Journal of Mathematical Research with Applications
Mar., 2019, Vol. 39, No. 2, pp. 141–152
DOI:10.3770/j.issn:2095-2651.2019.02.003
Http://jmre.dlut.edu.cn

Single-Valued Neutrosophic Lie Algebras

Muhammad AKRAM[1,*], Hina GULZAR[1], Kar Ping SHUM[2]
1. *Department of Mathematics, University of the Punjab, New Campus, Lahore*-54590*, Pakistan;*
2. *Institute of Mathematics, Yunnan University, Yunan* 650091*, P. R. China*

Abstract A single-valued neutrosophic (SVN) set is a powerful general formal framework that generalizes the concept of fuzzy set and intuitionistic fuzzy set. In SVN set, indeterminacy is quantified explicitly, and truth membership, indeterminacy membership, and falsity membership are independent. In this paper, we apply the notion of SVN sets to Lie algebras. We develop the concepts of SVN Lie subalgebras and SVN Lie ideals. We describe some interesting results of SVN Lie ideals.

Keywords Single-valued neutrosophic sets; Lie ideals; Lie algebras

MR(2010) Subject Classification 17B99; 03E72; 20N25

1. Introduction

The concept of Lie groups was first introduced by Sophus Lie in nineteenth century through his studies in geometry and integration methods for differential equations. Lie algebras were also discovered by him when he attempted to classify certain smooth subgroups of a general linear group. The importance of Lie algebras in mathematics and physics has become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. It is noted that Lie theory has applications not only in mathematics and physics but also in diverse fields such as continuum mechanics, cosmology and life sciences. Lie algebra has been used by electrical engineers, mainly in the mobile robot control [1]. Lie algebra has also been used to solve the problems of computer vision.

Fuzzy structures are associated with theoretical soft computing, especially Lie algebras and their different classifications, have numerous applications to the spectroscopy of molecules, atoms and nuclei. One of the key concepts in the application of Lie algebraic method in physics is that of spectrum generating algebras and their associated dynamic symmetries. The major advancements in the fascinating world of fuzzy sets started with the work of renowned scientist Zadeh [2] with new directions and ideas. Smarandache [3] and Wang et al. [4] defined SVN sets as a generalization of fuzzy sets [2] and intuitionistic fuzzy sets [5]. Algebraic structures have a vital place with vast applications in various disciplines. Neutrosophic set theory has been applied to algebraic structures [6]. Fuzzification of Lie algebras has been discussed in [7–15]. In

Received July 31, 2018; Accepted December 10, 2018
* Corresponding author
E-mail address: m.akram@pucit.edu.pk (Muhammad AKRAM); kpshum@ynu.edu.cn (Kar Ping SHUM)

this paper, we apply the notion of SVN sets to Lie algebras. We develop the concepts of SVN Lie subalgebras and SVN Lie ideals. We describe some interesting results of SVN Lie ideals.

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper. A Lie algebra is a vector space $\mathscr{L}$ over a field $\mathbb{F}$ (equal to $\mathbb{R}$ or $\mathbb{C}$) on which $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in \mathscr{L}$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathscr{L}$ (Jacobi identity).

Throughout this paper, $\mathscr{L}$ is a Lie algebra and $\mathbb{F}$ is a field. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is anti commutative, i.e., $[x, y] = -[y, x]$. A subspace H of $\mathscr{L}$ closed under $[\cdot, \cdot]$ will be called a Lie subalgebra.

A fuzzy set $\mu : \mathscr{L} \to [0, 1]$ is called a fuzzy Lie ideal [7] of $\mathscr{L}$ if

(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geq \mu(x)$,

(c) $\mu([x, y]) \geq \mu(x)$

hold for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$.

Definition 2.1([3, 4]) *Let X be a space of points (objects). A single-valued neutrosophic set (SVN set) N on a non-empty set X is characterized by a truth membership function $T_N : X \to [0, 1]$, indeterminacy membership function $I_N : X \to [0, 1]$ and a falsity membership function $F_N : X \to [0, 1]$. Thus, $N = \{< x, T_N(x), I_N(x), F_N(x) > | x \in X\}$.*

3. Single-valued neutrosophic Lie algebras

We define here single-valued neutrosophic Lie subalgebras and single-valued neutrosophic Lie ideal.

Definition 3.1 *An SVN set $N = (T_N, I_N, F_N)$ on Lie algebra $\mathscr{L}$ is called an SVN Lie subalgebra if the following conditions are satisfied:*

(1) $T_N(x + y) \geq \min(T_N(x), T_N(y))$, $I_N(x + y) \geq \min(I_N(x), I_N(y))$ and $F_N(x + y) \leq \max(F_N(x), F_N(y))$,

(2) $T_N(\alpha x) \geq T_N(x)$, $I_N(\alpha x) \geq I_N(x)$ and $F_N(\alpha x) \leq F_N(x)$,

(3) $T_N([x, y]) \geq \min\{T_N(x), T_N(y)\}$, $I_N([x, y]) \geq \min\{I_N(x), I_N(y)\}$ and $F_N([x, y]) \leq \max\{F_N(x), F_N(y)\}$

for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$.

