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Global Weak Solution to the Chemotaxis-Fluid System

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Abstract We investigate the existence of the global weak solution to the coupled Chemotaxis-fluid system

$$\begin{cases} n_t + u\cdot\nabla n = \triangle n - \nabla\cdot(n\nabla c) + rn - \mu n^2, & x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \triangle c + n - c, & x\in\Omega, t>0,\\ u_t + \nabla P = \triangle u + n\nabla\phi + g(x,t), & x\in\Omega, t>0,\\ \nabla\cdot u = 0, & x\in\Omega, t>0, \end{cases}$$

in a bounded smooth domain $\Omega\subset\mathbb{R}^2$. Here, $r\ge 0$ and $\mu>0$ are given constants, $\nabla\phi\in L^\infty(\Omega)$ and $g\in L^2((0,T);L^2_\sigma(\Omega))$ are prescribed functions. We obtain the local existence of the weak solution of the system by using the Schauder fixed point theorem. Furthermore, we study the regularity estimate of this system. Utilizing the regularity estimates, we obtain that the coupled Chemotaxis-fluid system with the initial-boundary value problem possesses a global weak solution.

Keywords Chemotaxis-fluid system; logistic source; global solution

MR(2010) Subject Classification 35A01; 35B45; 35D30

1. Introduction

Chemotaxis is a biological process of directed movement of cells in response to the gradient of a chemical signal substance [1, 2]. One of the first chemotaxis system was developed by Keller and Segel to describe the aggregation of bacteria [3, 4]. A standard Keller-Segel system assumes that the cells are attracted by higher concentration of the signal chemical and produce the chemical themselves. The system assumes that there is no interaction between the cells and their surrounding. But experiments show that the activities of the cells in the liquid are actually interacting with the liquid. In the paper [5], the authors consider the chemotaxis-fluid interaction on the basis of experimental observation. Another important example is that sperms are attracted to chemicals during the proliferation of organisms, which is the direct evidence of chemotaxis-fluid interaction [1, 6–8].

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In this paper, we consider the Chemotaxis-Stokes system with a logistic source

$$\begin{cases} n_t + u \cdot \nabla n = \triangle n - \nabla \cdot (n \nabla c) + rn - \mu n^2, & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \triangle c + n - c, & x \in \Omega,\ t > 0, \\ u_t + \nabla p = \triangle u + n \nabla \phi + g(x,t), & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $r \geq 0$, $\mu > 0$, $\nabla \phi \in L^\infty(\Omega)$, and $g \in L^2((0,T); L^2_\sigma(\Omega))$. Here $n = n(x,t)$ and $c = c(x,t)$ denote the density of the cell and the signal concentration, respectively. It is obvious that $n \geq 0$, $c \geq 0$. $u = u(x,t)$ and $p = p(x,t)$ represent the fluid velocity and the associated pressure, respectively. The gravitational potential ϕ is a given function. The model (1.1) presupposes that the motion of the fluid might be controlled by a given external force g.

There are some results which deal with chemotaxis-fluid interaction. In [9], the authors discussed the boundedness of a global classical solution in a three dimensional Chemotaxis-fluid system. Espejo and Suzuki proved the existence of a global weak solution when $r = 0$, $g = 0$ in system (1.1) [10]. Tao and Winkler have given the global existence and large time behavior of classical solution to a more complex Keller-Segel-Navier-Stokes system [11]. Zheng proved that the following Chemotaxis-Stokes system with rotational flux and a logistic source

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (nS(x,n,c) \nabla c) + an - bn^2, & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \triangle c + n - c, & x \in \Omega,\ t > 0, \\ u_t + \nabla P = \triangle u + n \nabla \phi + g(x,t), & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

admits a bounded weak solution through the Moser-type iteration [12].

In this paper, we consider this system (1.1) along with the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \text{ and } u = 0 \text{ for } x \in \partial\Omega \text{ and } t > 0, \tag{1.3}$$

and the initial conditions

$$n(x,0) = n_0(x),\ c(x,0) = c_0(x),\ u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.4}$$

We assume that the initial data satisfy

$$\begin{cases} n_0 \in L^\infty(\Omega), & n_0 \geq 0 \text{ in } \Omega, \\ c_0 \in H^1(\Omega), & c_0 \geq 0 \text{ in } \Omega, \\ u_0 \in H^1_0(\Omega) \cap D_2^{\frac{1}{2},2}(\Omega), & \nabla \cdot u_0 = 0, \end{cases} \tag{1.5}$$

where

$$D_q^{\alpha,s}(\Omega) = \{v \in L^q_\sigma(\Omega) : \|v\|_{D_q^{\alpha,s}(\Omega)} = \|v\|_{L^q(\Omega)} + \Big(\int_0^\infty \|t^{1-\alpha} A_q e^{-tA_q} v\|^s_{L^q(\Omega)} \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{s}} < \infty\}$$

regarding the Helmholtz projection $P_q : L^q \to L^q_\sigma$ and the Stokes operator $A^q = -P_q \triangle$ (see [13]).

We assume that

$$\nabla \phi \in L^\infty(\Omega) \text{ and } \triangle \phi = 0, \tag{1.6}$$

and

$$g \in L^2((0,T); L^2_\sigma(\Omega)). \tag{1.7}$$

This paper is organized as follows. In Section 2, we provide some preliminary lemmas which play a crucial role in the following proofs. We give a formal definition of weak solution for problem (1.1), (1.3) and (1.4) and present the main result. In Section 3, we prove the local existence of the weak solution by using the Schauder fixed point theorem and show the corresponding estimate to conclude global existence.

