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Hypergraphs with Spectral Radius at Most $\sqrt[r]{2+\sqrt{5}}$

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Abstract In this paper, we consider the r-uniform hypergraphs H with spectral radius at most $\sqrt[r]{2+\sqrt{5}}$. We show that H must have a quipus-structure, which is similar to the graphs with spectral radius at most $\frac{3}{2}\sqrt{2}$ [Woo-Neumaier, Graphs Combin. 2007].

Keywords r-uniform hypergraphs; spectral radius; α-normal

MR(2010) Subject Classification 05C50; 05C35; 05C65

1. Introduction

The spectral radius $\rho(G)$ of a graph G is the largest eigenvalue of its adjacency matrix. The (simple, undirected and connected) graphs with small spectral radius have been well-studied in the literature. In 1970 Smith classified all connected graphs with spectral radius at most 2. The graphs G with $\rho(G) < 2$ are simple Dynkin Diagrams A_n, D_n, E_6, E_7, and E_8, while the graphs G with $\rho(G) = 2$ simply extend Dynkin Diagrams $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$. Cvetković et al. [1] gave a nearly complete description of all graphs G with $2 < \rho(G) < \sqrt{2+\sqrt{5}}$. Their description was completed by Brouwer and Neumaier [2]. Namely, $E(1,b,c)$ for $b=2, c\geq 6$ or $b\geq 3, c\geq 4$, $E(2,2,c)$ for $c\geq 3$, and $G_{1,a:b:1,c}$ for $a\geq 3$, $c\geq 2$, $b > a+c$.

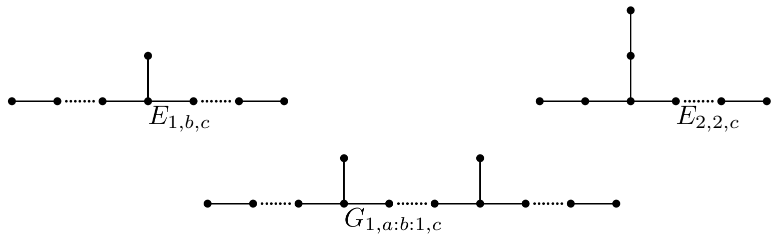

Figure 1 The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$

Wang et al. [3] studied some graphs with spectral radius close to $\frac{3}{2}\sqrt{2}$. Woo and Neumaier [4] proved that any connected graph G with $\sqrt{2+\sqrt{5}} < \rho(G) < \frac{3}{2}\sqrt{2}$ is one of the following graphs.

(1) If G has maximum degree at least 4, then G is a dagger (i.e., a tree obtained by attaching a path to a leaf vertex of the star S_5).

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(2) If G is a tree with maximum degree at most 3, then G is an open quipu (i.e., all vertices of degree 3 lie on a path).

(3) If G contains a cycle, then G is a closed quipu (i.e., a unicyclic graph with maximum degree at most 3 such that all vertices of degree 3 lie on a cycle).

In this paper, we would like to study the r-uniform hypergraphs H with small spectral radius. In our previous paper [5], we generalized Smith's theorem to hypergraphs and classified all connected r-uniform hypergraphs with the spectral radius at most $\rho_r = \sqrt[r]{4}$. The main method is using α-normal labeling. Roughly speaking, we can label all "corners of edges" by some numbers in $(0,1)$ such that for each vertex v the sum of these numbers at v is always equal to 1 while for each edge f the product of these numbers at f is always equal to α. The detail of the definition of α-normal labeling can be found in Section 2. If H has a "consistent" α-normal labeling, then $\rho(H) = \alpha^{-1/r}$. As an important corollary, any $(r-1)$-uniform hypergraph H' with $\rho(H') = \alpha^{-1/(r-1)}$ can be extended to an r-uniform hypergraph H with spectral radius $\rho(H) = \alpha^{-1/r}$ by simply extending each edge by adding one new vertex. If H is not extended from some H', then H is called irreducible. We use the following convention: if the notation $H^{(r')}$ is an r'-uniform hypergraph, then for each $r > r'$, $H^{(r)}$ means the unique r-uniform hypergraph extended from $H^{(r')}$ by a sequence of extension described above.

From [5], we show all r-uniform hypergraphs H with $\rho(H) = (r-1)!\sqrt[r]{4}$ listed as follows:

Extended from 2-graphs: $C_n^{(r)}$, $\tilde{D}_n^{(r)}$, $\tilde{E}_6^{(r)}$, $\tilde{E}_7^{(r)}$, and $\tilde{E}_8^{(r)}$.

Extended from 3-graphs: $\tilde{B}_n^{(r)}$, $\widetilde{BD}_n^{(r)}$, $C_2^{(r)}$, $S_4^{(r)}$, $F_{2,3,4}^{(r)}$, $F_{2,2,7}^{(r)}$, $F_{1,5,6}^{(r)}$, $F_{1,4,8}^{(r)}$, $F_{1,3,14}^{(r)}$, $G_{1,1:0:1,4}^{(r)}$, and $G_{1,1:6:1,3}^{(r)}$.

Extended from 4-graphs: $H_{1,1,2,2}^{(r)}$.

Similarly here are all r-uniform hypergraphs H with $\rho(H) < (r-1)!\sqrt[r]{4}$:

Extended from 2-graphs: $A_n^{(r)}$, $D_n^{(r)}$, $E_6^{(r)}$, $E_7^{(r)}$, and $E_8^{(r)}$.

Extended from 3-graphs: ${D'}_n^{(r)}$, $B_n^{(r)}$, ${B'}_n^{(r)}$, $\bar{B}_n^{(r)}$, $BD_n^{(r)}$, $F_{2,3,3}^{(r)}$, $F_{2,2,j}^{(r)}$ (for $2 \le j \le 6$), $F_{1,3,j}^{(r)}$ (for $3 \le j \le 13$), $F_{1,4,j}^{(r)}$ (for $4 \le j \le 7$), $F_{1,5,5}^{(r)}$, and $G_{1,1:j:1,3}^{(r)}$ (for $0 \le j \le 5$).

Extended from 4-graphs: $H_{1,1,1,1}^{(r)}$, $H_{1,1,1,2}^{(r)}$, $H_{1,1,1,3}^{(r)}$, $H_{1,1,1,4}^{(r)}$.

The details of these hypergraphs can be found in the paper [5].

It is natural to ask what structure the hypergraph with spectral radius slightly greater than ρ_r can have. Since $(2, \sqrt{2+\sqrt{5}})$ is the next interesting interval for the spectral radius of graphs, naturally we consider all connected r-uniform hypergraphs H with $\rho(H) \in (\sqrt[r]{4}, \sqrt[r]{2+\sqrt{5}})$. When $r=2$, these graphs are $E_{1,b,c}$, $E_{2,2,c}$, and $G_{1,a:b:1,c}$ with $b > a+c$ as shown by Cvetković et al. [1] and Brouwer-Neumaier [2]. The structures of these hypergraphs are slightly more complicated for $r \ge 3$. For $k \ge 3$, a vertex is called a degree-k vertex if it is incident to exactly k edges while an edge is called a k-branching edge if it contains no degree-k vertex but it is incident to exactly k edges (When $k=3$, we simply say branching edge instead of 3-branching edge). We have the following results.

Theorem 1.1 *Consider an irreducible connected 3-uniform hypergraph H. If the spectral radius of H satisfies $\rho(H) \le \sqrt[3]{2+\sqrt{5}}$, then no vertex (of H) can have degree more than three, no edge* *can be incident to more than 3 other edges, and each degree-3 vertex is not incident to any branching edges. Moreover, H belongs to one of the following two categories:*

(1) Open 3-quipu (see Figure 2): H is a hypertree with all branching vertices and all branching edges lying on a path. Moreover, there are at most 2 branching vertices. A branching vertex cannot lie between two branching edges, or between a branching edge and another branching vertex.

(2) Closed 3-quipu (see Figure 2): H contains a cycle C and no degree-3 vertex. All branching edges lie on C, and the vertices of the branching edges not on the cycle can be only incident to a path.

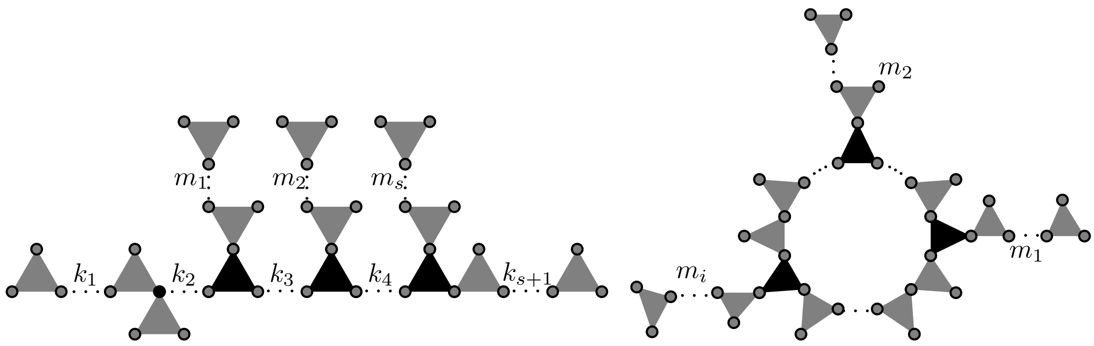


Figure 2 (Examples) Left: an open 3-quipu where the branching vertex/edges are filled in black. Right: a closed quipu where the branching edges are filled in black.