Definition 3.2 *An SVN set $N = (T_N, I_N, F_N)$ on $\mathscr{L}$ is called an SVN Lie ideal if it satisfies the conditions (1), (2) and the following additional condition:*

(4) $T_N([x,y]) \geq T_N(x)$, $I_N([x,y]) \geq I_N(x)$ *and* $F_N([x,y]) \leq F_N(x)$ *for all* $x, y \in \mathscr{L}$.

From (2) it follows that:

(5) $T_N(0) \geq T_N(x)$, $I_N(0) \geq I_N(x)$, $F_N(0) \leq F_N(x)$,

(6) $T_N(-x) \geq T_N(x)$, $I_N(-x) \geq I_N(x)$, $F_N(-x) \leq F_N(x)$.

Proposition 3.3 *Every SVN Lie ideal is an SVN Lie subalgebra.*

We note here that the converse of Proposition 3.3 does not hold in general as it can be seen in the following example.

Example 3.4 Consider $\mathbb{F} = \mathbb{R}$. Let $\mathscr{L} = \Re^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors which forms a Lie algebra and define

$$\Re^3 \times \Re^3 \rightarrow \Re^3$$

$$[x,y] \rightarrow x \times y,$$

where $\times$ is the usual cross product. We define an SVN set $N = (T_N, I_N, F_N) : \Re^3 \rightarrow [0,1] \times [0,1] \times [0,1]$ by

$$T_N(x,y,z) = \begin{cases} 1, & \text{if } x = y = z = 0, \\ 0.5, & \text{if } x \neq 0, y = z = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$I_N(x,y,z) = \begin{cases} 1, & \text{if } x = y = z = 0, \\ 0.5, & \text{if } x \neq 0, y = z = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$F_N(x,y,z) = \begin{cases} 0, & \text{if } x = y = z = 0, \\ 0.3, & \text{if } x \neq 0, y = z = 0, \\ 1, & \text{otherwise}. \end{cases}$$

Then $N = (T_N, I_N, F_N)$ is an SVN Lie subalgebra of $\mathscr{L}$ but $N = (T_N, I_N, F_N)$ is not an SVN Lie ideal of $\mathscr{L}$ since

$$T_N([(1,0,0)\ (1,1,1)]) = T_N(0,-1,1) = 0,$$

$$I_N([(1,0,0)\ (1,1,1)]) = I_N(0,-1,1) = 0,$$

$$F_N([(1,0,0)\ (1,1,1)]) = F_N(0,-1,1) = 1,$$

$$T_N(1,0,0) = 0.5, I_N(1,0,0) = 0.5, F_N(1,0,0) = 0.3.$$

That is,

$$T_N([(1,0,0)\ (1,1,1)]) \ngeq T_N(1,0,0),$$

$$I_N([(1,0,0)\ (1,1,1)]) \ngeq I_N(1,0,0),$$

$$F_N([(1,0,0)\ (1,1,1)]) \nleq F_N(1,0,0).$$

Proposition 3.5 *If N is an SVN Lie ideal of $\mathscr{L}$, then*

(1) $T_N(0) \geq T_N(x)$, $I_N(0) \geq I_N(x)$, $F_N(0) \leq F_N(x)$,
(2) $T_N([x,y]) \geq \max\{T_N(x), T_N(y)\}$,
(3) $I_N([x,y]) \geq \max\{I_N(x), I_N(y)\}$,
(4) $F_N([x,y]) \leq \min\{F_N(x), F_N(y)\}$,
(5) $T_N([x,y]) = T_N(-[y,x]) = T_N([y,x])$,
(6) $I_N([x,y]) = I_N(-[y,x]) = I_N([y,x])$,
(7) $F_N([x,y]) = F_N(-[y,x]) = F_N([y,x])$

for all $x, y \in \mathscr{L}$.

Proof The proof follows from Definition 3.2. □

Proposition 3.6 *If $\{N_i \mid i \in J\}$ is a family of SVN Lie ideals of $\mathscr{L}$, then $\bigcap N_i = (\bigwedge T_{N_i}, \bigvee F_{N_i})$ is an SVN Lie ideal of $\mathscr{L}$, where*

$$\bigwedge T_{N_i}(x) = \inf\{T_{N_i}(x) \mid i \in J,\, x \in \mathscr{L}\},$$

$$\bigwedge I_{N_i}(x) = \inf\{I_{N_i}(x) \mid i \in J,\, x \in \mathscr{L}\},$$

$$\bigvee F_{N_i}(x) = \sup\{F_{N_i}(x) \mid i \in J,\, x \in \mathscr{L}\}.$$

Proof The proof follows from Definition 3.2. □

Definition 3.7 *Let $N = (T_N, I_N, F_N)$ be an SVN set in a Lie algebra $\mathscr{L}$ and let $(\alpha, \beta, \gamma) \in [0,1] \times [0,1] \times [0,1]$ with $\alpha + \beta + \gamma \leq 3$. Then level subset of N is defined as:*

$$N^{(\alpha,\beta,\gamma)} = \{x \in \mathscr{L} \mid T_N(x) \geq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\},$$

are called (α, β, γ)-level subsets of SVN set N. The set of all $(\alpha, \beta, \gamma) \in \mathrm{Im}(T_N) \times \mathrm{Im}(I_N) \times \mathrm{Im}(I_N)$ such that $\alpha + \beta + \gamma \leq 3$ is known as image of $N = (T_N, I_N, F_N)$. Note that

$$N^{(\alpha,\beta,\gamma)} = \{x \in \mathscr{L} \mid T_N(x) \geq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\},$$
$$N^{(\alpha,\beta,\gamma)} = \{x \in \mathscr{L} \mid T_N(x) \geq \alpha\} \cap \{x \in \mathscr{L} \mid I_N(x) \geq \beta\} \cap \{x \in \mathscr{L} \mid F_N(x) \leq \gamma\},$$
$$N^{(\alpha,\beta,\gamma)} = \cup(T_N, \alpha) \cap \cup^{'}(I_N, \beta) \cap L(F_N, \gamma).$$