2. Preliminaries and main result

The main theorem of stokes equation that will be used in this paper is the next result.

Lemma 2.1 ([14]) *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary and $1 < p, m < \infty$ and $0 < T \le \infty$. Then for every $f \in L^p((0,T); L^m_\sigma(\Omega))$ and $u_0 \in D_q^{1-\frac{1}{p},p}(\Omega)$, there exists a unique solution $(u, \nabla p)$ of the nonstationary Stokes system*

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle u + \nabla p = f, & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega, \\ u(x,0) = u_0, & \text{in } \Omega, \end{cases} \tag{2.1}$$

such that

$$u \in L^p(0,T_0; W^{2,q}(\Omega)) \text{ for all } T_0 \le T \text{ and } T_0 < \infty,$$
$$\frac{\partial u}{\partial t}, \quad \nabla p \in L^p(0,T; L^m(\Omega)),$$
$$\int_0^T \|\frac{\partial u}{\partial t}\|^p_{L^m(\Omega)} \mathrm{d}t + \int_0^T \|\nabla^2 u(t)\|^p_{L^m(\Omega)} \mathrm{d}t + \int_0^T \|\nabla p\|^p_{L^m(\Omega)} \mathrm{d}t$$
$$\le C\Big(\int_0^T \|f(t)\|^p_{L^m(\Omega)} \mathrm{d}t + \|u_0\|_{D_m^{1-\frac{1}{p},p}(\Omega)}\Big),$$

with $C = C(p,q,\Omega)$.

Lemma 2.2 ([15, 16]) (Gagliardo-Nirenberg Interpolation Inequality) *Let j, k be any integers satisfying $0 \le j < k$. $R \in \mathbb{R}$ and let $1 \le S, Q \le \infty$ and $\frac{j}{k} \le \theta \le 1$ such that*

$$\frac{1}{R} = \frac{j}{n} + \theta(\frac{1}{S} - \frac{k}{n}) + (1-\theta)\frac{1}{Q}. \tag{2.2}$$

Then for all $h \in W^{k,S}(\Omega) \cap L^Q(\Omega)$ there exist two positive constants C_1, C_2 such that

$$\|D^j h\|_{L^R(\Omega)} \le C_1 \|D^k h\|^\theta_{L^S(\Omega)} \|h\|^{1-\theta}_{L^Q(\Omega)} + C_2 \|h\|_{L^Q(\Omega)}, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $C_2 = 0$ for any $h \in W_0^{k,S}(\Omega) \cap L^Q(\Omega)$.

Now we present the main result as follows.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and (1.5)–(1.7) be* *valid. The problems (1.1), (1.3) and (1.4) possess a global weak solution (n, c, u, p) in the sense of Definition 2.4 below.*

Definition 2.4 (Weak Solution) *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain and $T \in (0, \infty)$. A triple of functions (n, c, u) is called a weak solution of problem (1.1), (1.3) and (1.4) if it fulfills $n \geq 0$ and $c \geq 0$ as well as $\nabla \cdot u = 0$ a.e., in $\Omega \times (0, T)$, and*

$$\begin{cases} n \in L^2((0,T); H^1(\Omega)), \\ c \in L^2((0,T); H^1(\Omega)), \\ u \in L^2((0,T); H_0^1(\Omega)), \end{cases} \tag{2.4}$$

and for a.e., $t \in (0, T)$, we have

$$\int_\Omega n\varphi_1 \mathrm{d}x - \int_0^t \int_\Omega n\varphi_{1t} \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Omega [\nabla n \cdot \nabla \varphi_1 - (\nabla \varphi_1 \cdot u)n - n\nabla c \cdot \nabla \varphi_1 - (rn - \mu n^2)\varphi_1] \mathrm{d}x\mathrm{d}\tau = \int_\Omega n(x, 0)\varphi_1(x, 0)\mathrm{d}x$$

for any $\varphi_1 \in C_0^\infty(\overline{\Omega} \times [0, T))$, as well as

$$\int_\Omega c\varphi_2 \mathrm{d}x - \int_0^t \int_\Omega c\varphi_{2t} \mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega (\nabla \varphi_2 \cdot u)c\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Omega \nabla \varphi_2 \cdot \nabla c\mathrm{d}x\mathrm{d}\tau = \int_\Omega c(x, 0)\varphi_2(x, 0)\mathrm{d}x + \int_0^t \int_\Omega (n\varphi_2 - c\varphi_2)\mathrm{d}x\mathrm{d}\tau$$

for any $\varphi_2 \in C_0^\infty(\overline{\Omega} \times [0, T))$ and

$$\int_\Omega u \cdot \varphi_3 \mathrm{d}x - \int_0^t \int_\Omega u \cdot \varphi_{3t} \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Omega (\nabla u \cdot \nabla \varphi_3 - \nabla \varphi_3 \cdot \psi - g\varphi_3)\mathrm{d}x\mathrm{d}\tau = \int_\Omega u(x, 0) \cdot \varphi_3(x, 0)\mathrm{d}x$$

for any $\varphi_3 \in C_0^\infty(\overline{\Omega} \times [0, T); \mathbb{R}^2)$ that satisfies $\nabla \cdot \varphi_3 \equiv 0$ in $\Omega \times (0, T)$ and $\varphi_3 = 0$ on $\partial\Omega$.

