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Formation and Transition of Delta Shock in the Limits of Riemann Solutions to the Perturbed Chromatography Equations

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Abstract This paper is concerned with the formation and transition of delta shock solutions to the perturbed chromatography equations. We discuss the Riemann problem for the perturbed chromatography equations. By studying the limits of the Riemann solutions as the perturbation parameter tends to zero, we can observe two important phenomena. One is that a shock and a contact discontinuity coincide to form a delta shock. The second is that the transition from one kind of delta shock on which two state variables simultaneously contain the Dirac delta function, to another kind of delta shock on which only one state variable contains the Dirac delta function.

Keywords chromatography equations; perturbation; delta shock; formation; transition

MR(2010) Subject Classification 35L65; 35L67; 76L05; 76N10

1. Introduction

The nonlinear chromatography equations can be expressed as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\{(1+\dfrac{1}{1-u+v})u\} = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}\{(1+\dfrac{1}{1-u+v})v\} = 0, \end{cases} \tag{1.1}$$

where u and v are non-negative functions of the variables $(x,t) \in R \times R^{+}$, which express the concentrations of the two absorbing species, and $1-u+v>0$. Eq. (1.1) are a common analytical tool to study the preparative separations in the pharmaceutical, food, and agrochemical industries. Yang and Zhang [1], Cheng and Yang [2] studied the Riemann problem of Eq. (1.1) and proved the existence and uniqueness of the solution in recent years.

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Eq. (1.1) can be derived from a more general nonlinear chromatography system

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial t}\{(1 + \dfrac{a_1}{1-u+v})u\} = 0, \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial}{\partial t}\{(1 + \dfrac{a_2}{1-u+v})v\} = 0, \end{cases} \tag{1.2}$$

where a_1 and a_2 are constants with $a_2 > a_1 > 0$; u and v are non-negative functions of the variables $(x,t) \in R \times R^+$, and where $1 - u + v > 0$. The difference between (1.1) and (1.2) is that system (1.2) is hyperbolic in the region $(a_1(1+v) + a_2(1-u_1))^2 - 4a_1a_2(1-u-+v) > 0$ and elliptic in the complementary part of it in the (u,v) plane, while (1.1) is always hyperbolic in the whole composition space.

A distinctive feature of (1.1) and (1.2) is that the delta shock with Dirac delta function will appear in both u and v (see [2]). This fact was also captured numerically and experimentally by Mazzotti et al. [3, 4] for (1.2). This delta shock phenomenon originates in the synergistic-competitive behavior of the two species as described in [2].

Another system of nonlinear chromatography equations is introduced in [5–7]. The model reads

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(u + \dfrac{u}{1+u+v}) = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}(v + \dfrac{v}{1+u+v}) = 0, \end{cases} \tag{1.3}$$

where $u(x,t) \geq 0$, $v(x,t) \geq 0$ which express transformations of the concentrations of two solutes. System (1.3) is widely used by chemists and engineers to study the separation of two chemical components in a fluid phase. Different from (1.1) and (1.2), the delta shock does not develop in the solutions of (1.3).

Recently, Ambrosio et al. [5] have introduced the change of variables

$$\theta = u - v, \qquad \eta = u + v, \tag{1.4}$$

then the system (1.3) can be changed to

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + \dfrac{\partial}{\partial x}(\theta + \dfrac{\theta}{1+\eta}) = 0, \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}(\eta + \dfrac{\eta}{1+\eta}) = 0, \end{cases} \tag{1.5}$$

where $\eta \geq 0$. Because of the conditions $u \geq 0$ and $v \geq 0$, the change of variables (1.4) is not one-on-one, which implies that system (1.3) and system (1.5) are not equivalent. The existence and uniqueness of solutions to (1.5) are proven by employing the self-similar viscosity vanishing approach in [8]. The delta shock appears in the Riemann solution of (1.5). However, for this kind of delta shock, only one state variable θ contains the Dirac delta function and the other η has bounded variation.

From the above discussion, one can observe that the essential difference among the nonlinear chromatography equations (1.1), (1.3) and (1.5) is the coefficient value in front of the absorbing species. The structures of solutions for these nonlinear chromatography equations are quite different, in which the delta shock plays an important role.

A delta shock is a generalization of an ordinary shock. It is more compressive than an ordinary shock in the sense that more characteristics enter the discontinuity line. Mathematically, the delta shocks are new type singular solutions such that their components contain delta functions and their derivatives. Physically, they are interpreted as the process of formation of the galaxies in the universe, or the process of concentration of particles [1].

The theory of the delta shock has been intensively developed in the last 20 years. The delta shock solution and the corresponding Rankine-Hugoniot condition were presented by Zeldovich and Myshkis [9] in the case of the continuity equation. In 1999, Sheng and Zhang [10] discussed the Riemann problem for the zero-pressure gas dynamics, in which the delta shocks appear. For the previous delta shock, the investigations have mostly been focused on the case that only one state variable develops the Dirac delta function and the others have bounded variations. In 2012, Yang and Zhang [1] established a new theory of delta shock with Dirac delta functions developing in two state variables for a class of nonstrictly hyperbolic systems of conservation laws. As for delta shock, there are numerous excellent papers, for the related references we can see [1,8,10–18] and the references cited therein.