Theorem 3.8 *An SVN set $N = (T_N, I_N, F_N)$ of $\mathscr{L}$ is an SVN Lie ideal of $\mathscr{L}$ if and only if $N^{(\alpha,\beta,\gamma)}$ is a Lie ideal of $\mathscr{L}$ for every $(\alpha, \beta, \gamma) \in \mathrm{Im}(T_N) \times \mathrm{Im}(I_N) \times \mathrm{Im}(F_N)$ with $\alpha + \beta + \gamma \leq 3$.*

Proposition 3.9 *Let $N = (T_N, I_N, F_N)$ be an SVN Lie ideal of $\mathscr{L}$ and (r_1, s_1, t_1), $(r_2, s_2, t_2) \in \mathrm{Im}(T) \times \mathrm{Im}(I) \times \mathrm{Im}(F)$ with $r_i + s_i + t_i \leq 3$ for $i = 1, 2$. Then $\mathscr{L}_N^{(r_1,s_1,t_1)} = \mathscr{L}_N^{(r_2,s_2,t_2)}$ if and only if $(r_1, s_1, t_1) = (r_2, s_2, t_2)$.*

Theorem 3.10 *Let $P_0 \subset P_1 \subset P_2 \subset \ldots P_n = L$ be a chain of Lie ideals of a Lie algebra $\mathscr{L}$. Then there exists an SVN Lie ideal T_N of $\mathscr{L}$ for which level subsets $U(T_N, r)$, $\acute{U}(I_N, s)$ and $L(T_N, t)$ coincide with this chain.*

Proof Let $\{s_k \mid k = 0, 1, \ldots, n\}$ and $\{t_k \mid k = 0, 1, \ldots, n\}$ be finite decreasing and increasing sequences in $[0,1]$ such that $s_i + t_i \leq 1$, for $i = 0, 1, \ldots, n$. Let $N = (T_N, I_N, F_N)$ be an intuitionistic fuzzy set in $\mathscr{L}$ defined by $T_N(P_0) = r_0$, $I_N(P_0) = s_0$, $F_N(P_0) = t_0$, $T_N(P_k \setminus P_{k-1}) = r_k$, $I_N(P_k \setminus P_{k-1}) = s_k$ and $F_N(P_k \setminus P_{k-1}) = t_k$ for $0 < k \leq n$. Let $x, y \in \mathscr{L}$. If $x, y \in P_k \setminus P_{k-1}$, then $x + y, \alpha x, [x, y] \in P_k$ and

$$T_N(x+y) \geq r_k = \min\{T_N(x), T_N(y)\},$$

$$I_N(x+y) \geq s_k = \min\{T_N(x), T_N(y)\},$$

$$F_N(x+y) \leq t_k = \max\{F_N(x)), F_N(y)\},$$

$$T_N(\alpha x) \geq r_k = T_N(x),\ I_N(\alpha x) \geq s_k = I_N(x),\ F_N(\alpha x) \leq t_k = F_N(x),$$

$$T_N([x,y]) \geq r_k = T_N(x),\ I_N([x,y]) \geq s_k = I_N(x),\ F_N([x,y]) \leq t_k = F_N(x).$$

For $i > j$, if $x \in P_i \setminus P_{i-1}$ and $y \in P_j \setminus P_{j-1}$, then $T_N(x) = r_i = T_N(y)$, $I_N(x) = s_i = I_N(y)$, $F_N(x) = t_j = F_N(y)$ and $x + y, \alpha x, [x, y] \in P_i$. Thus

$$T_N(x+y) \geq r_i = \min\{T_N(x), T_N(y)\},$$

$$I_N(x+y) \geq s_i = \min\{I_N(x), I_N(y)\},$$

$$F_N(x+y) \leq t_j = \max\{F_N(x)), F_N(y)\},$$

$$T_N(\alpha x) \geq r_i = T_N(x),\ I_N(\alpha x) \geq s_i = I_N(x),\ F_N(\alpha x) \leq t_j = F_N(x),$$

$$T_N([x,y]) \geq r_i = T_N(x),\ I_N([x,y]) \geq s_i = I_N(x),\ F_N([x,y]) \leq t_j = F_N(x).$$

Thus, we conclude that $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of a Lie algebra $\mathscr{L}$ and all its nonempty level subsets are Lie ideals.