3. Global existence

In this section we shall establish the existence of the global weak solution to equation (1.1). We leave the proof of local existence for the last part of this paper. And we concentrate into getting a priori bounds that allow us to conclude global existence.

3.1. A priori estimate

Firstly, we plan to derive some appropriate estimates for n, c and u.

Lemma 3.1 *Let $0 \leq t < T$. Then*

$$\int_0^t \|n(\tau)\|_{L^2(\Omega)}^2 \mathrm{d}\tau \leq C. \tag{3.1}$$

Proof Integrating the first equation of (1.1) in Ω, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega n\mathrm{d}x = r \int_\Omega n\mathrm{d}x - \mu \int_\Omega n^2 \mathrm{d}x \leq r \int_\Omega n\mathrm{d}x. \tag{3.2}$$

Using the Gronwall's inequality, we have

$$\int_\Omega n\mathrm{d}x \le e^{\int_0^t r\mathrm{d}\tau}\int_\Omega n_0\mathrm{d}x. \tag{3.3}$$

Then we integrate (3.2) over $[0,t]$ to obtain

$$\int_\Omega n\mathrm{d}x - \int_\Omega n_0\mathrm{d}x = r\int_0^t\int_\Omega n\mathrm{d}x\mathrm{d}\tau - \mu\int_0^t\int_\Omega n^2\mathrm{d}x\mathrm{d}\tau. \tag{3.4}$$

Using the positive of n and (3.3), we have

$$\int_0^t\int_\Omega n^2\mathrm{d}x\mathrm{d}\tau \le C\Big(\int_0^t\int_\Omega n\mathrm{d}x\mathrm{d}\tau + \int_\Omega n_0\mathrm{d}x\Big) \le C. \ \square$$

Lemma 3.2 *We have*

$$\|u\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}\tau \le C. \tag{3.5}$$

Proof Multiplying the third equation of (1.1) with u and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\mathrm{d}x + \int_\Omega |\nabla u|^2\mathrm{d}x = \int_\Omega (n\nabla\phi u + gu)\mathrm{d}x.$$

From Hölder's inequality and Young's inequality, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\mathrm{d}x + \int_\Omega |\nabla u|^2\mathrm{d}x \le C_1\|n\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + C_2. \tag{3.6}$$

Multiplying (3.6) by e^{-t}, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}\|u\|^2_{L^2(\Omega)}) + e^{-t}\|\nabla u\|^2_{L^2(\Omega)} \le C_1e^{-t}\|n\|^2_{L^2(\Omega)} + C_2e^{-t} \le C_3\|n\|^2_{L^2(\Omega)} + C_3.$$

Then we integrate on $(0,t)$ with $t\le T$ and multiply by e^T and apply Lemma 3.1 to obtain that

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}\tau \le C_3e^T\int_0^t \|n\|^2_{L^2(\Omega)}\mathrm{d}\tau + \frac{1}{2}e^T\|u_0\|^2_{L^2(\Omega)} + C_3e^T \le C. \ \square$$

Lemma 3.3 ([10]) *There exists a positive constant C such that*

$$\int_\Omega c^2\mathrm{d}x \le C, \tag{3.7}$$

$$\int_0^t \|\nabla c\|^4_{L^2(\Omega)}\mathrm{d}\tau \le C\Big(\int_0^t \|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau + 1\Big), \ \textit{for } 0\le t\le T, \tag{3.8}$$

$$\|\nabla c\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |\triangle c|^2\mathrm{d}x\mathrm{d}\tau \le C, \ \textit{for } 0\le t\le T. \tag{3.9}$$

3.2. Bounded for the $L^p(\Omega)$ norm of n.

In order to pass from local to global existence we need to show that $\|n\|_{L^p(\Omega)}$ is bounded.

Lemma 3.4 *Let $t>0$, $2\le p<\infty$ and $n_0\in L^\infty(\Omega)$. Then there exists a constant C such that*

$$\int_\Omega n^p\mathrm{d}x < C, \ \textit{for all } 0<t<T, \tag{3.10}$$

and

$$\int_0^t\int_\Omega |\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x\mathrm{d}\tau \le C, \ \textit{for all } 0<t<T. \tag{3.11}$$

Proof We multiply the first equation of (1.1) with n^{p-1} and integrate by parts to see that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\mathrm{d}x=\int_\Omega[-(p-1)n^{p-2}|\nabla n|^2-n^p\triangle c-\frac{1}{p}\nabla c\cdot\nabla n^p+rn^p-\mu n^{p+1}]\mathrm{d}x.$$

Hence,

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\mathrm{d}x&\le-\int_\Omega\frac{4(p-1)}{p^2}|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x-\int_\Omega n^p\triangle c\mathrm{d}x-\frac{1}{p}\int_\Omega\nabla c\cdot\nabla n^p\mathrm{d}x+\int_\Omega rn^p\mathrm{d}x\\&\le-\frac{4(p-1)}{p^2}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x-\int_\Omega n^p\triangle c\mathrm{d}x+\frac{1}{p}\int_\Omega n^p\triangle c\mathrm{d}x+\int_\Omega rn^p\mathrm{d}x\\&\le-\frac{4(p-1)}{p^2}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x+\frac{1-p}{p}\int_\Omega n^p\triangle c\mathrm{d}x+r\int_\Omega n^p\mathrm{d}x.\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\mathrm{d}x\le-\frac{4(p-1)}{p}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x+(1-p)\int_\Omega n^p\triangle c\mathrm{d}x+pr\int_\Omega n^p\mathrm{d}x. \tag{3.12}$$