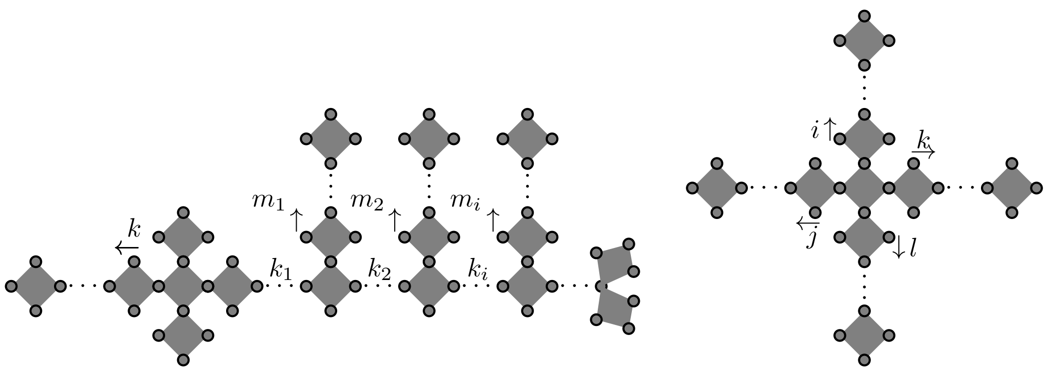


Figure 3 (Examples) Left: an open 4-quipu. Right: a 4-dagger $H^{(4)}_{i,j,k,l}$.

Theorem 1.2 *Suppose that H is an irreducible 4-uniform hypergraphs with $\rho(H) \leq \sqrt[4]{2+\sqrt{5}}$. Then H is a hypertree with no vertex (of H) having degree more than three and no edge incident to more than 4 other edges. The hypergraph H belongs to one of the following two categories:*

(1) Open 4-quipu (see Figure 3): H is a hypertree with all degree-3 vertices and all branching edges lying on a path. Moreover, there are at most two degree-3 vertices (or two 4-branching edges). A 4-branching edge (or a degree-3 vertex) cannot lie between two 3-branching edges, or between a 3-branching edge and another 4-branching edge (or a degree-3 vertex). In addition, each 4-branching edge is attached by three paths of length 1, 1, and k ($k = 1, 2, 3$), respectively.

(2) 4-dagger (see Figure 3): H is obtained by attaching 4-paths of length i, j, k, l to a 4-branching edge. Denote this hypergraph by $H^{(4)}_{i,j,k,l}$ with $i \le j \le k \le l$. Then H must be one of the following hypergraphs $H^{(4)}_{1,2,2,2}$, $H^{(4)}_{1,2,2,3}$, $H^{(4)}_{1,1,4,4}$, $H^{(4)}_{1,1,4,5}$, and $H^{(4)}_{1,1,k,l}$ ($1 \le k \le 3$, and $k \le l$).

Theorem 1.3 *For $r = 5$, there is only one irreducible 5-uniform hypergraph H with $\rho(H) \le \sqrt[r]{2+\sqrt{5}}$; namely the five edge-star as shown in Figure 4.*

For $r \ge 6$, all r-uniform hypergraphs H with $\rho(H) \le \sqrt[r]{2+\sqrt{5}}$ are reducible.

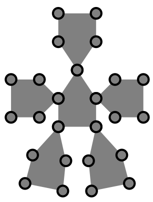


Figure 4 The five edge-star.

2. Notation and lemmas

Let us review some basic notation about hypergraphs. An r-uniform hypergraph H is a pair (V, E) where V is the set of vertices and $E \subset \binom{V}{r}$ is the set of edges. The degree of vertex v, denoted by d_v, is the number of edges incident to v. If $d_v = 1$, we say v is a leaf vertex. A walk on a hypergraph H is a sequence of vertices and edges: $v_0 e_1 v_1 e_2 \dots v_l$ satisfying that both v_{i-1} and v_i are incident to e_i for $1 \le i \le l$. The vertices v_0 and v_l are called the ends of the walk. The length of a walk is the number of edges on the walk. A walk is called a path if all vertices and edges on the walk are distinct. The walk is closed if $v_l = v_0$. A closed walk is called a cycle if all vertices and edges in the walk are distinct except for the first and last vertices. A hypergraph H is called connected if for any pair of vertices (u, v), there is a path connecting u and v. A hypergraph H is called a hypertree if it is connected and acyclic. A hypergraph H is called simple if every pair of edges intersects at most one vertex. In fact, any non-simple hypergraph contains at least a 2-cycle: $v_1 F_1 v_2 F_2 v_1$, i.e., $v_1, v_2 \in F_1 \cap F_2$. A hypertree is always simple.

The spectral radius $\rho(H)$ of an r-uniform hypergraph H is defined as

$$\rho(H) = r \max_{\substack{\mathbf{x} \in \mathbb{R}^n_{\ge 0} \\ \mathbf{x} \ne 0}} \frac{\sum_{\{i_1, i_2, \dots, i_r\} \in E(H)} x_{i_1} x_{i_2} \cdots x_{i_r}}{\sum_{i=1}^n x_i^r}. \tag{2.1}$$

Here $\mathbb{R}^n_{\ge 0}$ denotes the set of points with nonnegative coordinates in $\mathbb{R}^n$. This is a special case of p-spectral norm for $p = r$. The general p-spectral norm has been considered by various authors [6–8]. The following lemma has been proved in several papers.

Lemma 2.1 ([6–8]) *If G is a connected r-uniform hypergraph, and H is a proper subgraph of G, then $\rho(H) < \rho(G)$.*

In our previous paper [5], we discovered an efficient way to compute the spectral radius $\rho(H)$, in particular when H is a hypertree. The idea is using the method of α-normal labelling (or weighted matrix).

Definition 2.2 ([5]) *A weighted incidence matrix B of a hypergraph H is a $|V| \times |E|$ matrix such that for any vertex v and any edge e, the entry $B(v,e) > 0$ if $v \in e$ and $B(v,e) = 0$ if $v \notin e$.*

Definition 2.3 ([5]) *A hypergraph H is called α-normal if there exists a weighted incidence matrix B satisfying*

(1) $\sum_{e\colon v\in e} B(v,e) = 1$, for any $v \in V(H)$;

(2) $\prod_{v\in e} B(v,e) = \alpha$, for any $e \in E(H)$.

Moreover, the incidence matrix B is called consistent if for any cycle $v_0e_1v_1e_2\cdots v_l$ $(v_l = v_0)$

$$\prod_{i=1}^{l} \frac{B(v_i,e_i)}{B(v_{i-1},e_i)} = 1.$$

In this case, we call H consistently α-normal.

The following important lemma was proved in [5].

Lemma 2.4 ([5, Lemma 3]) *Let H be a connected r-uniform hypergraph. Then the spectral radius of H is $\rho(H)$ if and only if H is consistently α-normal with $\alpha = (\rho(H))^{-r}$.*

Often we need to compare the spectral radius with a particular value.

Definition 2.5 ([5]) *A hypergraph H is called α-subnormal if there exists a weighted incidence matrix B satisfying*

(1) $\sum_{e\colon v\in e} B(v,e) \leq 1$, for any $v \in V(H)$;

(2) $\prod_{v\in e} B(v,e) \geq \alpha$, for any $e \in E(H)$.

Moreover, H is called strictly α-subnormal if it is α-subnormal but not α-normal.

We have the following lemma.

Lemma 2.6 ([5, Lemma 4]) *Let H be an r-uniform hypergraph. If H is α-subnormal, then the spectral radius of H satisfies $\rho(H) \leq \alpha^{-\frac{1}{r}}$. Moreover, if H is strictly α-subnormal then $\rho(H) < \alpha^{-\frac{1}{r}}$.*

Definition 2.7 ([5]) *A hypergraph H is called α-supernormal if there exists a weighted incidence matrix B satisfying*

(1) $\sum_{e\colon v\in e} B(v,e) \geq 1$, for any $v \in V(H)$;

(2) $\prod_{v\in e} B(v,e) \leq \alpha$, for any $e \in E(H)$.

Moreover, the incidence matrix B is called consistent if for any cycle $v_0e_1v_1e_2\cdots v_l$ $(v_l = v_0)$

$$\prod_{i=1}^{l} \frac{B(v_i,e_i)}{B(v_{i-1},e_i)} = 1.$$

In this case, we call H consistently α-supernormal. In addition, H is called strictly α-supernormal if it is α-supernormal but not α-normal.

Lemma 2.8 ([5]) *Let H be an r-uniform hypergraph. If H is strictly and consistently α-supernormal, then the spectral radius of H satisfies $\rho(H) > \alpha^{-\frac{1}{r}}$.*

Because leaf vertices can be assigned the weight 1, we can note that if H is consistently α-normal and H is extended from H', then so is H'. This implies the following corollary.

Corollary 2.9 *For any $r \geq 3$ and $\alpha \in (0,1)$, if H extends H', then $\rho(H) = \alpha^{-1/r}$ (or $\rho(H) < \alpha^{-1/r}$) if and only if $\rho(H') = \alpha^{-1/(r-1)}$ (or $\rho(H') < \alpha^{-1/(r-1)}$, respectively).*

Example 2.10 We give an example regarding the labeling method and consider the following hypergraph H

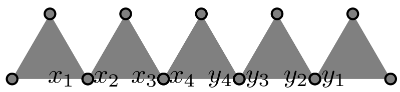


Figure 5 Hypergraph H and the labellings.