In the delta shock theories, there are still many open and complicated problems. Study of this area gives a new perspective in the theory of conservation law systems [19,20]. In this paper, we are interested in the research of internal mechanism of the above two different kinds of delta shocks, i.e., the formation and transition of two different kinds of delta shocks. For this purpose, we study the Riemann problem of the perturbed nonlinear chromatography equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\{(1 + \dfrac{1}{1+ku+v})u\} = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}\{(1 + \dfrac{1}{1+ku+v})v\} = 0, \end{cases} \tag{1.6}$$

with initial value

$$(u, v)(x, 0) = (u^{\pm}, v^{\pm}), \quad \pm x > 0. \tag{1.7}$$

Here k is a perturbation parameter, u and v are the non-negative functions of the variables $(x, t) \in R \times R^{+}$ and $1 + ku + v > 0$. $u^{\pm}$ and $v^{\pm}$ are constants with $(u^{-}, v^{-}) \neq (u^{+}, v^{+})$. System (1.6) can also be used to analyze the stability property of the nonlinear chromatography equations due to the perturbation on the absorbing species u. We show that, as the perturbation parameter k vanishes, there appear three phenomena:

(1) The transition from one kind of delta shock on which two state variables u and v simultaneously contain the Dirac delta function, to another kind of delta shock on which only one state variable u contains the Dirac delta function.

(2) The formation of delta shock. That is, a shock and a contact discontinuity coincide to form a delta shock.

(3) The transition from a rarefaction wave to a left contact discontinuity.

The paper is organized as follows. In Section 2, we present some preliminary knowledge about the perturbed chromatography Eq. (1.6). Then in Section 3, we discuss the Riemann problem of (1.6) and develop the limit behavior of Riemann solutions as the perturbation parameter k vanishes.

2. Preliminaries

The eigenvalues of (1.6) are

$$\lambda_1(u,v;k)=1+\frac{1}{1+ku+v},\quad \lambda_2(u,v;k)=1+\frac{1}{(1+ku+v)^2}, \tag{2.1}$$

with the corresponding left eigenvectors

$$l_1(u,v;k)=(v,-u),\quad l_2(u,v;k)=(k,1), \tag{2.2}$$

and the corresponding right eigenvectors

$$r_1(u,v;k)=(1,-k)^T,\quad r_2(u,v;k)=(u,v)^T. \tag{2.3}$$

By a direct calculation, we have

$$\nabla\lambda_1\cdot r_1=0,\quad \nabla\lambda_2\cdot r_2=\frac{-2(ku+v)}{(1+ku+v)^3}. \tag{2.4}$$

Therefore, λ_1 is always linearly degenerate, λ_2 is genuinely nonlinear if $ku\neq -v$, and linearly degenerate if $ku=-v$.

The Riemann invariants of system (1.6) along the characteristic fields are

$$\zeta(u,v;k)=ku+v,\quad \varsigma(u,v;k)=\frac{v}{u}. \tag{2.5}$$

Definition 2.1 *A pair (u,v) constitutes a solution of (1.6) in the sense of distributions if it satisfies*

$$\begin{cases}\displaystyle\int_0^{+\infty}\int_{-\infty}^{+\infty}(u\phi_t+(u+\frac{u}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t+\int_{-\infty}^{+\infty}u(x,0)\phi(x,0)\mathrm{d}x=0,\\ \displaystyle\int_0^{+\infty}\int_{-\infty}^{+\infty}(v\phi_t+(v+\frac{v}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t+\int_{-\infty}^{+\infty}v(x,0)\phi(x,0)\mathrm{d}x=0,\end{cases} \tag{2.6}$$

for all test functions $\phi\in C_0^\infty((-\infty,+\infty)\times[0,\infty))$.

Definition 2.2 *A two-dimensional weighted delta function $w(s)\delta_l$, supported on a smooth curve l parameterized as $x=x(s),t=t(s)$ $(c\le s\le d)$, is defined by*

$$\langle w(s)\delta_l,\phi\rangle=\int_c^d w(s)\phi(x(s),t(s))\mathrm{d}s \tag{2.7}$$

for all the test functions $\phi\in C_0^\infty((-\infty,+\infty)\times[0,\infty))$.

Besides the constant solution, it is easy to check that there are self-similar waves. Set $\xi=x/t$. For a given left state (u^-,v^-), all possible states (u,v) which can be connected to (u^-,v^-) on the right by a rarefaction wave must be located on the following curve

$$R(u^-,v^-):\quad\begin{cases}\xi=\lambda_2(u,v;k)=1+\dfrac{1}{(1+ku+v)^2},\\ uv^-=u^-v,\quad ku+v<ku^-+v^-.\end{cases} \tag{2.8}$$

In particular, we call it a backward rarefaction wave symbolized by $\overleftarrow{R}$ when $0 < ku + v < ku^- + v^-$, and a forward rarefaction wave symbolized by $\overrightarrow{R}$ when $ku + v < ku^- + v^- < 0$.

The state which can be connected to a given left state (u^-, v^-) on the right by a contact discontinuity must be located on the curve

$$J(u^-, v^-): \quad \sigma = 1 + \frac{1}{1 + ku + v} = 1 + \frac{1}{1 + ku^- + v^-}. \tag{2.9}$$

All possible states which can be connected to (u^-, v^-) on the right by a shock must be located on the curve

$$S(u^-, v^-): \quad \begin{cases} \sigma = 1 + \dfrac{1}{(1 + ku + v)(1 + ku^- + v^-)}, \\ uv^- = u^- v. \end{cases} \tag{2.10}$$

Here we notice that the shock curve coincides with the rarefaction wave curve in the phase plane, i.e., (1.6) belongs to “Temple class” [21]. In particular, we call it a backward shock symbolized by $\overleftarrow{S}$ when $0 < ku^- + v^- < ku + v$, and a forward shock symbolized by $\overrightarrow{S}$ when $ku^- + v^- < ku + v < 0$.