From Lemma 2.2, we can estimate

$$\begin{aligned}\Big|\int_\Omega n^p\triangle c\mathrm{d}x\Big|&\le\Big(\int_\Omega n^{2p}\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega|\triangle c|^2\mathrm{d}x\Big)^{\frac{1}{2}}\\&\le\|\triangle c\|_{L^2(\Omega)}(\int_\Omega n^{2p}\mathrm{d}x)^{\frac{1}{2}}\\&\le\|\triangle c\|_{L^2(\Omega)}\Big[C_3\Big(\int_\Omega n^p\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x\Big)^{\frac{1}{2}}+C_3\Big(\int_\Omega n^p\mathrm{d}x\Big)^{\frac{1}{2}}\Big]\\&\le C_4\int_\Omega n^p\mathrm{d}x\|\triangle c\|^2_{L^2(\Omega)}+\frac{2}{p}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x+C_5\int_\Omega n^p\mathrm{d}x\|\triangle c\|^2_{L^2(\Omega)}+C_6.\end{aligned}$$

Thus

$$\Big|\int_\Omega n^p\triangle c\mathrm{d}x\Big|\le C_7\|\triangle c\|^2_{L^2(\Omega)}\int_\Omega n^p\mathrm{d}x+\frac{2}{p}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x+C_7. \tag{3.13}$$

From (3.12) and (3.13), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^p\mathrm{d}x\le&-\frac{4(p-1)}{p}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x+(p-1)(C_7\|\triangle c\|^2_{L^2(\Omega)}\int_\Omega n^p\mathrm{d}x+\\&\frac{2}{p}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x+C_7)+pr\int_\Omega n^p\mathrm{d}x\\\le&-\frac{2(p-1)}{p}\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x+C(p-1)\|\triangle c\|^2_{L^2(\Omega)}\int_\Omega n^p\mathrm{d}x+pr\int_\Omega n^p\mathrm{d}x+C\\\le&[C(p-1)\|\triangle c\|^2_{L^2(\Omega)}+pr]\int_\Omega n^p\mathrm{d}x+C.\end{aligned}$$

By the Gronwall's inequality, we have

$$\begin{aligned}\int_\Omega n^p\mathrm{d}x&\le e^{\int_0^t[C(p-1)\|\triangle c\|^2_{L^2(\Omega)}+pr]\mathrm{d}\tau}\Big[\int_\Omega n_0^p\mathrm{d}x+\int_0^t C\mathrm{d}\tau\Big]\\&\le\Big(\int_\Omega n_0^p\mathrm{d}x+CT\Big)e^{C(p-1)\int_0^t\|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau+prT}\le C.\end{aligned}$$

Moreover, we regrate over $(0,T)$ and reorganise terms to obtain

$$\frac{2(p-1)}{p}\int_0^T\int_\Omega|\nabla(n^{\frac{p}{2}})|^2\mathrm{d}x\mathrm{d}\tau$$
$$\leq C(p-1)\int_0^T\|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau+pr\int_0^T\int_\Omega n^p\mathrm{d}x\mathrm{d}\tau+CT\leq C.\ \square$$

3.3. Local existence

The following Lemma is an adaptation from the Proposition of [17].

Lemma 3.5 *Suppose that*

$$u\in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))\ \textit{with}\ \nabla\cdot u=0\ \text{a.e.}\ t\geq 0,$$
$$0\leq n\in L^2(0,T;L^2(\Omega))\ \text{a.e.}\ t\geq 0$$

and

$$c(x,0)=c_0\in L^2(\Omega)$$

are known functions. Then the parabolic equation

$$\begin{cases} c_t+u\cdot\nabla c=\triangle c+n-c, & x\in\Omega,\ t>0,\\ \frac{\partial c}{\partial\nu}=0, & \textit{on}\ \partial\Omega,\\ c(x,0)=c_0, \end{cases}\tag{3.14}$$

has a unique weak solution $c\in L^2(0,T;H^1(\Omega))$ satisfying $c\geq 0$. There exists $c\in L^2(0,T;H^1(\Omega))$ such that

$$\int_\Omega c\xi_2\mathrm{d}x-\int_0^t\int_\Omega c\xi_{2t}\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_2\cdot u)c\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega\nabla\xi_2\cdot\nabla c\mathrm{d}x\mathrm{d}\tau$$
$$=\int_\Omega c(x,0)\xi_2(x,0)\mathrm{d}x+\int_0^t\int_\Omega(n\xi_2-c\xi_2)\mathrm{d}x\mathrm{d}\tau,$$

for any $\xi_2\in H^1((0,T)\times\Omega)$.

Proof Firstly, we prove the existence of the weak solution.

It is known that the solution to (3.14) will serve as the limit of the solution to the corresponding regularized system. Thus, we need to consider an appropriately regularized problem of (3.14) at first.