Starting from the first edge on the left, for the two leaf vertices can be assigned weight 1, we omit the label, and label x_1 on the left corner of the third vertex. We do like this for the other edges. If we want to check if this hypergraph is β-normal, we can set $x_1 = \beta$, and by Definition 2.3 we set $x_2 = 1 - \beta$, and thus this edge satisfies item 1 and 2 of Definition 2.3. For the same reason, we set $x_3 = \frac{\beta}{x_2} = \frac{\beta}{1-\beta}$, and $x_4 = 1 - x_3 = 1 - \frac{\beta}{1-\beta}$. On the right side, we do this by the same way, and we can get $y_4 = 1 - x_3 = 1 - \frac{\beta}{1-\beta}$. Therefore, by Definition 2.3, if $x_4 \cdot y_4 = \beta$, we can get H is β-normal. If $x_4 \cdot y_4 < \beta$, by Definition 2.7 we can get H is β-supernormal. If $x_4 \cdot y_4 > \beta$, by Definition 2.5, we can get H is β-subnormal.

Definition 2.11 *Given two r-uniform hypergraphs H_1 and H_2, a homomorphism from H_1 to H_2 is a map $f\colon V(H_1) \to V(H_2)$ which preserves the edges. If the natural map from $E(H_1)$ to $E(H_2)$ induced by f is injective, then is called a sub-homomorphism. In this case, we also say H_1 is a sub-homomorphic type of H_2.*

Every subhypergraph is a subhomorphic type. The reverse statement is not true. Consider the following example. Suppose that v_1 and v_2 are two vertices of H which are not contained in any common edge. We can form a new hypergraph H' from H by fusing v_1 and v_2 as one vertex that we call a fat vertex denoted by x. Now we define $f\colon V(H) \to V(H')$ to be the identity map everywhere but v_1 and v_2, and $f(v_1) = f(v_2) = x$. Then f is a sub-homomorphism. The following lemma generalizes Lemma 2.1.

Lemma 2.12 *Suppose H_1 and H_2 are two connected r-uniform hypergraphs. If H_1 is a sub-homomorphic type of H_2, then we have $\rho(H_1) \leq \rho(H_2)$ and the equality holds if and only if H_1 is isomorphic to H_2.*

Proof Let $f\colon V(H_1) \to V(H_2)$ be the sub-homomorphism. Setting $\alpha = (\frac{1}{\rho(H_2)})^r$, by Lemma 2.4, H_2 is consistently α-normal and let B_2 be the incidence matrix. We can define an incidence matrix B_1 of H_1 as follows:

$$B_1(v,e) = B_2(f(v), f(e)) \text{ for any } v \in V(H_1) \text{ and } e \in E(H_1).$$

For any fixed $e \in E(H_1)$, we have

$$\prod_{v\in e} B_1(v,e) = \prod_{v'\in f(e)} B_2(v', f(e)) = \alpha.$$

For any fixed $v \in E(H_1)$, the set $\{e \in E(H_1)\colon v \in e\}$ is a subset of $\{e \in E(H_1)\colon f(v) \in f(e)\}$. This observation implies

$$\sum_{e\colon v\in e} B_1(v,e) \le \sum_{e'\colon f(v)\in e'} B_2(f(v), e') = 1.$$

Therefore, H_1 is α-subnormal. This implies $\rho(H_1) \le \rho(H_2)$. When the inequality holds, $f(H_1) = H_2$ (Otherwise $\rho(H_1) \le \rho(f(H_1)) < \rho(H_2)$, $\rho(H_1) < \rho(H_2)$. Contradiction), and for any $v \in V(H_1)$ and $e \in E(H_1)$, $v \in e$ if and only if $f(v) \in f(e)$. This implies that f must be an injective map (Otherwise, we have $f(v_1) = f(v_2)$, then we can find an edge e_1 containing v_1. Since f is a homomorphism, v_2 is not in e_1, but $f(v_2) = f(v_1) \in f(e_1)$. Contradiction). Hence, f is an isomorphism. □

Often, we need to calculate the limit of the spectral radius of a sequence of hypergraphs. The following lemma is helpful.

Lemma 2.13 *For any fixed* $\beta \in (0, \frac{1}{4})$, *let* $f_\beta(x) = \frac{\beta}{1-x}$ *and* $f_\beta^n(x) = f(f_\beta^{n-1}(x))$ *for* $n \ge 2$.

(1) If $0 < x \le \frac{1-\sqrt{1-4\beta}}{2}$, *then* $f_\beta^n(x)$ *is increasing with respect to* n, $0 < f_\beta^n(x) \le \frac{1-\sqrt{1-4\beta}}{2}$ *and* $\lim_{n\to\infty} f_\beta^n(x) = \frac{1-\sqrt{1-4\beta}}{2}$. *Moreover, when* $x = \frac{1-\sqrt{1-4\beta}}{2}$, $f_\beta^n(x) = \frac{1-\sqrt{1-4\beta}}{2}, \forall n \ge 1$.

(2) If $\frac{1-\sqrt{1-4\beta}}{2} \le x \le \frac{1+\sqrt{1-4\beta}}{2}$, *then* $f_\beta^n(x)$ *is decreasing with respect to* n, *and* $\lim_{n\to\infty} f_\beta^n(x) = \frac{1-\sqrt{1-4\beta}}{2}$.

Proof We first prove item 1. Since $0 < x \le \frac{1-\sqrt{1-4\beta}}{2}$, the function $f_\beta(x) = \frac{\beta}{1-x}$ attains its maximum when $x = \frac{1-\sqrt{1-4\beta}}{2}$. So, $0 < f_\beta(x) \le \frac{1-\sqrt{1-4\beta}}{2}$. Similarly, $f_\beta^2(x) = \frac{\beta}{1-f_\beta(x)}$ attains its maximum when $f_\beta(x) = \frac{1-\sqrt{1-4\beta}}{2}$, so we get $0 < f_\beta^2(x) \le \frac{1-\sqrt{1-4\beta}}{2}$. Again, we get $0 < f_\beta^n(x) \le \frac{1-\sqrt{1-4\beta}}{2}$, for all $n \ge 3$. On the other hand, if $0 < f_\beta^n(x) < \frac{1-\sqrt{1-4\beta}}{2}$, we can easily check that

$$f_\beta^n(x) - f_\beta^{n-1}(x) = \frac{\beta}{1-f_\beta^{n-1}(x)} - f_\beta^{n-1}(x) = \frac{\beta - f_\beta^{n-1}(x) + (f_\beta^{n-1}(x))^2}{1-f_\beta^{n-1}(x)} > 0$$

for all $n \ge 2$. So, $f_\beta^{n-1}(x) < f_\beta^n(x)$ for all $n \ge 2$. Thus, the limit of $0 < f_\beta^n(x)$ exists and we let $\lim_{n\to\infty} f_\beta^n(x) = f_0(x)$. By $f_\beta^n(x) = \frac{\beta}{1-f_\beta^{n-1}(x)}$, we get $f_0(x) = \frac{1-\sqrt{1-4\beta}}{2}$. The proof of item 2 is very similar to the proof of item 1, so we omit the proof here. □

Lemma 2.14 *Let* $f_\beta(x) = \frac{\beta}{1-x}$ *and* $f_\beta^n(x) = f(f_\beta^{n-1}(x))$ *for* $n \ge 2$. *Then for any positive integer* n, *and any real* $\beta \in (0, \frac{1}{4})$, *there exists a unique* $x \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ *such that* $f_\beta^n(x) = 1 - x$.

Proof Consider the set $\mathcal{F}$ of functions f satisfying

(1) f is an increasing continuous function in $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$.

(2) Both $\frac{1-\sqrt{1-4\beta}}{2}$ and $\frac{1+\sqrt{1-4\beta}}{2}$ are fixed points of f.