3. Formation and transition

In this section, we study the formation and transition of two different kinds of delta shocks. We can capture this phenomenon by studying the limit behavior of Riemann solutions of (1.6) as the perturbation parameter $k \to 0$. First, we give some results on the Rieamnn solutions to system (1.6).

Let the left state (u^-, v^-) be fixed, and allow the right state (u^+, v^+) to vary. If (u^+, v^+) lies on any of the above curves (2.8)–(2.10), we have solved the problem. Assume that (u^+, v^+) is off the above curves in the rest of the paper. We put all of these curves together in the $u - v$ plane. Then the $u - v$ plane is divided into different disjoint regions. According to the right state (u^+, v^+) in the different region, one can construct the unique global Riemann solution to the problem (1.6) and (1.7).

Lemma 3.1 *The Riemann solutions to (1.6) and (1.7) are selected through the following conditions:*

(1) If $ku^- + v^- \le 0 \le ku^+ + v^+$, the solution is a delta shock δS;
(2) If $0 \le ku^+ + v^+ < ku^- + v^-$, the solution is $\overleftarrow{R} + J$;
(3) If $0 < ku^- + v^- < ku^+ + v^+$, the solution is $\overleftarrow{S} + J$;
(4) If $ku^+ + v^+ < 0 < ku^- + v^-$, the solution is $\overleftarrow{R}_1 + \overrightarrow{R}_2$;
(5) If $ku^+ + v^+ < ku^- + v^- \le 0$, the solution is $J + \overrightarrow{R}$;
(6) If $ku^- + v^- < ku^+ + v^+ < 0$, the solution is $J + \overrightarrow{S}$.

3.1. Formation of delta shock

Lemma 3.2 *Let $v^- = 0 < v^+$. For arbitrarily small $k > 0$, the Riemann solution of (1.6) is*

$\overleftarrow{S}+J$:

$$(u,v)(x,t)=\begin{cases}(u^-,0), & x<\sigma(k)t,\\ (u^*,v^*), & \sigma(k)t<x<\lambda_1(u^+,v^+;k)t,\\ (u^+,v^+), & x>\lambda_1(u^+,v^+;k)t,\end{cases} \tag{3.1}$$

where the intermediate state can be calculated as $(u^*,v^*)=(\dfrac{ku^++v^+}{k},0)$, *and* $\sigma(k)=1+\frac{1}{(1+ku^-)(1+ku^++v^+)}$ *is the propagation speed of the shock* $\overleftarrow{S}$.

Proof Due to $v^-=0<v^+$, there is a $k_0>0$, such that if $0<k<k_0$, then $0<ku^-+v^-<ku^++v^+$. This, together with (2.9), (2.10) and Lemma 3.1 yield the lemma. □

Theorem 3.3 *Let* $v^-=0<v^+$. *As* $k\to 0^+$, *the Riemann solution (3.1) converges to*

$$(\theta,\eta)(x,t)\triangleq\lim_{k\to 0^+}(u,v)(x,t)=\begin{cases}(u^-,0), & x<\tilde{\sigma}_\delta t,\\ (\tilde{\omega}(t)\delta(x-\tilde{x}(t)),v^+), & x=\tilde{\sigma}_\delta t,\\ (u^+,v^+), & x>\tilde{\sigma}_\delta t,\end{cases} \tag{3.2}$$

in the sense of distributions, which forms a δ*-shock solution of (1.5) with the same initial data* $(u^\pm,v^\pm)$. *Here* $\delta(\cdot)$ *is the standard Dirac measure,* $\tilde{\omega}(t)=\frac{u^-v^+}{1+v^+}t$ *and* $\tilde{\sigma}_\delta=1+\frac{1}{1+v^+}$ *are strength and velocity of the delta shock* $x=\tilde{x}(t)$, *respectively (see Figure 1).*

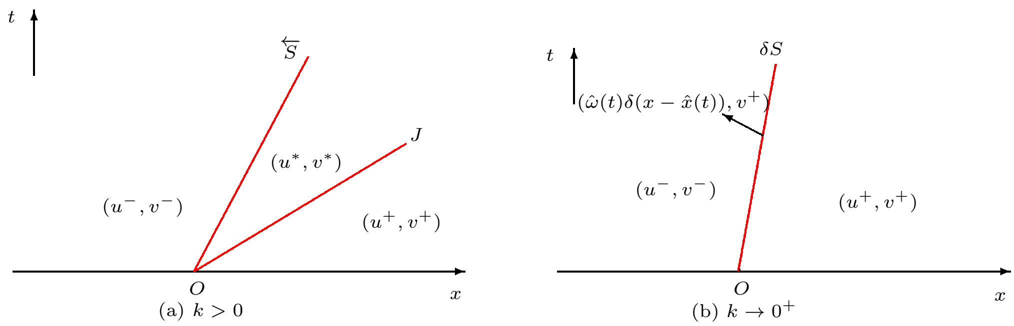

Figure 1 Formation of delta shock

Proof Letting $k\to 0^+$, it is obvious to see that the intermediate state (u^*,v^*) satisfies

$$\lim_{k\to 0^+}u^*=\lim_{k\to 0^+}\frac{ku^++v^+}{ku^-}\cdot u^-=\infty,\quad \lim_{k\to 0^+}v^*=\lim_{k\to 0^+}0=0. \tag{3.3}$$

At the same time, we have

$$\lim_{k\to 0}\sigma(k)=\lim_{k\to 0}\lambda_1(u^+,v^+;k)=1+\frac{1}{1+v^+}. \tag{3.4}$$

That is, the propagation speed of shock $\overleftarrow{S}$ tends to that of contact discontinuity J. Combining (3.3) and (3.4), we deduce $\overleftarrow{S}$ and J coincide to form a new type of nonlinear hyperbolic wave, which is called delta shock in [18]. The propagation speed of the delta shock is $\tilde{\sigma}_\delta=1+\frac{1}{1+v^+}$.