Let $\{u_k\}_{k\geq 1}$ be a sequence of bounded functions which satisfies

$$u_k(x,t)\in C_0^\infty(\Omega)\ \text{and}\ \operatorname{div}u_k(x,t)=0\ \text{for a.e.}\ t\geq 0\ \text{and for all}\ k\geq 1,$$

such that

$$u_k\to u\ \text{in}\ L^2(\Omega\times(0,T)).$$

Next we consider the following regularized problem

$$\begin{cases} c_{kt}+u_k\cdot\nabla c_k=\triangle c_k+n-c_k, & x\in\Omega,\ t>0,\\ \frac{\partial c_k}{\partial\nu}=0, & \text{on}\ \partial\Omega,\\ c_k(x,0)=c_0. \end{cases}\tag{3.15}$$

Similar to the proof process of Lemma 3.3, we can obtain the following estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_k^2\mathrm{d}x+\|\nabla c_k\|_{L^2(\Omega)}^2+\frac{1}{2}\int_\Omega c_k^2\mathrm{d}x\le\frac{1}{2}\int_\Omega n^2\mathrm{d}x.$$

Therefore,

$$\frac{1}{2}\int_\Omega c_k^2\mathrm{d}x+\int_0^t\|\nabla c_k\|_{L^2(\Omega)}^2\mathrm{d}\tau+\frac{1}{2}\int_0^t\|c_k\|_{L^2(\Omega)}^2\mathrm{d}\tau\le\frac{1}{2}\int_0^t\int_\Omega n^2\mathrm{d}x\mathrm{d}\tau+\frac{1}{2}\int_\Omega c_0\mathrm{d}x\le C.$$

In consequence there exists $c\in L^2((0,T),H^1(\Omega))$ and a subsequence of $\{c_k\}_{k\ge1}$ such that

$$\begin{aligned}&c_k\to c\text{ weakly}^*\text{ in }L^\infty(0,T;L^2(\Omega)),\\&c_k\to c\text{ weakly in }L^2(\Omega\times(0,T)),\\&\nabla c_k\to\nabla c\text{ weakly in }L^2(\Omega\times(0,T)).\end{aligned}$$

We can conclude that c_k satisfies

$$\begin{aligned}&\int_\Omega c_k\xi_2\mathrm{d}x-\int_0^t\int_\Omega c_k\xi_{2t}\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_2\cdot u_k)c_k\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega\nabla\xi_2\cdot\nabla c_k\mathrm{d}x\mathrm{d}\tau\\&\quad=\int_\Omega c(x,0)\xi_2(x,0)\mathrm{d}x+\int_0^t\int_\Omega(n\xi_2-c_k\xi_2)\mathrm{d}x\mathrm{d}\tau\end{aligned}\tag{3.16}$$

for any $\xi_2\in H^1((0,T)\times\Omega)$.

For the nonlinear term (3.16) we have

$$\begin{aligned}&\int_0^t\int_\Omega(\nabla\xi_2\cdot u_k)c_k\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_2\cdot u)c\mathrm{d}x\mathrm{d}\tau\\&\quad=\int_0^t\int_\Omega(\nabla\xi_2\cdot(u_k-u))c_k\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_2\cdot u)(c-c_k)\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Using the convergence of u_k and c_k, we conclude

$$\int_0^t\int_\Omega(\nabla\xi_2\cdot u_k)c_k\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_2\cdot u)c\mathrm{d}x\mathrm{d}\tau\to0\text{ as }k\to\infty.$$

Hence, we can get the following equality by taking the limit in (3.16)

$$\begin{aligned}&\int_\Omega c\xi_2\mathrm{d}x-\int_0^t\int_\Omega c\xi_{2t}\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_2\cdot u)c\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega\nabla\xi_2\cdot\nabla c\mathrm{d}x\mathrm{d}\tau\\&\quad=\int_\Omega c(x,0)\xi_2(x,0)\mathrm{d}x+\int_0^t\int_\Omega(n\xi_2-c\xi_2)\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Then we prove the uniqueness of the weak solution.

Let c_1 and c_2 be two weak solutions of (3.14). Then taking the difference of the two equations, multiplying it by c_1-c_2 and using $\nabla\cdot u=0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|c_1-c_2|^2\mathrm{d}x=-\int_\Omega|\nabla(c_1-c_2)|^2\mathrm{d}x-\int_\Omega|c_1-c_2|^2\mathrm{d}x\le\int_\Omega|c_1-c_2|^2\mathrm{d}x.$$

Using the Gronwall's inequality, we have

$$\int_\Omega|c_1-c_2|^2\mathrm{d}x\le0.$$

Thus we conclude that $c_1=c_2$. □

Lemma 3.6 *Suppose that*

$$f \in L^s((0,T)\times\Omega) \text{ for some } s \geq 1,$$

$$u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \text{ with } \nabla\cdot u = 0 \text{ a.e } t \geq 0,$$

and

$$c \in H^1((0,T)\times\Omega), \quad n(x,0) = n_0 \in L^2(\Omega)$$

are known functions. Then the parabolic equation

$$\begin{cases} n_t + u\cdot\nabla n = \triangle n - \nabla\cdot(n\nabla c) + rn - \mu f, & x\in\Omega,\ t>0, \\ \frac{\partial n}{\partial \nu} = 0, & \text{on } \partial\Omega, \\ n(x,0) = n_0, & x \in \Omega, \end{cases} \tag{3.17}$$

has a unique weak solution $n \in L^2(0,T;H^1(\Omega))$. There exists $n \in L^2(0,T;H^1(\Omega))$ such that

$$\int_\Omega n\xi_1 \mathrm{d}x - \int_0^t\int_\Omega n\xi_{1t}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega [\nabla n\cdot\nabla\xi_1 - (\nabla\xi_1\cdot u)n - n\nabla c\cdot\nabla\xi_1 -$$
$$(rn-\mu f)\xi_1]\mathrm{d}x\mathrm{d}\tau = \int_\Omega n(x,0)\xi_1(x,0)\mathrm{d}x,$$

for any $\xi_1 \in H^1((0,T)\times\Omega)$.