We claim that for any $f \in \mathcal{F}$ there exists a unique $x \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ such that $f(x) =$

$1-x$. This is because $g(x) := f(x)+x$ is a strictly increasing and continuous function in $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ and

$$g(\frac{1-\sqrt{1-4\beta}}{2}) = 1-\sqrt{1-4\beta} < 1, \text{ and } g(\frac{1+\sqrt{1-4\beta}}{2}) = 1+\sqrt{1-4\beta} > 1.$$

It suffices to show $f_\beta^m(x) \in \mathcal{F}$ for any positive integer m. This can be proved by induction on m. For $m=1$, $f_\beta^1(x) = f_\beta(x) \in \mathcal{F}$ can be easily verified. Now we assume $f_\beta^m \in \mathcal{F}$. Note both f_β and f_β^m map $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ to $(\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ increasingly and continuously to itself. So is their composition, $f_\beta \circ f_\beta^m = f_\beta^{m+1}$. We complete the proof. $\square$

Lemma 2.15 *Let the following graph be denoted by* $F^{(3)}_{m,n,k}$

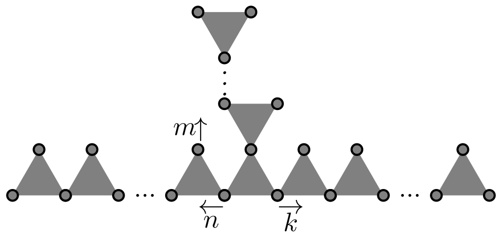


Figure 6 $F^{(3)}_{m,n,k}$

and the spectral radius of $F^{(3)}_{m,n,k}$ *by* $\rho(F^{(3)}_{m,n,k})$. *Then, when* $m,n,k \to \infty$,

$$\lim_{m,n,k\to\infty} \rho(F^{(3)}_{m,n,k}) = \sqrt[3]{2+\sqrt{5}}.$$

Proof We label this graph as follows

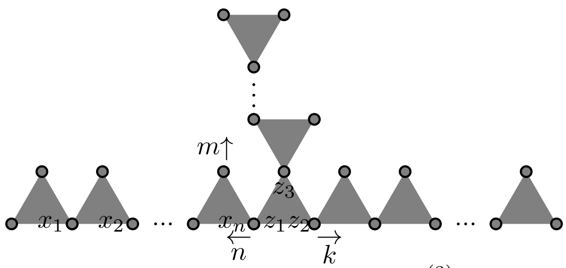


Figure 7 The labellings of $F^{(3)}_{m,n,k}$

Let β be a real number in $(0, \frac{1}{4})$, chosen later. Set $z_1 = 1 - f_\beta^{n-1}(\beta)$, $z_2 = 1 - f_\beta^{k-1}(\beta)$, and $z_3 = 1 - f_\beta^{m-1}(\beta)$. Note that $(\sqrt{5}-2)^3 = \sqrt{5}-2$. By setting $\beta = \sqrt{5}-2$, we get $(\sqrt{5}-2)$-subnormal labeling of $F^{(3)}_{m,n,k}$. Thus, for all m,n,k,

$$\rho(F^{(3)}_{m,n,k}) < \sqrt[3]{2+\sqrt{5}}.$$

Now let $\beta_{m,n,k}$ be the solution of $z_1z_2z_3 = \beta$. We get a $\beta_{n,m,k}$-normal labeling. Thus $\rho(F^{(3)}_{m,n,k}) = \beta_{m,n,k}^{-1/3}$ and $\beta < \sqrt[3]{2+\sqrt{5}}$. By the first item of Lemma 2.13, for a fixed β, note that all z_i's are strictly increasingly approaching to $\frac{1+\sqrt{1-4\beta}}{2}$. We conclude that $\beta_{m,n,k}$ are increasing functions of each m, n, and k. The limit $\lim_{m,n,k\to\infty}\beta_{m,n,k}$ must exist and is the solution of

$$\left(\frac{1+\sqrt{1-4\beta}}{2}\right)^3=\beta.$$

By simple calculus, we get this limit $\beta=\sqrt{5}-2$. By Lemma 2.4, we get

$$\lim_{m,n,k\to\infty}\rho(F^{(3)}_{m,n,k})=\sqrt[3]{2+\sqrt{5}}.\ \square$$

Taking $\rho'_r=\sqrt[r]{2+\sqrt{5}}$, we have the following lemma.

Lemma 2.16 *For $r\geq 3$, let H be an r-uniform hypergraph with spectral radius $\rho(H)\leq\rho'_r$. If H is not simple, then $H=C_2^{(r)}$ (i.e., the hypergraph consists of two edges sharing two common vertices).*

Proof In [5], we have shown that $\rho_r(C_2^{(r)})=\sqrt[r]{4}<\rho'_r$.

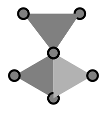
Since H is not simple, H contains two edges F_1 and F_2 sharing s vertices for some $s\geq 2$.

If $s\geq 3$, denote by $C_{s+}^{(r)}$ the subgraph consisting of the two edges F_1,F_2. Define a weighted incidence matrix B of $C_{s+}^{(r)}$ as follows: for any vertex v and edge e (called the other edge e'),

$$B(v,e)=\begin{cases}\frac{1}{2}, & \text{if } v\in e\cap e',\\ 1, & \text{if } v\in e\setminus e',\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to check that when $s\geq 3$ we have $(\frac{1}{2})^s<0.1251<\beta$, so $C_{s+}^{(r)}$ is consistently β-supernormal (The choice of labeling guarantees the labeling is consistent) and thus $\rho(H)\geq\rho(C_{s+}^{(r)})>\rho'_r$. Contradiction.

Thus, F_1 and F_2 can only share 2-common vertices. Since H is connected and $H\neq C_2^{(r)}$, there is a third edge F_3 having non-empty intersection with $F_1\cup F_2$. Since identifying the vertices will not change the sub-homomorphic type, we can only consider the two sub-homomorphic types: $C_{2+}^{(r)}$ and $C'^{(r)}_{2+}$. Here both the hypergraphs $C_{2+}^{(r)}$ and $C'^{(r)}_{2+}$ consist of three edges F_1,F_2,F_3 where $|F_1\cap F_2|=2$ and $|F_3\cap(F_1\cup F_2)|=1$. The difference is that in $C_{2+}^{(r)}$, $F_3\cap(F_1\cup F_2)\in F_1\cap F_2$ while in $C'^{(r)}_{2+}$, $F_3\cap(F_1\cup F_2)\in F_1\Delta F_2$ the symmetric difference of F_1 and F_2. The below are the drawings of $C_{2+}^{(3)}$ and $C'^{(3)}_{2+}$.



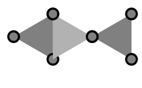


Figure 8 Left: $C_{2+}^{(3)}$. Right: $C'^{(3)}_{2+}$

To draw the contradiction, it is sufficient to show $\rho_r(C_{2+}^{(r)})>\rho'_r$ and $\rho_r(C'^{(r)}_{2+})>\rho'_r$ (This implies $\rho(H)>\rho'_r$ by Lemma 2.12). Observe that $C_{2+}^{(r)}$ is extended from $C_{2+}^{(3)}$ and $C'^{(r)}_{2+}$ is extended from $C'^{(3)}_{2+}$. We only need to show that both $C_{2+}^{(3)}$ and $C'^{(3)}_{2+}$ are consistently strict β-supernormal. We label the two hypergraphs as follows:

Figure 9 The labellings of $C_{2+}^{(3)}$ and $C'^{(3)}_{2+}$.

In $C_{2+}^{(3)}$, we set the labels $y_1=\beta$, $y_2=y_3=\frac{1-\beta}{2}$, and $y_4=y_5=\frac{2\beta}{1-\beta}$. Since $y_4+y_5\approx 1.2361>1$, this is a consistently β-supernormal labelling.

In $C'^{(3)}_{2+}$, we set $x_1=\beta$, $x_2=1-\beta$, $x_3=x_6=\sqrt{\frac{\beta}{1-\beta}}$, and $x_4=x_5=\sqrt{\beta}$. Since $x_3+x_4=x_5+x_6\approx 1.0418>1$, this is a consistently β-supernormal labelling. $\square$

3. Proof of Theorem 1.1

By the lemmas in Section 2, we give the proof of Theorem 1.1. It suffices to consider irreducible hypergraphs. We consider 3 cases for the proof.

Proof Assume that H is an irreducible 3-uniform hypergraph with $\rho(H)\le\sqrt[3]{2+\sqrt{5}}$. We need to show that H has certain forbidden structures. The idea is to show these forbidden subgraphs have some (consistently, if not a hypertree) $(\sqrt{5}-2)$-supernormal labelings. To simplify our notation, we write $\beta=\sqrt{5}-2$ in this proof. By Lemma 2.16, when $r=3$, we only need to consider H is simple.

Case 1 If $\exists\, v\in V(H)$, such that $d_v\ge 5$, then H contains $S_5^{(3)}$ that has been labeled as follows.

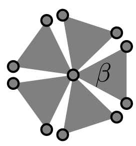


Figure 10 Hypergraph $S_5^{(3)}$

For the status of each edge is the same, we only label one edge. We can check $5\beta\approx 1.1803>1$, so, by Lemmas 2.1 and 2.8, we get $\rho(H)>\rho_3'$. Thus we can assume that every vertex in H has degree at most 4. If $\exists\, v\in V(H)$, such that $d_v=4$, and H contains graph $S_{4+}^{(3)}$ that has been labeled as follows,

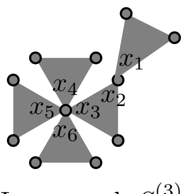


Figure 11 Hypergraph $S_{4+}^{(3)}$ and the labelings

where $x_1=\beta$, $x_2=1-\beta$, $x_3=\frac{\beta}{1-\beta}$, $x_4=x_5=x_6=\beta$. We can check that $x_3+x_4+x_5+x_6\approx 1.0172>1$, so, by Lemmas 2.1 and 2.8, we get $\rho(H)>\rho(S_{4+}^{(3)})>\rho_3'$. Thus, since $\rho(S_4^{(3)})=\rho_3$ and $\rho(S_{4+}^{(3)})>\rho_3'$, so if H is irreducible, we can assume that every vertex in H has degree at most 3.