It is not hard to prove that the delta shock satisfies the δ-entropy condition

$$\lambda_2(u^+, v^+; 0) \leq \lambda_1(u^+, v^+; 0) \leq \tilde{\sigma_\delta} \leq \lambda_1(u^-, 0; 0) \leq \lambda_2(u^-, 0; 0),$$

which means none of the four characteristic lines on both side of the delta shock $x = \tilde{\sigma_\delta} t$ is outgoing.

Now let us calculate the total quantities of u between $\overleftarrow{S}$ and J as $k \to 0^+$. Since both (1.6) and (1.7) are invariant under the transformation $(x,t) \to (\alpha x, \alpha t)$ with the constant $\alpha > 0$, we consider the solution of the form

$$(u,v)(x,t) = (U,V)(\xi) = (U,V)(\frac{x}{t}).$$

Substituting the above equation into (1.6), one can see that the system (1.6) becomes

$$\begin{cases} -\xi U_\xi + (U + \dfrac{U}{1+kU+V})_\xi = 0, \\ -\xi V_\xi + (V + \dfrac{V}{1+kU+V})_\xi = 0. \end{cases} \tag{3.5}$$

We define the quantities $a = 1 + \frac{1}{1+ku^+ + v^+}$ and $b = 1 + \dfrac{1}{(1+ku^-)(1+ku^+ + v^+)}$. From the first equation of (3.5), it follows

$$\begin{aligned} 0 &= \int_{\xi=b-0}^{\xi=a+0} -\xi \mathrm{d}U + \mathrm{d}(U + \frac{U}{1+kU+V}) \\ &= -(\xi U)\Big|_{\xi=b-0}^{\xi=a+0} + \int_{\xi=b-0}^{\xi=a+0} U \mathrm{d}\xi + (U + \frac{U}{1+kU+V})\Big|_{\xi=b-0}^{\xi=a+0}. \end{aligned} \tag{3.6}$$

An easy computation leads to

$$\lim_{k\to 0^+} \int_{\xi=b-0}^{\xi=a+0} U(\xi)\mathrm{d}\xi = \frac{u^- v^+}{1+v^+}, \tag{3.7}$$

which shows that $u(x,t) = U(\xi)$ has the same singularity as a weighted Dirac delta function at $\xi = 1 + \frac{1}{1+v^+}$. In view of (3.4) and (3.7), we verify that the Riemann solution (3.1) converges to (3.2) as $k \to 0^+$.

In the following, we prove that the delta shock defined by (3.2) satisfies (1.5) in the sense of distributions. For any test functions $\phi \in C_0^\infty((-\infty, +\infty) \times [0, \infty))$, since $v^- = 0$, we have

$$\begin{aligned} &\int_0^{+\infty} \int_{-\infty}^{+\infty} (\theta\phi_t + (\theta + \frac{\theta}{1+\eta})\phi_x)\mathrm{d}x\mathrm{d}t \\ &= \Big(\int_0^{+\infty}\int_{-\infty}^{\tilde{x}(t)} + \int_0^{+\infty}\int_{\tilde{x}(t)}^{+\infty}\Big)((\theta\phi)_t + ((\theta + \frac{\theta}{1+\eta})\phi)_x)\mathrm{d}x\mathrm{d}t + \int_0^{+\infty} (\phi_t + \tilde{\sigma}_\delta \phi_x)\tilde{\omega}\mathrm{d}t \\ &= \Big(\oint_1 + \oint_2\Big)((\theta + \frac{\theta}{1+\eta})\phi \mathrm{d}t - \theta\phi\mathrm{d}x) + \int_0^{+\infty} \tilde{\omega}\mathrm{d}\phi \\ &= \int_0^{+\infty} \phi(u^+ + \frac{u^+}{1+v^+} - u^- - \frac{u^-}{1+v^-} - (u^+ - u^-)\tilde{\sigma}_\delta)\mathrm{d}t - \int_0^{+\infty} \phi \mathrm{d}\tilde{\omega} \\ &= \int_0^{+\infty} \phi(u^+ + \frac{u^+}{1+v^+} - u^- - \frac{u^-}{1+v^-} - (u^+ - u^-)\tilde{\sigma}_\delta - \frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}t})\mathrm{d}t = 0 \end{aligned}$$

and

$$\begin{aligned}&\int_0^{+\infty}\int_{-\infty}^{+\infty}(\eta\phi_t+(\eta+\frac{\eta}{1+\eta})\phi_x)\mathrm{d}x\mathrm{d}t\\&=\Big(\int_0^{+\infty}\int_{-\infty}^{\tilde{x}(t)}+\int_0^{+\infty}\int_{\tilde{x}(t)}^{+\infty}\Big)((\eta\phi)_t+((\eta+\frac{\eta}{1+\eta})\phi)_x)\mathrm{d}x\mathrm{d}t\\&=\Big(\oint_1+\oint_2\Big)((\eta+\frac{\eta}{1+\eta})\phi\mathrm{d}t-\eta\phi\mathrm{d}x)\\&=\int_0^{+\infty}\phi(v^++\frac{v^+}{1+v^+}-v^+\tilde{\sigma}_\delta)\mathrm{d}t=0.\ \Box\end{aligned}$$