Proof Firstly, we prove the existence of the weak solution.

Let $\{u_k\}_{k\geq 1}$ be a sequence of bounded functions which satisfies

$$u_k(x,t) \in C_0^\infty(\Omega) \text{ and } \operatorname{div} u_k(x,t) = 0 \text{ for a.e. } t \geq 0 \text{ and for all } k \geq 1,$$

such that

$$u_k \to u \text{ in } L^2(\Omega\times(0,T)).$$

Next we consider the following regularized problem

$$\begin{cases} n_{kt} + u_k\cdot\nabla n_k = \triangle n_k - \nabla\cdot(n_k\nabla c) + rn_k - \mu f, & x\in\Omega,\ t>0, \\ \frac{\partial n_k}{\partial \nu} = 0, & \text{on } \partial\Omega, \\ n_k(x,0) = n_0, & x \in \Omega. \end{cases} \tag{3.18}$$

Similar to the proof process of Lemma 3.4, we can obtain the following estimate

$$\int_\Omega n_k^p \mathrm{d}x < C, \text{ for all } 0<t<T,$$
$$\int_0^t\int_\Omega |\nabla(n_k^{\frac{p}{2}})|^2\mathrm{d}x\mathrm{d}\tau \leq C, \text{ for all } 0<t<T,$$
$$\int_0^T\int_\Omega n_k^2\mathrm{d}x\mathrm{d}\tau < C, \text{ for all } 0<t<T.$$

In consequence there exists $n \in L^2((0,T),H^1(\Omega))$ and a subsequence of $\{n_k\}_{k\geq 1}$ such that

$$n_k \to n \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)),$$
$$n_k \to n \text{ weakly in } L^2(\Omega\times(0,T)),$$
$$\nabla n_k \to \nabla n \text{ weakly in } L^2(\Omega\times(0,T)).$$

We can conclude that n_k satisfies

$$\int_\Omega n_k\xi_1\mathrm{d}x-\int_0^t\int_\Omega n_k\xi_{1t}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega[\nabla n\cdot\nabla\xi_1-(\nabla\xi_1\cdot u_k)n_k-n_k\nabla c\cdot\nabla\xi_1-$$
$$(rn_k-\mu f)\xi_1]\mathrm{d}x\mathrm{d}\tau=\int_\Omega n(x,0)\xi_1(x,0)\mathrm{d}x \tag{3.19}$$

for any $\xi_1\in H^1((0,T)\times\Omega)$.

For the nonlinear term (3.19) we have

$$\begin{aligned}&\int_0^t\int_\Omega(\nabla\xi_1\cdot u_k)n_k\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_1\cdot u)n\mathrm{d}x\mathrm{d}\tau\\&=\int_0^t\int_\Omega(\nabla\xi_1\cdot(u_k-u))n_k\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_1\cdot u)(n-n_k)\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Using the convergence of u_k and n_k, we conclude

$$\int_0^t\int_\Omega(\nabla\xi_1\cdot u_k)n_k\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega(\nabla\xi_1\cdot u)n\mathrm{d}x\mathrm{d}\tau\to0\text{ as }k\to\infty.$$

Hence, we can get the following equality by taking the limit in (3.19)

$$\int_\Omega n\xi_1\mathrm{d}x-\int_0^t\int_\Omega n\xi_{1t}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega[\nabla n\cdot\nabla\xi_1-(\nabla\xi_1\cdot u)n-n\nabla c\cdot\nabla\xi_1-$$
$$(rn-\mu f)\xi_1]\mathrm{d}x\mathrm{d}\tau=\int_\Omega n(x,0)\xi_1(x,0)\mathrm{d}x.$$

Then we prove the uniqueness of the weak solution.

Let n_1 and n_2 be two weak solutions of (3.17). Then taking the difference of the two equations, multiplying it by n_1-n_2 and using $\nabla\cdot u=0$, we get