Case 2 The hypergraph H contains a cycle denoted by $C_n^{(3)}$. Since $\rho(C_n^{(3)}) = \sqrt[3]{4}$ (see [5]), we may assume H contains at least one edge F not on the cycle $C_n^{(3)}$ (But attached to $C_n^{(3)}$). First we prove that F can be only attached to the cycle through a branching edge, not a branching vertex, otherwise, H contains a sub-homomorphic type $C_{n+}^{(3)}$ shown as follows:

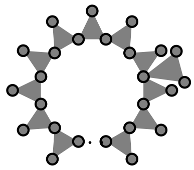


Figure 12 Hypergraph $C_{n+}^{(3)}$

This graph is reducible and can be extended from the following 2-graph $C_{n+}^{(2)}$:

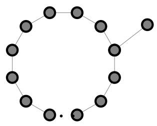


Figure 13 Hypergraph $C_{n+}^{(2)}$

The graph $C_{n+}^{(2)}$ is not in the list of Brouwer and Neumaier [2]. Thus, $\rho(C_{n+}^{(2)}) > \sqrt{2+\sqrt{5}}$. Applying Corollary 2.9, we get $\rho(C_{n+}^{(3)}) > \sqrt[3]{2+\sqrt{5}}$. Contradiction.

Thus, F must be attached to the cycle through a branching edge. Considering that we walk away from the cycle through this edge F, we have the following subcases.

(1) Eventually, the path at F reaches a degree-3 vertex. In this subcase, H contains the following sub-homomorphic type ${C'}_{n+}^{(3)}$:

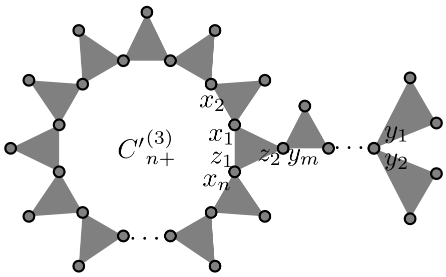


Figure 14 Hypergraph ${C'}_{n+}^{(3)}$ and the labelings

By Lemma 2.14, there exists $x_1 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ satisfying $f_\beta^n(x_1) = 1 - x_1$. Now $x_n = f_\beta^n(x_1) = 1 - x_1$ (This labeling guarantees the consistency). So, $z_1 = 1 - x_n = x_1$. We set $y_1 = y_2 = \beta$, $y_i = f_\beta^{i-2}(2\beta)$ for $3 \le i \le m$. Since $\frac{1-\sqrt{1-4\beta}}{2} \le 2\beta \le \frac{1+\sqrt{1-4\beta}}{2}$, by Lemma 2.13, we get that y_i is decreasing and $\lim_{i\to\infty} y_i = \frac{1-\sqrt{1-4\beta}}{2}$. In particular, $y_m \ge \frac{1-\sqrt{1-4\beta}}{2}$. This implies $z_2 = 1 - y_m \le \frac{1+\sqrt{1-4\beta}}{2}$. Therefore, we have

$$x_1 \cdot z_1 \cdot z_2 < (\frac{1+\sqrt{1-4\beta}}{2})^3 = \beta.$$

Thus, ${C'}^{(3)}_{n+}$ is consistently β-supernormal. So, we have $\rho(H) \geq \rho({C'}^{(3)}_{n+}) > \rho'_3$. Contradiction!

(2) If the path leaving at F reaches a branching edge, in this subcase, H contains the following sub-homomorphic type ${C''}^{(3)}_{n+}$:

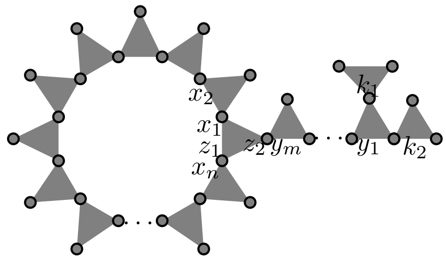


Figure 15 Hypergraph ${C''}^{(3)}_{n+}$ and the labelings

This is similar to the previous subcase. By Lemma 2.14, there exists $x_1 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ satisfying $f^n_\beta(x_1) = 1 - x_1$. Now $x_n = f^n_\beta(x_1) = 1 - x_1$ (The choice of labeling guarantees the labeling is consistent). So, $z_1 = 1 - x_n = x_1$. We set $k_1 = k_2 = \beta$, $y_1 = \frac{\beta}{(1-\beta)^2}$, and $y_i = f^{i-1}_\beta(y_1)$ for $2 \leq i \leq m$. Since $\frac{1-\sqrt{1-4\beta}}{2} \leq \frac{\beta}{(1-\beta)^2} \leq \frac{1+\sqrt{1-4\beta}}{2}$, by Lemma 2.13, we get that y_i is decreasing and the limit goes to $\frac{1-\sqrt{1-4\beta}}{2}$. In particular, $y_m \geq \frac{1-\sqrt{1-4\beta}}{2}$. This implies $z_2 = 1 - y_m \leq \frac{1+\sqrt{1-4\beta}}{2}$. Therefore, we have

$$x_1 \cdot z_1 \cdot z_2 < (\frac{1+\sqrt{1-4\beta}}{2})^3 = \beta.$$

Thus, ${C''}^{(3)}_{n+}$ is consistently β-supernormal. So, we have $\rho(H) \geq \rho({C''}^{(3)}_{n+}) > \rho'_3$. Contradiction.

(3) Eventually, the path leaving at F returns to the cycle. In this subcase, H contains subgraph $\Theta(m_1, m_2, m_3)$, which can be obtained by connecting three pairs of vertices between two branching edges using three paths of lengths m_1, m_2, and m_3, respectively.

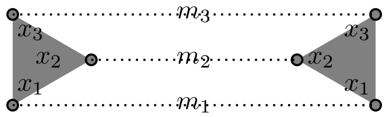


Figure 16 $\Theta(m_1, m_2, m_3)$ and its labeling

By Lemma 2.14, for $i = 1, 2, 3$, there exists $x_i \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ satisfying $f^{m_i}_\beta(x_i) = 1 - x_i$. We label x_1, x_2, and x_3 on the $\Theta(m_1, m_2, m_3)$ and extend these labels on path P_{m_i} naturally. The choice of x_i guarantees the labeling is consistent. Note

$$x_1 x_2 x_3 < (\frac{1+\sqrt{1-4\beta}}{2})^3 = \beta.$$

The labeling is therefore consistently β-supernormal and this implies

$$\rho(H) \geq \rho(\Theta(m_1, m_2, m_3)) > \rho'_3.$$

Contradiction.

(4) This is the remaining subcase: H contains a cycle C with several paths attached to C. So H is a closed quipu as stated in the theorem.

Case 3 We assume that H is a hypertree, and let the following partial hypergraphs (That can be glued together to form a hypergraph) be denoted by $H_1^{(3)}$ and $H_2^{(3)}$ that correspond to the degree-3 vertex and the branching edge structure, respectively.

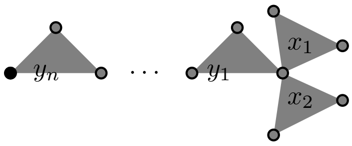


Figure 17 $H_1^{(3)}(n)$ and its labeling

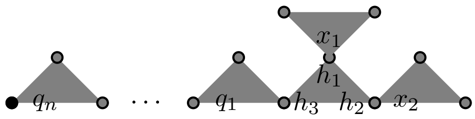


Figure 18 $H_2^{(3)}(n)$ and its labeling

In graph $H_1^{(3)}(n)$, we set $x_1 = x_2 = \beta$, $y_1 = \frac{\beta}{1-2\beta} = f_\beta(2\beta)$, $y_n = f_\beta^n(2\beta)$. Since $2\beta \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$, by Lemma 2.13, we get that $y_n = f_\beta^n > \frac{1-\sqrt{1-4\beta}}{2}$.

In graph $H_2^{(3)}(n)$, we set $x_1 = x_2 = \beta$, $h_1 = h_2 = 1-\beta$, $h_3 = \frac{\beta}{(1-\beta)^2}$. We can check that $h_3 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$. Since $q_n = f_\beta^n(h_3)$, by Lemma 2.13 we get $q_n > \frac{1-\sqrt{1-4\beta}}{2}$.

To show H must be an open quipu as stated in the theorem, we need exclude the following structures. First, suppose that there is a degree-3 vertex with black color in H, and H contains the following subgraph,

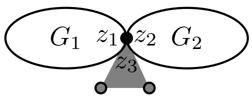


Figure 19 Subgraph 1

where G_1 and G_2 are chosen from $H_1^{(3)}(n)$ and $H_2^{(3)}(n)$ (for some $n \geq 0$) and pieces are united together by gluing the black nodes together. We can get $z_1 = y_n$ or $z_1 = q_n$, and so is z_2. Just as before, we can get

$$z_1 + z_2 + \beta > \frac{1-\sqrt{1-4\beta}}{2} + \frac{1-\sqrt{1-4\beta}}{2} + \beta = 1.$$

This is a supernormal labeling of this subgraph. Thus, $\rho(H) > \rho_3'$. Contradiction.

If H contains one branching edge, whose all three branches are not paths, then H contains the following subgraph.