3.2. Transition of two different kinds of delta shocks

Lemma 3.4 *Let $v^-=0<v^+$. For $k<0$ with $|k|$ arbitrarily small, the Riemann solution of (1.6) and (1.7) is a delta shock δS:*

$$(u,v)(x,t)=\begin{cases}(u^-,0), & x<\sigma_\delta(k)t,\\(\omega(t;k)\delta(x-\sigma_\delta(k)t),-k\omega(t;k)\delta(x-\sigma_\delta(k)t)), & x=\sigma_\delta(k)t,\\(u^+,v^+), & x>\sigma_\delta(k)t,\end{cases}\tag{3.8}$$

where $\omega(t;k)=\frac{u^-v^+}{(1+ku^-)(1+ku^++v^+)}t$ and $\sigma_\delta(k)=1+\frac{1}{(1+ku^-)(1+ku^++v^+)}$ are strength and velocity of the delta shock $x=x(t)$, respectively. The strength $\omega(t;k)$ and the velocity $\sigma_\delta(k)$ of the delta shock have the following properties:

$$\lim_{k\to0^-}\omega(t;k)=\tilde{\omega}(t)\ \ \text{and}\ \ \lim_{k\to0^-}\sigma_\delta(k)=\tilde{\sigma_\delta}.\tag{3.9}$$

Proof Due to $v^-=0<v^+$, there exists a $k_0<0$ such that $ku^-+v^-\le0\le ku^++v^+$ for $k_0<k<0$. Comparing this with Lemma 3.1, we get the Riemann solution of (1.6) and (1.7) is a delta shock. An easy calculation shows

$$\lim_{k\to0^-}\omega(t;k)=\lim_{k\to0^-}\frac{u^-v^+}{(1+ku^-)(1+ku^++v^+)}t=\frac{u^-v^+}{1+v^+}t=\tilde{\omega}(t)$$

and

$$\lim_{k\to0^-}\sigma_\delta(k)=\lim_{k\to0^-}(1+\frac{1}{(1+ku^-)(1+ku^++v^+)})=1+\frac{1}{1+v^+}=\tilde{\sigma_\delta}.\ \Box$$

Theorem 3.5 *Let $v^-=0<v^+$. As $k\to0^-$, the δ-shock solution (3.8) converges to (3.2), which is a δ-shock solution of the Riemann problem (1.5) with the same initial data $(u^\pm,v^\pm)$; it shows how the Dirac delta function in v disappears when the parameter k vanishes (see Figure 2).*

Proof Since the delta shock solution (3.8) satisfies (1.6) and (1.7) in the sense of distributions, for all the test functions $\phi\in C_0^\infty((-\infty,+\infty)\times[0,\infty))$, we have

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}(u\phi_t+(u+\frac{u}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t=0\tag{3.10}$$

and

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}(v\phi_t+(v+\frac{v}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t=0. \tag{3.11}$$

For the left-hand side of (3.10), using Green's formulation and integration by parts, we calculate

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}(u\phi_t+(u+\frac{u}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t\\
&=\Big(\int_0^{+\infty}\int_{-\infty}^{x(t)}+\int_0^{+\infty}\int_{x(t)}^{+\infty}\Big)((u\phi)_t+((u+\frac{u}{1+ku+v})\phi)_x)\mathrm{d}x\mathrm{d}t-\\
&\quad\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi(u_t+(u+\frac{u}{1+ku+v})_x)\mathrm{d}x\mathrm{d}t+\int_0^{+\infty}(\phi_t+\sigma_\delta\phi_x)\omega\mathrm{d}t\\
&=\Big(\oint_1+\oint_2\Big)((u+\frac{u}{1+ku+v})\phi\mathrm{d}t-u\phi\mathrm{d}x)+\int_0^{+\infty}\omega\mathrm{d}\phi\\
&=\int_0^{+\infty}\phi([u+\frac{u}{1+ku+v}]-[u]\sigma_\delta)\mathrm{d}t-\int_0^{+\infty}\phi\mathrm{d}\omega\\
&=\int_0^{+\infty}\phi(([u+\frac{u}{1+ku+v}]-[u]\sigma_\delta)\mathrm{d}t-\mathrm{d}\omega)\\
&=\int_0^{+\infty}\phi([u+\frac{u}{1+ku+v}]-[u]\sigma_\delta-\frac{\mathrm{d}\omega}{\mathrm{d}t})\mathrm{d}t\\
&=\int_0^{+\infty}\phi([u+\frac{u}{1+ku+v}]-[u]\sigma_\delta-\frac{u^-v^+}{(1+ku^-)(1+ku^++v^+)})\mathrm{d}t.
\end{aligned}$$

Here and below, we use the usual notation $[u]=u^r-u^l$ with u^l and u^r the values of the function u on the left-hand and right-hand sides of a discontinuity, etc. Taking the limit in (3.10), in view of (3.9) and the above expression, we have

$$\begin{aligned}
&\lim_{k\to0^-}\int_0^{+\infty}\int_{-\infty}^{+\infty}(u\phi_t+(u+\frac{u}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t\\
&=\lim_{k\to0^-}\int_0^{+\infty}\phi([u+\frac{u}{1+ku+v}]-[u]\sigma_\delta-\frac{u^-v^+}{(1+ku^-)(1+ku^++v^+)})\mathrm{d}t\\
&=\int_0^{+\infty}\phi([\theta+\frac{\theta}{1+\eta}]-[\theta]\tilde{\sigma_\delta}-\frac{u^-v^+}{1+v^+})\mathrm{d}t\\
&=\int_0^{+\infty}\phi([\theta+\frac{\theta}{1+\eta}]-[\theta]\tilde{\sigma_\delta}-\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}t})\mathrm{d}t=0.
\end{aligned}\tag{3.12}$$