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|n_1-n_2|^2\mathrm{d}x\\
&=-\int_\Omega|\nabla(n_1-n_2)|^2\mathrm{d}x-\int_\Omega(n_1-n_2)\nabla\cdot((n_1-n_2)\nabla c)\mathrm{d}x+r\int_\Omega|n_1-n_2|^2\mathrm{d}x\\
&=-\int_\Omega|\nabla(n_1-n_2)|^2\mathrm{d}x-\frac12\int_\Omega\triangle c(n_1-n_2)^2\mathrm{d}x+r\int_\Omega|n_1-n_2|^2\mathrm{d}x\\
&\le-\int_\Omega|\nabla(n_1-n_2)|^2\mathrm{d}x+\frac12\Big(\int_\Omega|n_1-n_2|^4\mathrm{d}x\Big)^{\frac12}\Big(\int_\Omega|\triangle c|^2\mathrm{d}x\Big)^{\frac12}+r\int_\Omega|n_1-n_2|^2\mathrm{d}x\\
&\le-\|\nabla(n_1-n_2)\|^2_{L^2(\Omega)}+\frac12\|\nabla(n_1-n_2)\|^2_{L^2(\Omega)}+\\
&\quad\frac12\|(n_1-n_2)\|^2_{L^2(\Omega)}\|\triangle c\|^2_{L^2(\Omega)}+r\|(n_1-n_2)\|^2_{L^2(\Omega)}\\
&\le-\frac12\|\nabla(n_1-n_2)\|^2_{L^2(\Omega)}+(r+\frac12\|\triangle c\|^2_{L^2(\Omega)})\|(n_1-n_2)\|^2_{L^2(\Omega)}\\
&\le(r+\frac12\|\triangle c\|^2_{L^2(\Omega)})\|(n_1-n_2)\|^2_{L^2(\Omega)}.\end{aligned}$$

Using the Gronwall's inequality, we have

$$\int_\Omega|n_1-n_2|^2\mathrm{d}x\le0.$$

Thus we conclude that $n_1=n_2$. $\square$

3.4. Fixed point argument

We consider the Banach space $(Y, |\cdot|)$, where

$$Y := L^4((0,T)\times\Omega)$$

with the natural norm

$$|\tilde{n}|_Y = \Big(\int_0^T\int_\Omega |\tilde{n}|^4 \mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{4}}. \tag{3.20}$$

We will show the local existence of the weak solution through the Schauder fixed point theorem [18]. We first define the convex set

$$B_Y(0,R) := \tilde{n} : |\tilde{n}|_Y \le R.$$

Next we define the functional $\Gamma : Y \to Y$ as follows. We take $\tilde{n} \in B_Y(0,R)$ and construct $n = \Gamma(\tilde{n})$ through the next steps:

Step 1. We put $\tilde{n}$ to take place of n in the third equation of (1.1) and then obtain the solution u. The existence and uniqueness of the solution for the third equation of (1.1) are guaranteed by Lemma 2.1.

Step 2. Next we obtain c from the second equation of (1.1) and Lemma 3.5.

Step 3. Finally, we put c and u into the first equation of (1.1) and apply Lemma 3.6 to define n as the solution of the linear problem

$$n_t + u\cdot\nabla n = \triangle n - \nabla\cdot(n\nabla c) + rn - \mu\tilde{n}^2,$$

with initial data $n(x,0) = n_0$ and the Neumann boundary conditions. Then a solution of the nonlinear system (1.1), (1.3) and (1.4) corresponds to a fixed point of the map Γ.

Theorem 3.7 *If $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$, $c \in H^1((0,T)\times\Omega)$ and $n(x,0) = n_0 \in L^2(\Omega)$. The following problem*

$$\begin{cases} n_t + u\cdot\nabla n = \triangle n - \nabla\cdot(n\nabla c) + rn - \mu n^2, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = 0, & on\ \partial\Omega,\\ n(x,0) = n_0, & x\in\Omega, \end{cases}$$

has a unique weak solution $n \in L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^1(\Omega))$ and $n_t \in L^2((0,T);H^{-1}(\Omega))$.

The following Lemma will be useful later when we prove the hypothesis of Schauder's point theorem.

Similar to the proof process of Lemmas 3.2–3.4, we can obtain the following conclusions.

Lemma 3.8 *In the construction of $n = \Gamma(\tilde{n})$ as described above, the next estimations are satisfied:*

(1) $\int_\Omega c^2 \mathrm{d}x < \infty$,

(2) $\|u\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)}\mathrm{d}\tau \le C\int_0^t \|\tilde{n}\|^2_{L^2(\Omega)}\mathrm{d}\tau + C$,

(3) $\int_0^t \|\nabla c\|^4_{L^2(\Omega)}\mathrm{d}\tau \le C(\int_0^t \|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau + 1)$,

(4) $\|\nabla c\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |\triangle c|^2\mathrm{d}x\mathrm{d}\tau \le C\int_0^T (1 + \|\tilde{n}\|^2_{L^2(\Omega)})\mathrm{d}\tau$,

(5) $\int_\Omega n^p\mathrm{d}x \le (\int_\Omega n_0^p\mathrm{d}x + CT)e^{C(p-1)\int_0^t \|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau + prT}$.