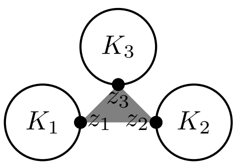


Figure 20 Subgraph 2

Where K_1, K_2 and K_3 are chosen from $H_1^{(3)}(n)$ and $H_2^{(3)}(n)$ (for some $n \geq 0$) and pieces are glued through black nodes. Similar to the previous case, for $i = 1, 2, 3$, by Lemma 2.13, we can get $z_i < \frac{1+\sqrt{1-4\beta}}{2}$. Thus, $z_1 \cdot z_2 \cdot z_3 < (\frac{1+\sqrt{1-4\beta}}{2})^3 = \beta$. This is a supernormal labeling of this subgraph. So, we have $\rho(H) > \rho'_3$. Contradiction. Therefore, H must be an open quipu as stated in the theorem. $\square$

4. Proof of Theorem 1.2

Now, we give the proof of Theorem 1.2. We also only need to consider irreducible hypergraphs.

Proof Let H be an irreducible 4-uniform hypergraph with $\rho(H) \le \rho'_4 = \sqrt[4]{2+\sqrt{5}}$. If H is not simple, then it must be $C_2^{(4)}$ by Lemma 2.16. Now we suppose H is simple.

Case 1 H contains a cycle C. Since H is irreducible, it also has an edge F which contains no leaf vertex. We consider the following two subcases.

(1) The edge F is on the cycle C. The H contains the following sub-isomorphic type:

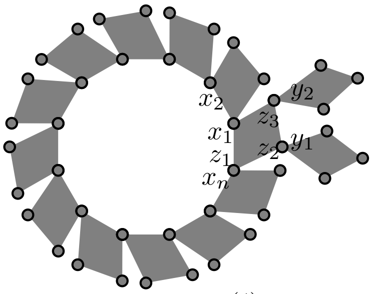


Figure 21 Hypergraph $C_{n+}^{(4)}$ and its labeling

By Lemma 2.14, there exists a $x_1 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ satisfying $f_\beta^n(x_1) = 1 - x_1$. Now $x_n = f_\beta^n(x_1) = 1 - x_1$ (This labeling guarantees the consistency). So, $z_1 = 1 - x_n = x_1$. We set $y_1 = y_2 = \beta$, $z_2 = z_3 = 1 - \beta$, and we can check that $x_1 \cdot z_1 \cdot z_2 \cdot z_3 < (\frac{1+\sqrt{1-4\beta}}{2})^2 \cdot (1-\beta)^2 \approx 0.2229 < \beta$, and thus $C_{n+}^{(4)}$ is β-supernormal. So we have $\rho(H) \ge \rho(C_{n+}^{(4)}) > \rho'_r$.

(2) If F is not on C, there is a path connecting F to C. Thus, H has the following sub-homomorphic type:

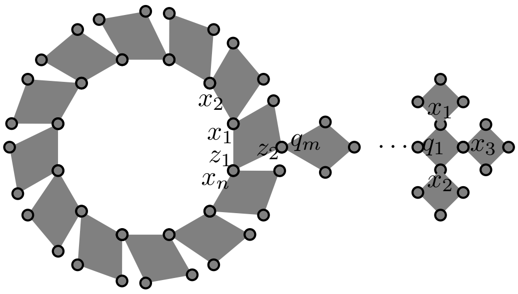


Figure 22 Hypergraph ${C'}_{n+}^{(4)}$ and its labeling

As above, there exists a $x_1 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$ and $z_1 = x_1$. We set $x_1 = x_2 = x_3 = \beta$, $q_1 = \frac{\beta}{(1-\beta)^3}$, and we can check $q_1 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$. We set $q_m = f_\beta^{m-1}(q_1)$, and thus by Lemma 2.13, we get q_m decreases with m, and when $m \to \infty$, we get $q_m > \frac{1-\sqrt{1-4\beta}}{2}$. So

$z_2 = 1 - q_m < \frac{1+\sqrt{1-4\beta}}{2}$. We can check that $x_1 \cdot z_1 \cdot z_2 < (\frac{1+\sqrt{1-4\beta}}{2})^3 = \beta$, and thus ${C'}^{(4)}_{n+}$ is β-supernormal (The labeling guarantees the consistency). So we have $\rho(H) \geq \rho({C'}^{(4)}_{n+}) > \rho'_r$.

Case 2 H is a hypertree but not a 4-dagger. To get the open quipu structures, we need to forbid certain subhypergraphs.

The following partial hypergraphs $H_1^{(4)}(n)$ and $H_2^{(4)}(n,j)$ (for $j = 0,1,2,3$) correspond to the branching vertex and the branching edge structure, respectively.

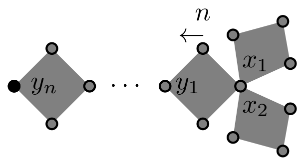


Figure 23 $H_1^{(4)}(n)$ and its labeling

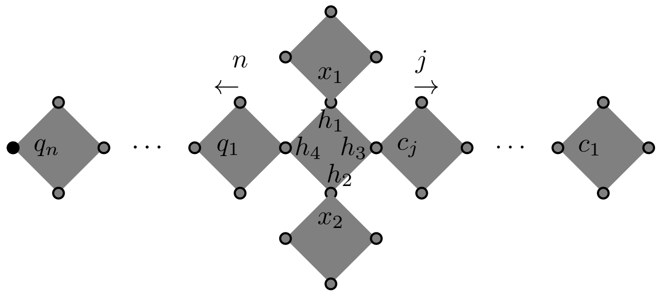


Figure 24 $H_2^{(4)}(n,j)$ for $j = 0,1,2,3$.

Claim (a) Both $H_1^{(4)}(n)$ and $H_2^{(4)}(n,j)$ (for $j = 0,1,2,3$) admit a β-supernormal labeling such that the label at the corner of the black vertex is greater than $\frac{1-\sqrt{1-4\beta}}{2}$.

Proof of Claim (a) We will label the partial graphs so that the β-normal properties hold except at the corner of the black vertex. In graph $H_1^4(n)$, we set $x_1 = x_2 = \beta$, $y_1 = \frac{\beta}{1-2\beta} = f_\beta(2\beta)$, $y_i = f_\beta^i(2\beta)$, $i = 2,3,\ldots,n$. Since $2\beta \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$, by Lemma 2.13, we get that $y_n = f_\beta^n > \frac{1-\sqrt{1-4\beta}}{2}$.

In graph $H_2^4(n,j)$, we set $x_1 = x_2 = c_1 = \beta$, $h_1 = h_2 = 1-\beta$, $c_j = f_\beta^{j-1}(\beta)$. When $j = 0$, we have $h_3 = 1$ and $h_4 = \frac{\beta}{h_1h_2h_3} = \frac{\beta}{(1-\beta)^2}$. When $j = 1$, we have $h_3 = 1-\beta$ and $h_4 = \frac{\beta}{h_1h_2h_3} = \frac{\beta}{(1-\beta)^3}$. When $j = 2$, we have $h_3 = 1 - c_2 = \frac{1-2\beta}{1-\beta}$ and $h_4 = \frac{\beta}{h_1h_2h_3} = \frac{\beta}{(1-\beta)(1-2\beta)}$. When $j = 3$, we set $h_3 = 1 - c_3 = \frac{1-3\beta+\beta^2}{1-2\beta}$ and $h_4 = \frac{\beta}{h_1h_2h_3} = \frac{\beta(1-2\beta)}{(1-3\beta+\beta^2)(1-\beta)^2}$. We can check directly that for all $j = 0,1,2,3$, the value $h_4 \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$. Since $q_n = f_\beta^n(h_4)$, and thus by Lemma 2.13, we get $q_n > \frac{1-\sqrt{1-4\beta}}{2}$.

To show H must be an open quipu as stated in the theorem, we need exclude the following structures.

(1) We first show that all branching vertices and branching edges lie on the same path denoted by P. Otherwise, H contains the following subhypergraph.

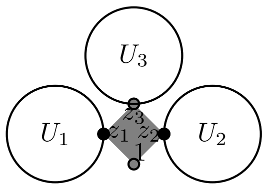


Figure 25 Subhypergraph 1 and its labeling

Where U_1, U_2 and U_3 are chosen from $H_1^4(n)$ (for some $n \geq 0$) and $H_2^4(n, j)$ (for some $n \geq 0$ and $j = 0, 1, 2, 3$) and pieces are glued through black nodes.

From Claim (a), we have

$$z_1 \cdot z_2 \cdot z_3 \cdot 1 < (\frac{1 + \sqrt{1 - 4\beta}}{2})^3 \cdot 1 = \beta.$$

So, this subhypergraph is β-supernormal. This implies $\rho(H) > \rho'_4$.

(2) Now we show that any branch vertex must lie at the end of that path P. Otherwise, H contains the following subhypergraph.

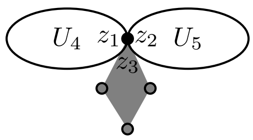


Figure 26 Subhypergraph 2 and its labeling

Where U_4 and U_5 are chosen from $H_1^4(n)$ (for some $n \geq 0$) and $H_2^4(n, j)$ (for some $n \geq 0$ and $j = 0, 1, 2, 3$) and pieces are glued through black nodes. From Claim (a), we have

$$z_1 + z_2 + z_3 > \frac{1 - \sqrt{1 - 4\beta}}{2} + \frac{1 - \sqrt{1 - 4\beta}}{2} + \beta = 1.$$

So, this subhypergraph is β-supernormal. This implies $\rho(H) > \rho'_4$.

(3) Now we show that any branch edge must also lie at the end of that path P. Otherwise, H contains the following subhypergraph.