Since $\mathrm{Im}(T_N) = \{r_0, r_1, \ldots, r_n\}$, $\mathrm{Im}(I_N) = \{s_0, s_1, \ldots, s_n\}$, $\mathrm{Im}(F_N) = \{t_0, t_1, \ldots, t_n\}$, level subsets of N form chains:

$$U(T_N, r_0) \subset U(T_N, r_1) \subset \cdots \subset U(T_N, r_n) = L,$$

$$\acute{U}(I_N, s_0) \subset \acute{U}(I_N, s_1) \subset \cdots \subset \acute{U}(I_N, s_n) = L,$$

$$L(F_N, t_0) \subset L(F_N, t_1) \subset \cdots \subset L(F_N, t_n) = L,$$

respectively. Indeed,

$$U(T_N, r_0) = \{x \in \mathscr{L} \mid T_N(x) \geq r_0\} = P_0,$$

$$\acute{U}(I_N, s_0) = \{x \in \mathscr{L} \mid I_N(x) \geq s_0\} = P_0,$$

$$L(F_N, t_0) = \{x \in \mathscr{L} \mid F_N(x) \leq t_0\} = P_0.$$

We now prove that

$$U(T_N, r_k) = U(I_N, s_k) = P_k = L(F_N, t_k) \ \text{ for } 0 < k \leq n.$$

Clearly, $P_k \subseteq U(T_k, r_k)$, $P_k \subseteq U(I_k, s_k)$ and $P_k \subseteq L(F_N, t_k)$.

If $x \in U(T_N, r_k)$, then $T_N(x) \geq r_k$ and so $x \notin P_i$ for $i > k$. Hence

$$T_N(x) \in \{r_0, r_1, \ldots, r_k\},$$

which implies $x \in P_i$ for some $i \leq k$. Since $P_i \subseteq P_k$, it follows that $x \in P_k$. Consequently, $U(T_N, r_k) = P_k$ for some $0 < k \leq n$. If $x \in \acute{U}(I_N, s_k)$, then $I_N(x) \geq s_k$ and so $x \notin P_i$ for $i > k$. Hence

$$I_N(x) \in \{s_0, s_1, \ldots, s_k\},$$

which implies $x \in P_i$ for some $i \leq k$. Since $P_i \subseteq P_k$, it follows that $x \in P_k$. Consequently, $\acute{U}(I_N, s_k) = P_k$ for some $0 < k \leq n$. Now if $x \in \mathscr{L}(F_N, t_k)$, then $F_N(x) \leq t_k$ and so $x \notin P_i$ for $j \leq k$. Thus

$$F_N(x) \in \{t_0, t_1, \ldots, t_k\},$$

which implies $x \in P_j$ for some $j \leq k$. Since $P_j \subseteq P_k$, it follows that $x \in P_k$. Consequently, $L(F_N, t_k) = P_k$ for some $0 < k \leq n$. This completes the proof. □

Theorem 3.11 *If $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of a Lie algebra $\mathscr{L}$, then*

$$T_N(x) = \sup\{r \in [0,1] \mid x \in U(T_N, r)\},$$

$$I_N(x) = \sup\{s \in [0,1] \mid x \in \acute{U}(I_N, s)\},$$

$$F_N(x) = \inf\{t \in [0,1] \mid x \in \mathscr{L}(F_N, t)\}$$

for every $x \in \mathscr{L}$.

Proof The proof follows from Definition 3.2. □

Definition 3.12 *Let f be a map from a set $\mathscr{L}_1$ to a set $\mathscr{L}_2$. If $N = (T_N, I_N, F_N)$ and $M = (T_M, I_M, F_M)$ are SVN sets in $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, then the preimage of M under f, denoted by $f^{-1}(M)$, is an SVN set defined by*

$$f^{-1}(M) = (f^{-1}(T_M), f^{-1}(F_M)).$$

Theorem 3.13 *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be an onto homomorphism of Lie algebras. If $M = (T_M, I_M, F_M)$ is an SVN Lie ideal of $\mathscr{L}_2$, then the preimage $f^{-1}(M) = (f^{-1}(T_M), f^{-1}(F_M))$ of M under f is an SVN Lie ideal of $\mathscr{L}_1$.*

Proof The proof follows from Definitions 3.2 and 3.12. □

Theorem 3.14 *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be an epimorphism of Lie algebras. If $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}_2$, then $f^{-1}(N^c) = (f^{-1}(N))^c$.*

Proof The proof follows from Definitions 3.2 and 3.12. □

Theorem 3.15 *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be an epimorphism of Lie algebras. If $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}_2$ and $M = (T_M, I_M, F_M)$ is the preimage of $N = (T_N, F_N)$ under f. Then $M = (T_M, I_M, F_M)$ is an SVN Lie ideal of $\mathscr{L}_1$.*

Proof The proof follows from Definitions 3.2 and 3.12. □

Definition 3.16 *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two Lie algebras and f be a mapping of $\mathscr{L}_1$ into $\mathscr{L}_2$. If*

$N = (T_N, I_N, F_N)$ is an SVN set of $\mathscr{L}_1$, then the image of $N = (T_N, I_N, F_N)$ under f is the SVN set in $\mathscr{L}_2$ defined by

$$f(T_N)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} T_N(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

$$f(I_N)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} I_N(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

$$f(F_N)(y) = \begin{cases} \inf_{x\in f^{-1}(y)} F_N(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise} \end{cases}$$

for each $y \in \mathscr{L}_2$.