As for the left-hand side of (3.11), with the same reason as before, we arrive at

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{+\infty}(v\phi_t+(v+\frac{v}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t\\
&=\Big(\int_0^{+\infty}\int_{-\infty}^{x(t)}+\int_0^{+\infty}\int_{x(t)}^{+\infty}\Big)((v\phi)_t+((v+\frac{v}{1+ku+v})\phi)_x)\mathrm{d}x\mathrm{d}t-\\
&\quad\int_0^{+\infty}\int_{-\infty}^{+\infty}\phi(v_t+(v+\frac{v}{1+ku+v})_x)\mathrm{d}x\mathrm{d}t+\int_0^{+\infty}k(\phi_t+\sigma_\delta\phi_x)\omega\mathrm{d}t\\
&=\Big(\oint_1+\oint_2\Big)((v+\frac{v}{1+ku+v})\phi\mathrm{d}t-v\phi\mathrm{d}x)+\int_0^{+\infty}k\omega\mathrm{d}\phi
\end{aligned}$$

$$= \int_0^{+\infty} \phi([v + \frac{v}{1+ku+v}] - [v]\sigma_\delta)\mathrm{d}t - \int_0^{+\infty} k\phi \mathrm{d}\omega$$
$$= \int_0^{+\infty} \phi(([v + \frac{v}{1+ku+v}] - [v]\sigma_\delta)\mathrm{d}t - k\mathrm{d}\omega)$$
$$= \int_0^{+\infty} \phi([v + \frac{v}{1+ku+v}] - [v]\sigma_\delta - k\frac{\mathrm{d}\omega}{\mathrm{d}t})\mathrm{d}t$$
$$= \int_0^{+\infty} \phi([v + \frac{v}{1+ku+v}] - [v]\sigma_\delta - \frac{ku^-v^+}{(1+ku^-)(1+ku^++v^+)})\mathrm{d}t$$

and

$$\lim_{k\to 0^-}\int_0^{+\infty}\int_{-\infty}^{+\infty} (v\phi_t + (v+\frac{v}{1+ku+v})\phi_x)\mathrm{d}x\mathrm{d}t$$
$$= \lim_{k\to 0^-}\int_0^{+\infty} \phi([v + \frac{v}{1+ku+v}] - [v]\sigma_\delta - \frac{ku^-v^+}{(1+ku^-)(1+ku^++v^+)})\mathrm{d}t$$
$$= \int_0^{+\infty} \phi([\eta + \frac{\eta}{1+\eta}] - [\eta]\tilde{\sigma_\delta})\mathrm{d}t = 0. \tag{3.13}$$

Due to (3.9), (3.12) and (3.13), as $k \to 0^-$, we verify that the limit of (3.8) is (3.2), which is the Riemann solution of the problem (1.5) with the same initial data $(u^\pm, v^\pm)$. It is interesting to see from (3.12) that the Dirac delta function in u remains as $k \to 0^-$. However, from (3.13), we can see that the Dirac delta function in v disappears. □

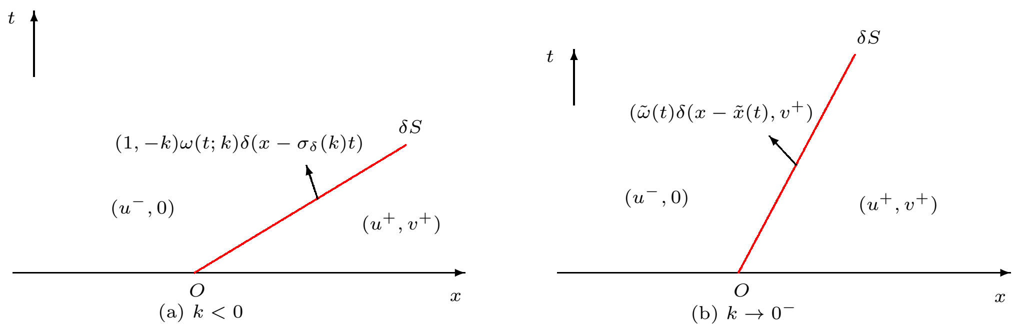

Figure 2 Transition of delta shock

Remark 3.6 From the proof of Theorem 3.5, we can see the transition between two different kinds of delta shocks as $k \to 0^-$. That is, the transition from one kind of delta shock on which both state variables u and v simultaneously contain the Dirac delta function, to another kind of delta shock on which only one state variable θ contains the Dirac delta function.

3.3. Transition from rarefaction wave to contact discontinuity

Lemma 3.7 *Let $k < 0$ with $|k|$ arbitrarily small. If $v^- > v^+ > 0$, the Riemann solution of (1.6) and (1.7) is $\overleftarrow{R} + J$:*

$$(u,v)(x,t)=\begin{cases}(u^-,v^-), & x<\lambda_2(u^-,v^-;k)t,\\ \overleftarrow{R}, & \lambda_2(u^-,v^-;k)t\le x\le \lambda_2(u^*,v^*;k)t,\\ (u^*,v^*), & \lambda_2(u^*,v^*;k)t<x<\lambda_1(u^+,v^+;k)t,\\ (u^+,v^+), & x>\lambda_1(u^+,v^+;k)t,\end{cases} \tag{3.14}$$

where the intermediate state (u^*,v^*) *can be calculated as* $(u^*,v^*)=(\frac{ku^++v^+}{ku^-+v^-}u^-,\frac{ku^++v^+}{ku^-+v^-}v^-)$, *and the rarefaction wave* $\overleftarrow{R}$ *is given by*

$$(u,v)(x,t)=(\frac{u^-}{ku^-+v^-}(\sqrt{\frac{1}{\xi-1}}-1),\frac{v^-}{ku^-+v^-}(\sqrt{\frac{1}{\xi-1}}-1)),\quad \xi=\frac{x}{t}. \tag{3.15}$$

