$

So $X$ has relative Property A with respect to $Y$ and $\rho_{X,Y}$. □

Acknowledgements We thank the referees for their time and comments.

References