Theorem 3.17 *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be an epimorphism of Lie algebras. If $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}_1$, then $f(N)$ is an SVN Lie ideal of $\mathscr{L}_2$.*

Proof The proof follows from Definitions 3.2 and 3.16. □

Definition 3.18 *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a homomorphism of Lie algebras. For any SVN set $N = (T_N, I_N, F_N)$ in a Lie algebra $\mathscr{L}_2$, we define an SVN set $N^f = (T_N^f, F_N^f)$ in $\mathscr{L}_2$ by*

$$T_N^f(x) = T_N(f(x)),\ I_N^f(x) = I_N(f(x)),\ F_N^f(x) = F_N(f(x))$$

for all $x \in \mathscr{L}_1$. Clearly, $N^f(x_1) = N^f(x_2) = A(x)$ for all $x_1, x_2 \in f^{-1}(x)$.

Lemma 3.19 *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a homomorphism of Lie algebras. If $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}_2$, then $N^f = (T_N^f, I_N^f, F_N^f)$ is an SVN Lie ideal of $\mathscr{L}_1$.*

Proof Let $x, y \in \mathscr{L}_1$ and $\alpha \in \mathbb{F}$. Then

$$T_N^f(x+y) = T_N(f(x+y)) = T_N(f(x)+f(y)) \geq \min\{T_N(f(x)), T_N(f(y))\} = \min\{T_N^f(x), T_N^f(y)\},$$

$$I_N^f(x+y) = I_N(f(x+y)) = I_N(f(x)+f(y)) \geq \min\{I_N(f(x)), I_N(f(y))\} = \min\{I_N^f(x), I_N^f(y)\},$$

$$F_N^f(x+y) = F_N(f(x+y)) = F_N(f(x)+f(y)) \leq \max\{F_N(f(x)), F_N(f(y))\} = \max\{F_N^f(x), F_N^f(y)\},$$

$$T_N^f(\alpha x) = T_N(f(\alpha x)) = T_N(\alpha f(x)) \geq T_N(f(x)) = \alpha_N^f(x),$$

$$I_N^f(\alpha x) = I_N(f(\alpha x)) = I_N(\alpha f(x)) \geq I_N(f(x)) = \alpha_N^f(x),$$

$$F_N^f(\alpha x) = F_N(\alpha f(x)) \leq F_N(f(x)) = F_N^f(x).$$

Similarly,

$$T_N^f([x,y]) = T_N(f([x,y])) = T_N([f(x), f(y)]) \geq T_N(f(x)) = T_N^f(x),$$

$$I_N^f([x,y]) = I_N(f([x,y])) = I_N([f(x), f(y)]) \geq I_N(f(x)) = I_N^f(x),$$

$$F_N^f([x,y]) = F_N([f(x,y)]) = F_N([f(x), f(y)]) \leq F_N(f(x)) = F_N^f(x).$$

This proves that $N^f = (T_N^f, I_N^f, F_N^f)$ is an SVN Lie ideal of $\mathscr{L}_1$. □

We now characterize the SVN Lie ideals of Lie algebras.

Theorem 3.20 *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be an epimorphism of Lie algebras. Then $N^f = (T_N^f, I_N^f, F_N^f)$ is an SVN Lie ideal of $\mathscr{L}_1$ if and only if $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}_2$.*

Proof The sufficiency follows from Lemma 3.19. In proving the necessity, since f is a surjective mapping, for any $x, y \in \mathscr{L}_2$ there are $x_1, y_1 \in \mathscr{L}_1$ such that $x = f(x_1)$, $y = f(y_1)$. Thus $T_N(x) = T_N^f(x_1)$, $T_N(y) = T_N^f(y_1)$, $I_N(x) = I_N^f(x_1)$, $I_N(y) = I_N^f(y_1)$, $F_N(x) = F_N^f(x_1)$, $F_N(y) = F_N^f(y_1)$, whence

$$\begin{aligned}
T_N(x+y) &= T_N(f(x_1)+f(y_1)) = T_N(f(x_1+y_1)) = T_N^f(x_1+y_1)\\
&\geq \min\{T_N^f(x_1), T_N^f(y_1)\} = \min\{T_N(x), T_N(y)\},\\
I_N(x+y) &= I_N(f(x_1)+f(y_1)) = I_N(f(x_1+y_1)) = I_N^f(x_1+y_1)\\
&\geq \min\{I_N^f(x_1), I_N^f(y_1)\} = \min\{I_N(x), I_N(y)\},\\
F_N(x+y) &= F_N(f(x_1)+f(y_1)) = F_N(f(x_1+y_1)) = F_N^f(x_1+y_1)\\
&\leq \max\{F_N^f(x_1), F_N^f(y_1)\} = \max\{F_N(x), F_N(y)\},
\end{aligned}$$

$$T_N(\alpha x) = T_N(\alpha f(x_1)) = T_N(f(\alpha x_1)) = T_N^f(\alpha x_1) \geq T_N^f(x_1) = T_N(x),$$

$$I_N(\alpha x) = i_N(\alpha f(x_1)) = I_N(f(\alpha x_1)) = I_N^f(\alpha x_1) \geq I_N^f(x_1) = I_N(x),$$

$$F_N(\alpha x) = F_N(\alpha f(x_1)) = F_N(f(\alpha x_1)) = F_N^f(\alpha x_1) \leq F_N^f(x_1) = F_N(x).$$