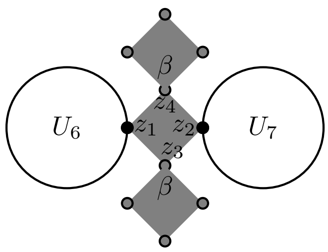


Figure 27 Subhypergraph 3 and its labeling

Where U_6 and U_7 are chosen from $H_1^4(n)$ (for some $n \geq 0$) and $H_2^4(n, j)$ (for some $n \geq 0$ and $j = 0, 1, 2, 3$) and pieces are glued through black nodes. We have

$$z_1 \cdot z_2 \cdot z_3 \cdot z_4 < (\frac{1 + \sqrt{1 - 4\beta}}{2})^2 \cdot (1 - \beta)^2 \approx 0.2229 < \beta.$$

This subhypergraph is β-supernormal. Thus we have $\rho(H) > \rho'_r$. Contradiction.

(4) It remains to show that each 4-branching edge is attached by three paths of length 1, 1, and k ($k = 1, 2, 3$), respectively, if it is not a 4-dagger. Otherwise, it contains one of the following two hypergraphs as a subhypergraph.

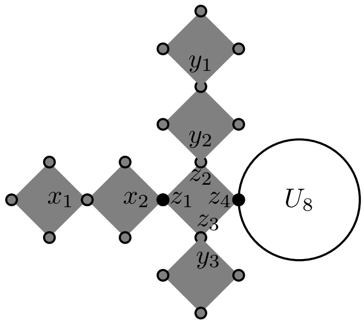


Figure 28 Subhypergraph 4

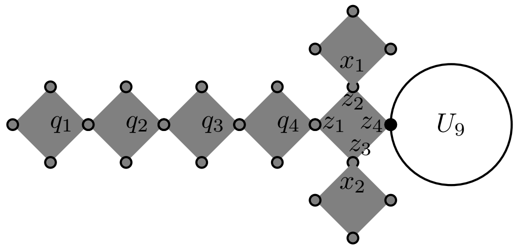


Figure 29 Subhypergraph 5

Where U_8 and U_9 are chosen from $H_1^4(n)$ (for some $n \geq 0$) and $H_2^4(n,j)$ (for some $n \geq 0$ and $j = 0,1,2,3$) and pieces are glued through black nodes.

For the left hypergraph, we set $x_1 = y_1 = y_3 = \beta$, $x_2 = y_2 = \frac{\beta}{1-\beta}$, $z_3 = 1-\beta$, and $z_1 = z_2 = \frac{1-2\beta}{1-\beta}$, and $z_4 < \frac{1+\sqrt{1-4\beta}}{2}$ (from Claim (a)). Thus, the product of labels on the branching edge is

$$z_1 \cdot z_2 \cdot z_3 \cdot z_4 < (\frac{1+\sqrt{1-4\beta}}{2}) \cdot (\frac{1-2\beta}{1-\beta})^2 \cdot (1-\beta) \approx 0.2254 < \beta.$$

For the right hypergraph, we set $q_1 = x_1 = x_2 = \beta$, $q_i = f_\beta^{i-1}(\beta)$ $(i = 2,3,4)$, $z_1 = 1 - q_4 = \frac{1-4\beta+3\beta^2}{1-3\beta+\beta^2}$, $z_2 = z_3 = 1-\beta$, and $z_4 < \frac{1+\sqrt{1-4\beta}}{2}$ (from Claim (a)). Thus, the product of labels on the branching edge is

$$z_1 \cdot z_2 \cdot z_3 \cdot z_4 < \frac{1+\sqrt{1-4\beta}}{2} \cdot \frac{1-4\beta+3\beta^2}{1-3\beta+\beta^2} \cdot (1-\beta)^2 \approx 0.2314 < \beta.$$

Thus the both hypergraphs above are β-supernormal. Thus we have $\rho(H) > \rho'_r$. Contradiction. Therefore, H must be an open quipu as stated in the theorem.

Case 3 H is the 4-dagger $H^{(4)}_{i,j,k,l}$ for $1 \leq i \leq j \leq k \leq l$.

We try to label $H^{(4)}_{i,j,k,l}$ so that the β-normal properties hold except the product of the labels at the branching edge. Not that the product of the labels at the branching edge, denoted by $g(i,j,k,l)$, is given by $g(i,j,k,l) = f_\beta^{i-1}(\beta) f_\beta^{j-1}(\beta) f_\beta^{k-1}(\beta) f_\beta^{l-1}(\beta)$. It is easy to verify that $g(i,j,k,l) < \beta$ for $(i,j,k,l) = (2,2,2,2)$, $(1,2,2,4)$, $(1,2,3,3)$, $(1,1,5,5)$, $(1,1,4,6)$. H cannot contain those 4-daggers as a subhypergraph. Therefore, H must be one of the following hypergraphs $H^{(4)}_{1,2,2,2}$, $H^{(4)}_{1,2,2,3}$, $H^{(4)}_{1,1,4,4}$, $H^{(4)}_{1,1,4,5}$, and $H^{(4)}_{1,1,k,l}$ $(1 \leq k \leq 3$, and $k \leq l)$. It is also easy to verify that those 4-daggers are β-subnormal. So this is a complete list of 4-daggers with $\rho(H) < \rho'_4$. $\square$

5. Proof of Theorem 1.3

Now, we give the proof of Theorem 1.3.

Proof Let the *edge-star* $S_r^{(r)}$ be the r-uniform hypergraph consisting of $r+1$ edges: $F_0 = \{v_1, v_2, \ldots, v_r\}$, $F_1, \ldots, F_r$, where each $F_i \cap F_0 = \{v_i\}$ for $1 \leq i \leq r$, and $F_i \cap F_j = \emptyset$ for $1 \leq i \leq j \leq r$. (See the picture of $S_5^{(5)}$ at Theorem 1.3.)

We first show that $\rho_r(S_r^{(r)}) > \rho'_r$ for $r \geq 6$. This can be done by assigning $B(v_i, F_i) = \beta$ and $B(v_i, F_0) = 1-\beta$, for $1 \leq i \leq r$. Note that the product of labels on F_0 is $(1-\beta)^r < \beta$ for all $r \geq 6$. Thus, $S_r^{(r)}$ is β-supernormal. If there is an irreducible r-uniform hypergraph H with $\rho(H) \leq \rho'_r$ for $r \geq 6$, then H contains a sub-homomorphic type $S_r^{(r)}$. By Lemma 2.12, we have $\rho(H) \geq \rho(S_r^{(r)}) > \rho'_r$. Contradiction.

The same argument shows that $S_5^{(5)}$ is β-subnormal. Let H be an irreducible 5-uniform hypergraph H with $\rho(H) \leq \rho'_5$. If H is not $S_5^{(5)}$, H contains one of the following sub-homomorphic types ${S'}_5^{(5)}$ and $S_{5+}^{(5)}$.

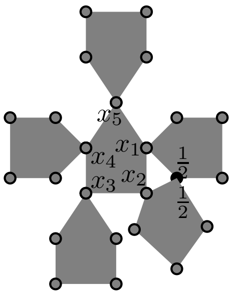


Figure 30 ${S'}_5^{(5)}$ and its labeling

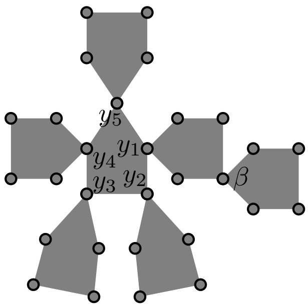


Figure 31 $S_{5+}^{(5)}$ and its labeling

For ${S'}_5^{(5)}$, we can label the corner of the only black vertex not on the branching edge by $\frac{1}{2}$, and set $x_1 = x_2 = 1-2\beta$, $x_3 = x_4 = x_5 = 1-\beta$. We can check that the product of labels on the branching edge is

$$x_1x_2x_3x_4x_5 = (1-2\beta)^2(1-\beta)^3 \approx 0.1242 < \beta.$$

For $S_{5+}^{(5)}$, we can set $y_1 = 1 - f_\beta(\beta) = \frac{1-2\beta}{1-\beta}$, $y_2 = y_3 = y_4 = y_5 = 1-\beta$. We can check that the product of labels on the branching edge is

$$y_1y_2y_3y_4y_5 = (1-2\beta)(1-\beta)^4 \approx 0.1798 < \beta.$$

Thus, both ${S'}_5^{(5)}$ and $S_{5+}^{(5)}$ are consistently β-supernormal. This implies that $\rho(H) > \rho'_5$, a contradiction. Thus H must be the five edge-star. $\square$

6. Constructing open quipus and closed quipus with $\rho(H) \leq \sqrt[r]{2+\sqrt{5}}$

In this section, we give a description of the connected r-uniform hypergraphs with spectral radius at most $\sqrt[r]{2+\sqrt{5}}$: they are extended from the irreducible ones listed in Theorems 1.1-1.3 and the 2-graphs listed by Cvetković et al [1] and Brouwer-Neumaier [2]. This is not a complete description for $r \geq 3$, but rather a coarse description. The scenario is similar to the results of Woo and Neumaier on the graphs with spectral radius at most $\frac{3}{2}\sqrt{2}$ (see [4]). Our method is very different from the linear algebra method used by Woo and Neumaier. In fact, it is possible to make the proof of Woo-Neumaier's result simply using our new method but we will omit it here.