If $v^->v^+=0$, *the Riemann solution is either (3.14) or* $\overleftarrow{R}_1+\overrightarrow{R}_2$*:*

$$(u,v)(x,t)=\begin{cases}(u^-,v^-), & x<\lambda_2(u^-,v^-;k)t,\\ \overleftarrow{R}_1, & \lambda_2(u^-,v^-;k)t\le x<2t,\\ (0,0), & x=2t,\\ \overrightarrow{R}_2, & 2t<x<\lambda_2(u^+,0;k)t,\\ (u^+,0), & x>\lambda_2(u^+,0;k)t,\end{cases} \tag{3.16}$$

where the rarefaction wave $\overleftarrow{R}_1$ *can be given by (3.15) and the rarefaction wave* $\overrightarrow{R}_2$ *is defined by*

$$(u,v)(x,t)=(\frac{1}{k}(\sqrt{\frac{1}{\xi-1}}-1),0),\ \xi=\frac{x}{t}. \tag{3.17}$$

Furthermore,

$$\lim_{k\to 0}\lambda_1(u,v;k)=\lambda_1(u,v;0),\quad \lim_{k\to 0}\lambda_2(u,v;k)=\lambda_2(u,v;0), \tag{3.18}$$

and

$$\lim_{k\to 0^-}u^*=\lim_{k\to 0^-}\frac{ku^++v^+}{ku^-+v^-}u^-=\frac{v^+u^-}{v^-}\triangleq\hat{u}^*,\quad \lim_{k\to 0^-}v^*=\lim_{k\to 0^-}\frac{ku^++v^+}{ku^-+v^-}v^-=v^+\triangleq\hat{v}^* \tag{3.19}$$

hold true.

Proof If $v^->v^+>0$, then there exists $k_0<0$ such that $0\le ku^++v^+<ku^-+v^-$ for all $0<k<k_0$. This together with Lemma 3.1 implies that the Riemann solution of (1.6) and (1.7) is $\overleftarrow{R}+J$.

If $v^->v^+=0$, then there exists $k_1<0$ such that $0\le ku^++v^+<ku^-+v^-$ or $ku^++v^+<0<ku^-+v^-$ for all $0<k<k_1$. From Lemma 3.1, it follows that the Riemann solution of (1.6) and (1.7) is $\overleftarrow{R}+J$ or $\overleftarrow{R}_1+\overrightarrow{R}_2$. The calculations for (3.18) and (3.19) are straightforward, and we omit them. □

Theorem 3.8 *Let* $v^->v^+\ge 0$. *As* $k\to 0^-$, *both the Riemann solution (3.14) and (3.16)* *converge to*

$$(\theta,\eta)(x,t)=\begin{cases}(u^-,v^-), & x<\lambda_2(u^-,v^-;0)t,\\ \overleftarrow{R}, & \lambda_2(u^-,v^-;0)t\le x\le \lambda_2(\hat{u}^*,\hat{v}^*;0)t,\\ (\hat{u}^*,\hat{v}^*), & \lambda_2(\hat{u}^*,\hat{v}^*;0)t<x<\lambda_1(u^+,v^+;0)t,\\ (u^+,v^+), & x>\lambda_1(u^+,v^+;0)t,\end{cases}\tag{3.20}$$

where the intermediate state $(\hat{u}^*,\hat{v}^*)$ *is defined by (3.19), and the rarefaction wave* $\overleftarrow{R}$ *is given by*

$$(\theta,\eta)(x,t)=\Big(\frac{u^-}{v^-}\Big(\sqrt{\frac{1}{\xi-1}}-1\Big),\sqrt{\frac{1}{\xi-1}}-1\Big),\quad \xi=\frac{x}{t};\tag{3.21}$$

it shows how a rarefaction wave degenerates into a left contact discontinuity (see Figure 3). Furthermore, the limit (3.20) is a solution of Riemann problem (1.5) with the same initial data $(u^\pm,v^\pm)$*.*