Similarly,

$$\begin{aligned}
T_N([x,y]) &= T_N([f(x_1), f(y_1)]) = T_N(f([x_1, y_1])) = T_N^f([x_1, y_1])\\
&\geq T_N^f(x_1) = T_N(x),\\
I_N([x,y]) &= I_N([f(x_1), f(y_1)]) = I_N(f([x_1, y_1])) = I_N^f([x_1, y_1])\\
&\geq I_N^f(x_1) = I_N(x),\\
F_N([x,y]) &= F_N([f(x_1), f(y_1)]) = F_N(f([x_1, y_1])) = F_N^f([x_1, y_1])\\
&\leq F_N^f(x_1) = F_N(x).
\end{aligned}$$

This shows that $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}_2$. □

Definition 3.21 *Let $N = (T_N, I_N, F_N)$ be an SVN Lie ideal in $\mathscr{L}$. Define inductively a sequence of SVN Lie ideals in $\mathscr{L}$ by*

$$N^0 = N,\ N^1 = [N^0, N^0],\ N^2 = [N^1, N^1], \ldots, N^n = [N^{n-1}, N^{n-1}].$$

N^n is called the nth derived SVN Lie ideal of $\mathscr{L}$. A series

$$N^0 \supseteq N^1 \supseteq N^2 \supseteq \cdots \supseteq N^n \supseteq \cdots$$

is called derived series of an SVN Lie ideal N in $\mathscr{L}$.

Definition 3.22 *An SVN Lie ideal N in $\mathscr{L}$ is called a solvable SVN Lie ideal, if there exists a positive integer n such that*

$$N^0 \supseteq N^1 \supseteq N^2 \supseteq \cdots \supseteq N^n = (0, 0, 0).$$

Theorem 3.23 *Homomorphic images of solvable SVN Lie ideals are solvable SVN Lie ideals.*

Proof Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a homomorphism of Lie algebras. Suppose that $N = (T_N, I_N, F_N)$ is an SVN Lie ideal in $\mathscr{L}_1$. We prove by induction on n that $f(N^n) \supseteq [f(N)]^n$, where n is any positive integer. First we claim that $f([N, A]) \supseteq [f(N), f(N)]$. Let $y \in \mathscr{L}_2$. Then

$$\begin{aligned}
f(\ll T_N, T_N \gg)(y) =& \sup\{\ll T_N, T_N \gg (x) \mid f(x) = y\} \\
=& \sup\{\sup\{\min(T_N(a), T_N(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x, f(x) = y\}\} \\
=& \sup\{\min(T_N(a), T_N(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x, f(x) = y\} \\
=& \sup\{\min(T_N(a), T_N(b)) \mid a, b \in \mathscr{L}_1, [f(a), f(b)] = x\} \\
=& \sup\{\min(T_N(a), T_N(b)) \mid a, b \in \mathscr{L}_1, f(a) = u, f(b) = v, [u, v] = y\} \\
\geq& \sup\{\min(\sup_{a \in f^{-1}(u)} T_N(a), \sup_{b \in f^{-1}(v)} T_N(b)) \mid [u, v] = y\} \\
=& \sup\{\min(f(T_N)(u), f(T_N)(v)) \mid [u, v] = y\} \\
=& \ll f(T_N), f(T_N) \gg (y), \\
f(\ll I_N, I_N \gg)(y) =& \sup\{\ll I_N, I_N \gg (x) \mid f(x) = y\} \\
=& \sup\{\sup\{\min(I_N(a), I_N(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x, f(x) = y\}\} \\
=& \sup\{\min(I_N(a), I_N(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x, f(x) = y\} \\
=& \sup\{\min(I_N(a), I_N(b)) \mid a, b \in \mathscr{L}_1, [f(a), f(b)] = x\} \\
=& \sup\{\min(I_N(a), I_N(b)) \mid a, b \in \mathscr{L}_1, f(a) = u, f(b) = v, [u, v] = y\} \\
\geq& \sup\{\min(\sup_{a \in f^{-1}(u)} I_N(a), \sup_{b \in f^{-1}(v)} I_N(b)) \mid [u, v] = y\} \\
=& \sup\{\min(f(I_N)(u), f(I_N)(v)) \mid [u, v] = y\} \\
=& \ll f(I_N), f(I_N) \gg (y), \\
f(\ll F_N, F_N \gg)(y) =& \inf\{\ll F_N, F_N \gg)(x) \mid f(x) = y\} \\
=& \inf\{\inf\{\max(F_N(a), F_N(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x, f(x) = y\}\} \\
=& \inf\{\max(F_N(a), F_N(b)) \mid a, b \in \mathscr{L}_1, [a, b] = x, f(x) = y\} \\
=& \inf\{\max(F_N(a), F_N(b)) \mid a, b \in \mathscr{L}_1, [f(a), f(b)] = x\} \\
=& \inf\{\max(F_N(a), F_N(b)) \mid a, b \in \mathscr{L}_1, f(a) = u, f(b) = v, [u, v] = y\} \\
\leq& \inf\{\max(\inf_{a \in f^{-1}(u)} F_N(a), \inf_{b \in f^{-1}(v)} F_N(b)) \mid [u, v] = y\} \\
=& \inf\{\max(f(F_N)(u), f(F_N)(v)) \mid [u, v] = y\} \\
=& \ll f(F_N), f(F_N) \gg (y).
\end{aligned}$$

Thus

$$f([N, N]) \supseteq f(\ll A, A \gg) \supseteq \ll f(N), f(N) \gg = [f(N), f(N)].$$