In the rest of this section, we will construct many examples with $\rho(H) \le \sqrt[r]{2+\sqrt{5}}$. This shows that the descriptions in Theorems 1.1–1.3 are somewhat tight.

The 4-daggers are completely classified so no construction is needed. We only need to construct closed 3- quipus, open 3-quipus and open 4-quipus first. The idea is to present some partial hypergraphs, which can be glued together to form a hypergraph with $\rho(H) \le \sqrt[r]{2+\sqrt{5}}$. A partial r-uniform hypergraph is an r-uniform hypergraph together with (one or two) designated vertex/vertices. A partial hypergraph H is called α-subnormal if there exists a weighted incidence matrix B satisfying

(1) $\prod_{v\in e} B(v,e) \ge \alpha$, for any $e \in E(H)$;

(2) $\sum_{e:\, v\in e} B(v,e) \le \frac{1}{2}$, for any designated vertex v;

(3) $\sum_{e:\, v\in e} B(v,e) \le 1$, for any non-designated vertex.

Lemma 6.1 *Consider the following partial hypergraphs* $G_1^{(3)}(m,k_1,k_2)$, $G_2^{(2)}(m,k)$, *and* $G_3^{(4)}(t,k)$ *(with designated vertices colored in black). We have*

(1) For any $m \ge 1$, *there exists a* k_0 *such that for any* $k_1, k_2 \ge k_0$, $G_1^{(3)}(m,k_1,k_2)$ *is* $(\sqrt{5}-2)$*-subnormal.*

(2) For any $m \ge 1$, *there exists a* k_0 *such that for any* $k \ge k_0$, $G_2^{(2)}(m,k)$ *is* $(\sqrt{5}-2)$*-subnormal.*

(3) For any $t = 1,2,3$, *there exists a* k_t *such that for any* $k \ge k_t$, $G_3^{(4)}(t,k)$ *is* $(\sqrt{5}-2)$*-subnormal.*

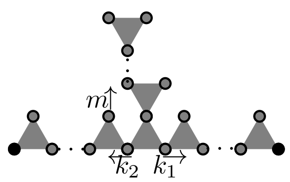


Figure 32 $G_1^{(3)}(m,k_1,k_2)$

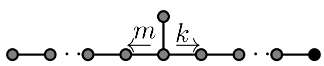


Figure 33 $G_2^{(2)}(m,k)$

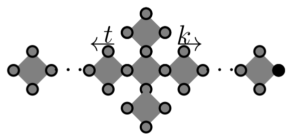


Figure 34 $G_3^{(4)}(t,k)$ (for $t = 1,2,3$)

Proof We label the corner of the designated vertices by $\frac{1}{2}$ and the corner of other leaf-vertices by 1. We try to maintain the properties that the product of all labels in one edge is β where $\beta = \sqrt{5}-2$ and the sum of all labels at one vertex is 1 except at the branching vertex or at the branching edge. We get the labels of the three partial graphs as follows

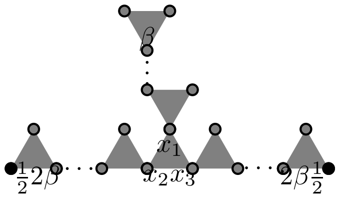


Figure 35 $G_1^{(3)}(m, k_1, k_2)$

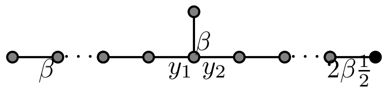


Figure 36 $G_2^{(2)}(m, k)$

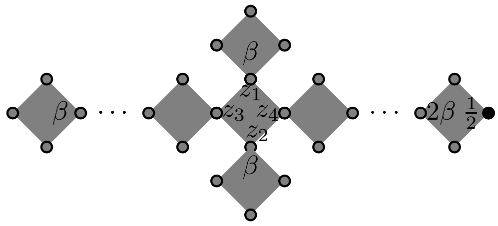


Figure 37 $G_3^{(4)}(t, k)$

Now we consider the first partial hypergraph $G_1^{(3)}$. Using the function f_β, we have $x_1 = 1 - f_\beta^{m-1}(\beta)$, $x_2 = 1 - f_\beta^{k_1-1}(2\beta)$, and $x_3 = 1 - f_\beta^{k_2-1}(2\beta)$. The product of the labels on the central branching edges, denoted by $g(m, k_1, k_2)$, satisfies

$$g(m, k_1, k_2) = x_1 x_2 x_3 = (1 - f_\beta^{m-1}(\beta))(1 - f_\beta^{k_1-1}(2\beta))(1 - f_\beta^{k_2-1}(2\beta)).$$

By Lemma 2.13, $1 - f_\beta^{m-1}(\beta) > \frac{1+\sqrt{1-4\beta}}{2}$, and $\lim_{k_1\to\infty}(1 - f_\beta^{k_1-1}(2\beta)) = \lim_{k_2\to\infty}(1 - f_\beta^{k_2-1}(2\beta)) = \frac{1+\sqrt{1-4\beta}}{2}$ since $2\beta \in (\frac{1-\sqrt{1-4\beta}}{2}, \frac{1+\sqrt{1-4\beta}}{2})$. Thus,

$$\lim_{k_1,k_2\to\infty} g(m, k_1, k_2) > (\frac{1+\sqrt{1-4\beta}}{2})^3 = \beta.$$

There exists a k_0 such that for $k_1, k_2 \geq k_0$, $g(m, k_1, k_2) > \beta$. i.e., $G_1^{(3)}$ is β-subnormal.

A similar argument works for the graph $G_2^{(2)}$. We have $y_1 = f_\beta^{m-1}(\beta)$ and $y_2 = f_\beta^{k-1}(2\beta)$. The sum of the labels at the branching vertex is

$$\beta + y_1 + y_2 = \beta + f_\beta^{m-1}(\beta) + f_\beta^{k-1}(2\beta).$$

Note that the limit of this sum as k goes to infinity satisfies

$$\lim_{k\to\infty}(\beta + f_\beta^{m-1}(\beta) + f_\beta^{k-1}(2\beta)) < \beta + \frac{1-\sqrt{1-4\beta}}{2} + \frac{1-\sqrt{1-4\beta}}{2} = 1.$$

Thus, there exists a $k_0 = k_0(m)$ such that for any $k \geq k_0$, we get $y_1 + y_2 + \beta < 1$. So $G_2^{(2)}$ is β-subnormal.

In graph $G_3^{(4)}(t, k)$, we have $z_1 = z_2 = 1 - \beta$, $z_3 = 1 - f_\beta^{t-1}(\beta)$, $z_4 = 1 - f_\beta^{k-1}(2\beta)$. The product of the labels at the branching edge is

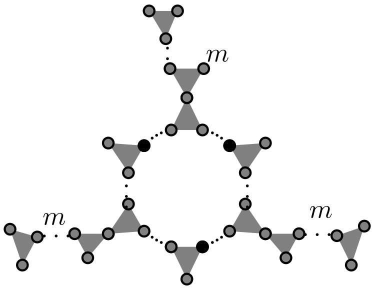
$$z_1 z_2 z_3 z_4 = (1-\beta)^2 (1 - f_\beta^{t-1}(\beta))(1 - f_\beta^{k-1}(2\beta)).$$

For each $t = 1, 2, 3$, it is easy to check

$$(1-\beta)^2 (1 - f_\beta^{t-1}(\beta)) \frac{1+\sqrt{1-4\beta}}{2} < \beta.$$

There exists a k_t such that for any $k \geq k_t$, $G_3^{(4)}$ is β-subnormal. $\Box$

Extension also works for partial hypergraphs: add one vertex to each edge while keep the designated vertices. Observe that if a partial hypergraph H is α-subnormal then so is the extension of H. For any $r \geq 4$, we can extend $G_1^{(3)}(m, k_1, k_2)$ to $G_1^{(r)}(m, k_1, k_2)$, $G_2^{(2)}(m, k)$ to $G_2^{(r)}(m, k)$, and $G_3^{(4)}(t, k)$ to $G_3^{(r)}(t, k)$, glue $G_1^{(r)}$, $G_2^{(r)}$ and $G_3^{(r)}$ together via the designated vertices, and get a new graph H that is still $(\sqrt{5}-2)$-subnormal. We can get many examples of H with $\rho(H) < \rho'_r$ as follows:



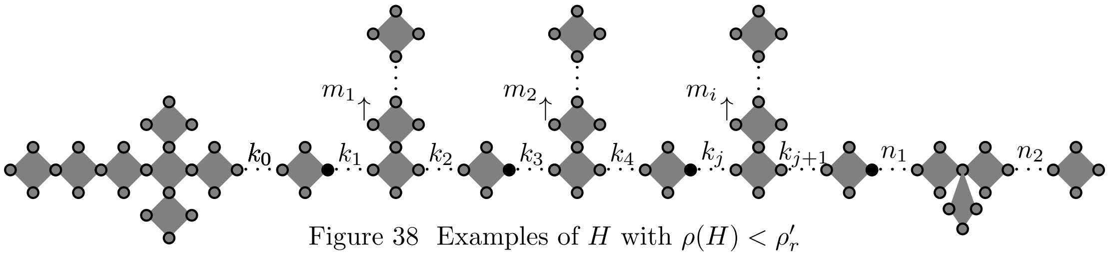


Figure 38 Examples of H with $\rho(H) < \rho'_r$

Acknowledgements We thank the referees for their time and comments.

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