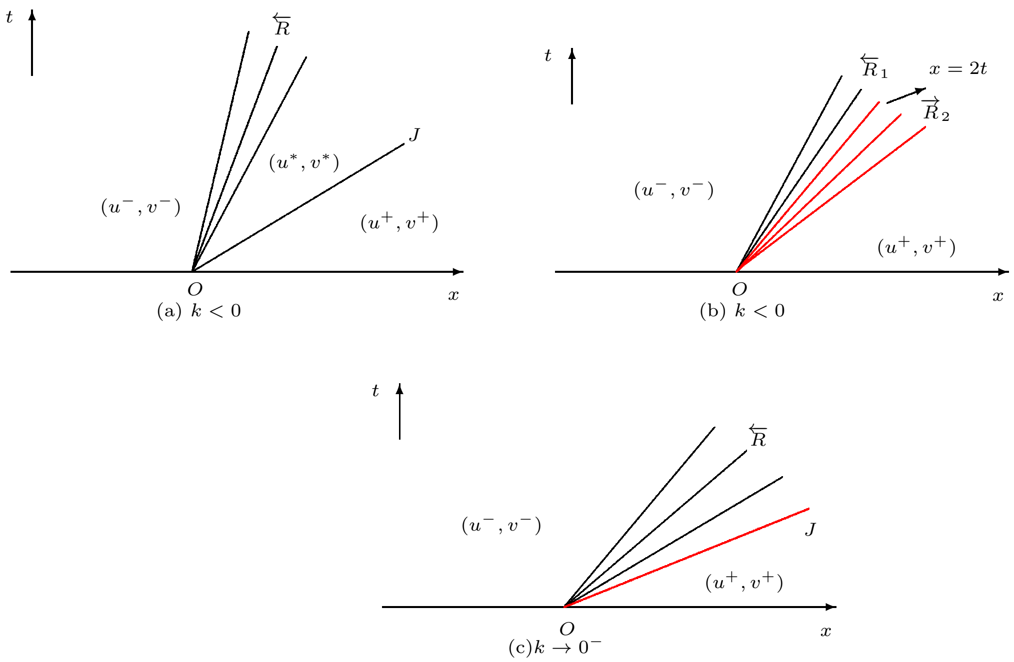

Figure 3 Transition from rarefaction wave to contact discontinuity

Proof If $v^->v^+>0$, for $k<0$ with $|k|$ arbitrarily small, we deduce from Lemma 3.7 that the Riemann solution of (1.6) is given by (3.14). Taking the limit $k\to 0^-$ in (3.15), we have

$$\begin{aligned}&\lim_{k\to 0^-}\frac{u^-}{ku^-+v^-}\Big(\sqrt{\frac{1}{\xi-1}}-1\Big)=\frac{u^-}{v^-}\Big(\sqrt{\frac{1}{\xi-1}}-1\Big),\\ &\lim_{k\to 0^-}\frac{v^-}{ku^-+v^-}\Big(\sqrt{\frac{1}{\xi-1}}-1\Big)=\sqrt{\frac{1}{\xi-1}}-1.\end{aligned}\tag{3.22}$$

From (3.18), (3.19) and (3.22), it yields that the Riemann solution (3.14) converges to (3.20) as $k \to 0^-$.

If $v^- > v^+ = 0$, for $k < 0$ with $|k|$ arbitrarily small, the Riemann solution of (1.6) is either (3.14) or (3.16). First, assume that the Riemann solution of (1.6) is given by (3.14). As in the proof of subcase $v^- > v^+ > 0$, we can prove that the Riemann solution (3.14) converges to (3.20) when the parameter k vanishes.

Secondly, assume that the Riemann solution of (1.6) is (3.16). Substituting the left boundary $\xi = 2$ of rarefaction wave $\overrightarrow{R}_2$ into (3.17) and taking the limit in it, we have

$$\lim_{k\to 0^-} u(x,t) = \lim_{k\to 0^-} \frac{1}{k}\Big(\sqrt{\frac{1}{2-1}} - 1\Big) = 0. \tag{3.23}$$

Similarly, substituting the right boundary $\xi = \lambda_2(u^+, 0; k)$ of rarefaction wave $\overrightarrow{R}_2$ into (3.17) and taking the limit in it, we get

$$\lim_{k\to 0^-} u(x,t) = \lim_{k\to 0^-} \frac{1}{k}\Big(\sqrt{\frac{1}{1+\frac{1}{(1+ku^+)^2}-1}} - 1\Big) = u^+. \tag{3.24}$$

Furthermore,

$$\lim_{k\to 0^-} \lambda_2(u^+, 0; k) = 2 = \lambda_1(u^+, 0; 0). \tag{3.25}$$

From (3.23)–(3.25), it is easily seen that the left boundary and the right boundary of the rarefaction wave $\overrightarrow{R}_2$ in (3.16), coincide to form a left contact discontinuity with speed $\lambda_1(u^+, 0; 0) = 2$ as $k \to 0^-$. However, the rarefaction wave $\overrightarrow{R}_1$ in (3.16) can retain its form after the limit. So we verify that the limit of (3.16) is (3.20).

For any test function $\phi \in C_0^\infty((-\infty, +\infty) \times [0, \infty))$, it is not hard to show that (3.20) implies the following equations:

$$\Big(\int_{-\infty}^{\lambda_2(u^-,v^-;0)} + \int_{\lambda_2(u^-,v^-;0)}^{\lambda_2(\hat{u}^*,\hat{v}^*;0)} + \int_{\lambda_2(\hat{u}^*,\hat{v}^*;0)}^{\lambda_1(u^+,v^+;0)} + \int_{\lambda_1(u^+,v^+;0)}^{+\infty}\Big)(-\xi\theta_\xi + (\theta + \frac{\theta}{1+\eta})_\xi)\phi \mathrm{d}\xi = 0, \tag{3.26}$$

$$\Big(\int_{-\infty}^{\lambda_2(u^-,v^-;0)} + \int_{\lambda_2(u^-,v^-;0)}^{\lambda_2(\hat{u}^*,\hat{v}^*;0)} + \int_{\lambda_2(\hat{u}^*,\hat{v}^*;0)}^{\lambda_1(u^+,v^+;0)} + \int_{\lambda_1(u^+,v^+;0)}^{+\infty}\Big)(-\xi\eta_\xi + (\eta + \frac{\eta}{1+\eta})_\xi)\phi \mathrm{d}\xi = 0. \tag{3.27}$$

In view of (3.26) and (3.27), we conclude that the limit function (3.20) is a solution to the Riemann problem (1.5) with the same initial data $(u^\pm, v^\pm)$. □

Theorems 3.3 and 3.5 provide a detailed description of formation and transition of different kinds of delta shocks, which allow us to better investigate instability and internal mechanism of delta shocks.

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