Now for $n > 1$, we get

$$f(N^n) = f([N^{n-1}, N^{n-1}]) \supseteq [f(N^{n-1}), f(N^{n-1})]$$

$$\supseteq [(f(N))^{n-1}, (f(N))^{n-1}] = (f(N))^n.$$

This completes the proof. □

Definition 3.24 *Let $N = (T_N, I_N, F_N)$ be an SVN Lie ideal in $\mathscr{L}$. We define inductively a sequence of SVN Lie ideals in $\mathscr{L}$ by*

$$N_0 = N,\ N_1 = [N, N_0],\ N_2 = [N, N_1], \ldots, N_n = [N, N_{n-1}].$$

A series

$$N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$$

is called descending central series of an intuitionistic fuzzy Lie ideal N in $\mathscr{L}$. An SVN Lie ideal N in $\mathscr{L}$ is called a nilpotent SVN Lie ideal, if there exists a positive integer n such that

$$N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n = (0, 0, 0).$$

Theorem 3.25 *Homomorphic image of a nilpotent SVN Lie ideal is a nilpotent SVN Lie ideal.*

Proof Straightforward. □

Theorem 3.26 *Let J be a Lie ideal of a Lie algebra $\mathscr{L}$. If $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}$, then the SVN set $\overline{N} = (\overline{T}_N, \overline{F}_N)$ of $\mathscr{L}/J$ defined by*

$$\overline{T}_N(a + J) = \sup_{x \in J} T_N(a + x),$$

$$\overline{I}_N(a + J) = \sup_{x \in J} I_N(a + x),$$

$$\overline{F}_N(a + J) = \inf_{x \in J} F_N(a + x)$$

is an SVN Lie ideal of the quotient Lie algebra $\mathscr{L}/J$ of $\mathscr{L}$ with respect to J.

Proof Clearly, $\overline{N}$ is well-defined. Let $x + J,\ y + J \in \mathscr{L}/J$. Then

$$\begin{aligned}
\overline{I}_N((x + J) + (y + J)) =& \overline{I}_N((x + y) + J) = \sup_{z \in J} T_N((x + y) + z) \\
=& \sup_{z = s + t \in J} T_N((x + y) + (s + t)) \\
\geq& \sup_{s,\ t \in J} \min\{T_N(x + s), T_N(y + t)\} \\
=& \min\{\sup_{s \in J} T_N(x + s), \sup_{t \in J} T_N(y + t)\} \\
=& \min\{\overline{I}_N(x + J), \overline{I}_N(y + J)\}, \\
\overline{I}_N(\alpha(x + J)) =& \overline{I}_N(\alpha x + J) = \sup_{z \in J} T(\alpha x + z) \geq \sup_{z \in J} T(x + z) = \overline{I}_N(x + J), \\
\overline{I}_N([x + J, y + J]) =& \overline{I}_N([x, y] + J) = \sup_{z \in J} T_N([x, y] + z) \geq \sup_{z \in J} T_N(x + z) = \overline{I}_N(x + J).
\end{aligned}$$

Thus $\overline{I}_N$ is an SVN Lie ideal of $\mathscr{L}/J$. In a similar way we can verify that $\overline{T}_N$ and $\overline{F}_N$ are SVN Lie ideals of $\mathscr{L}/J$. Hence $\overline{N} = (\overline{T}_N, \overline{I}_N, \overline{F}_N)$ is an SVN Lie ideal of $\mathscr{L}/J$. □

Theorem 3.27 *Let J be a Lie ideal of a Lie algebra $\mathscr{L}$. Then there is a one-to-one correspondence between the set of SVN Lie ideals $N = (T_N, I_N, F_N)$ of $\mathscr{L}$ such that $N(0) = A(s)$ for all $s \in J$ and the set of all SVN Lie ideals $\overline{N} = (\overline{T}_N, \overline{I}_N, \overline{F}_N)$ of $\mathscr{L}/J$.*

Proof Let $N = (T_N, I_N, F_N)$ be an SVN Lie ideal of $\mathscr{L}$. Using Theorem 3.26, we prove that $\overline{T}_N$ and $\overline{F}_N$ defined by

$$\overline{T}_N(a+J) = \sup_{x\in J} T_N(a+x),$$

$$\overline{I}_N(a+J) = \sup_{x\in J} I_N(a+x),$$

$$\overline{F}_N(a+J) = \inf_{x\in J} F_N(a+x),$$

are SVN Lie ideals of $\mathscr{L}/J$. Since $T_N(0) = T_N(s)$, $F_N(0) = F_N(s)$ for all $s \in J$,

$$T_N(a+s) \geq \min(T_N(a), T_N(s)) = T_N(a),$$

$$I_N(a+s) \geq \min(I_N(a), I_N(s)) = I_N(a),$$

$$F_N(a+s) \leq \max(F_N(a), F_N(s)) = F_N(a).$$

Again,

$$T_N(a) = T_N(a+s-s) \geq \min(T_N(a+s), T_N(s)) = T_N(a+s),$$
$$I_N(a) = I_N(a+s-s) \geq \min(I_N(a+s), I_N(s)) = I_N(a+s),$$
$$F_N(a) = F_N(a+s-s) \leq \max(F_N(a+s), F_N(s)) = F_N(a+s).$$

Thus $N(a+s) = N(a)$ for all $s \in J$. Hence the correspondence $N \mapsto \overline{N}$ is one-to-one. Let $\overline{N}$ be an SVN Lie ideal of $\mathscr{L}/J$ and define an SVN set $N = (T_N, I_N, F_N)$ in $\mathscr{L}$ by $T_N(a) = \overline{T}_N(a+J)$, $I_N(a) = \overline{I}_N(a+J)$, $F_N(a) = \overline{F}_N(a+J)$ for all $a \in J$. For x, $y \in \mathscr{L}$, we have

$$\begin{aligned}
T_N(x+y) =& \overline{T}_N((x+y)+J) = \overline{T}_N((x+J)+(y+J)) \\
\geq& \min\{\overline{T}_N(x+J), \overline{T}_N(y+J)\} \\
=& \min\{T_N(x), T_N(y)\}, \\
T_N(\alpha x) =& \overline{T}_N(\alpha x+J) \geq \overline{T}_N(x+J) = T_N(x), \\
T_N([x,y]) =& \overline{T}_N([x,y]+J) = \overline{T}_N([x+J, y+J]) \\
\geq& \overline{T}_N(x+J) = T_N(x).
\end{aligned}$$

Thus T_N is an SVN Lie ideal of $\mathscr{L}$. In a similar way we can verify that I_N and F_N are SVN Lie ideal of $\mathscr{L}$. Hence $N = (T_N, I_N, F_N)$ is an SVN Lie ideal of $\mathscr{L}$. Note that $T_N(z) = \overline{T}_N(z+J) = T_N(J)$, $I_N(z) = \overline{I}_N(z+J) = I_N(J)$, $F_N(z) = \overline{F}_N(z+J) = F_N(J)$ for all $z \in J$, which shows that $N(z) = N(0)$ for all $z \in J$. This completes the proof. □

References

[1] P. COELHO, U. NUNES. *Lie algebra application to mobile robot control: a tutorial.* Robotica, 2003, **21**: 483–493.

[2] L. A. ZADEH. *Fuzzy sets*. Information and Control, 1965, **8**: 338–353.

[3] F. SMARANDACHE. *Neutrosophy Neutrosophic Probability, Set, and Logic*. Amer Res Press, Rehoboth, USA, 1998.

[4] Haibin WANG, F. SMARANDACHE, Yanqing ZHANG, et al. *Single valued neutrosophic sets*. Multispace and Multistruct, 2010, **4**: 410–413.

[5] K. ATANASSOV. *Intuitionistic fuzzy sets*. Fuzzy Sets and Systems, 1986, **20**(1): 87–96.

[6] R. A. BORZOOEI, H. FARAHANI, M. MONIRI. *Neutrosophic deductive filters on BL-algebras*. J. Intell. Fuzzy Systems, 2014, **26**(6): 2993–3004.

[7] M. AKRAM. *Anti fuzzy Lie ideals of Lie algebras*. Quasigroups Related Systems, 2006, **14**(2): 123–132.

[8] M. AKRAM. *Intuitionistic fuzzy Lie ideals of Lie algebras*. J. Fuzzy Math., 2008, **16**(4): 991–1008.

[9] M. AKRAM. *Generalized fuzzy Lie subalgebras*. J. Gen. Lie Theory Appl., 2008, **2**(4): 261–268.

[10] M. AKRAM. *Fuzzy Lie ideals of Lie algebras with interval-valued membership function*. Quasigroups Related Systems, 2008, **16**(1): 1–12.

[11] M. AKRAM. *Intuitionistic (S, T)-fuzzy Lie ideals of Lie algebras*. Quasigroups Related Systems, 2007, **15**(2): 201–218.

[12] M. AKRAM, K. P. SHUM. *Intuitionistic fuzzy Lie algebras*. Southeast Asian Bull. Math., 2007, **31**(5): 843–855.

[13] C. G. KIM, D. S. LEE. *Fuzzy Lie ideals and fuzzy Lie subalgebras*. Fuzzy Sets and Systems, 1998, **94**: 101–107.

[14] S. E. YEHIA. *Fuzzy ideals and fuzzy subalgebras of Lie algebras*. Fuzzy Sets and Systems, 1996, **80**(2): 237–244.

[15] S. E. YEHIA. *The adjoint representation of fuzzy Lie algebras*. Fuzzy Sets and Systems, 2001, **119**(3): 409–417.

[16] M. AKRAM, H. GULZAR, K. P. SHUM. *Certain notions of single-valued neutrosophic K-algebras*. Italian Journal of Pure and Applied Mathematics, 2019, **41**(1), In press.

[17] M. AKRAM, H. GULZAR, F. SMARANDACHE, et al. *Certain notions of neutrosophic topological K-algebras*. Mathematics, 2018, **6**(11): 234.

[18] M. AKRAM, H. GULZAR, F. SMARANDACHE, et al. *Application of neutrosophic soft sets to K-algebras*. Axioms, 2018, **7**(4): 83.

[19] J. E. HUMPHREYS. *Introduction to Lie Algebras and Representation Theory*. Springer, New York, 1972.

[20] M. KONDO, W. A. DUDEK. *On the transfer principle in fuzzy theory*. Mathware Soft Comput., 2005, **12**(1): 41–55.