Lemma 3.9 *For T small enough, Γ is a map from $B_Y(0,R)$ into $B_Y(0,R)$.*

Proof From Lemma 3.8 (5), we have

$$\begin{aligned}\Big(\int_0^T\int_\Omega n^p\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{p}} &\le \Big[\int_0^T\Big(\int_\Omega n_0^p\mathrm{d}x + CT\Big)e^{C(p-1)\int_0^t \|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau + prT}\mathrm{d}\tau\Big]^{\frac{1}{p}}\\ &\le T^{\frac{1}{p}}\Big[\Big(\int_\Omega n_0^p\mathrm{d}x + CT\Big)e^{C(p-1)\int_0^t \|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau + prT}\Big]^{\frac{1}{p}}.\end{aligned}\tag{3.21}$$

By Lemma 3.8 (4) and the definition of $|\cdot|_Y$, we obtain

$$\int_0^t \|\triangle c\|^2_{L^2(\Omega)}\mathrm{d}\tau \le C\int_0^T(1+\|\tilde{n}\|^2_{L^2(\Omega)})\mathrm{d}\tau \le C(T+R^2).\tag{3.22}$$

From (3.21) and (3.22), we have

$$\Big(\int_0^T\int_\Omega n^p\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{p}} \le T^{\frac{1}{p}}\Big[\Big(\int_\Omega n_0^p\mathrm{d}x + CT\Big)e^{C(p-1)(T+R^2)+prT}\Big]^{\frac{1}{p}}.\tag{3.23}$$

Therefore, the right hand side of (3.23) tends to zero as the variable T goes to zero. Thus we can take $p=4$ and T enough small to conclude

$$|\tilde{n}|_Y = \Big(\int_0^T\int_\Omega |\tilde{n}|^4\mathrm{d}x\mathrm{d}\tau\Big)^{\frac{1}{4}} \le R.\ \Box$$

Lemma 3.10 *The map $\Gamma : (B_Y(0,R), |\cdot|_Y) \to (B_Y(0,R), |\cdot|_Y)$ is continuous.*

Proof Suppose that $\{\tilde{n}_k\}_{k\ge1}$ is a sequence of functions in $B_Y(0,R)$ satisfying $\tilde{n}_k \to \tilde{n}$ in the norm $|\cdot|_Y$. Then we prove that solutions $\{n_k\}_{k\in N}$ of the linear equations

$$\begin{cases} n_{kt} + u\cdot\nabla n_k = \triangle n_k - \nabla\cdot(n_k\nabla c_k) + rn_k - \mu\tilde{n}_k^2, & x\in\Omega,\ t>0,\\ \frac{\partial n_k}{\partial\nu} = 0, & \text{on } \partial\Omega,\\ n_k(x,0) = n_0(x), & x\in\Omega,\end{cases}\tag{3.24}$$

converge in the norm $|\cdot|_Y$ to the unique solution of

$$\begin{cases} n_t + u\cdot\nabla n = \triangle n - \nabla\cdot(n\nabla c) + rn - \mu\tilde{n}^2, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = 0, & \text{on } \partial\Omega,\\ n(x,0) = n_0(x), & x\in\Omega.\end{cases}\tag{3.25}$$

Following the same proof process of Lemma 3.4, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_k^p\mathrm{d}x &\le -\frac{2(p-1)}{p}\int_\Omega|\nabla(n_k^{\frac{p}{2}})|^2\mathrm{d}x + C(p-1)\|\triangle c_k\|^2_{L^2(\Omega)}\int_\Omega n_k^p\mathrm{d}x + pr\int_\Omega n_k^p\mathrm{d}x + C\\ &\le -\frac{2(p-1)}{p}\int_\Omega|\nabla(n_k^{\frac{p}{2}})|^2\mathrm{d}x + [C(p-1)\|\triangle c_k\|^2_{L^2(\Omega)} + pr]\int_\Omega n_k^p\mathrm{d}x + C.\end{aligned}\tag{3.26}$$

In addition from Lemma 3.8 (4), we have

$$\int_0^t \|\triangle c_k\|^2_{L^2(\Omega)}\mathrm{d}\tau \le C\int_0^T(1+\|\tilde{n}_k\|^2_{L^2(\Omega)})\mathrm{d}\tau \le C(T+R^2).\tag{3.27}$$

Using the Gronwall's inequality, we have

$$\int_\Omega n_k^p \mathrm{d}x < C_T \text{ and } \int_0^T \int_\Omega |\nabla(n_k^{\frac{p}{2}})|^2 \mathrm{d}x\mathrm{d}\tau < C_T \text{ for all } p \geq 2. \tag{3.28}$$

Taking $p = 2$ in (3.28), we find

$$\|n_k\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{3.29}$$

For any $\omega \in H^1(\Omega)$ with $\|\omega\|_{H^1(\Omega)} \leq 1$, we have

$$\begin{aligned}
&\|n_{kt}\|_{H^1(\Omega)^*}^2 = \| - u \cdot \nabla n_k + \triangle n_k - \nabla \cdot (n_k \nabla c_k) + rn_k - \mu \tilde{n}_k^2\|_{H^1(\Omega)^*}^2 \\
&\quad \leq C(\|u \cdot \nabla n_k\|_{H^1(\Omega)^*}^2 + \|\triangle n_k\|_{H^1(\Omega)^*}^2 + \|\nabla \cdot (n_k \nabla c_k)\|_{H^1(\Omega)^*}^2 + \\
&\qquad \|rn_k\|_{H^1(\Omega)^*}^2 + \|\mu \tilde{n}_k^2\|_{H^1(\Omega)^*}^2) \\
&\quad \leq C(\sup_{\|\omega\|_{H^1(\Omega)} \leq 1} (u \cdot \nabla n_k, \omega)_{(H^1(\Omega)^*, H^1(\Omega))})^2 + C(\sup_{\|\omega\|_{H^1(\Omega)} \leq 1} (\triangle n_k, \omega)_{(H^1(\Omega)^*, H^1(\Omega))})^2 + \\
&\qquad C(\sup_{\|\omega\|_{H^1(\Omega)} \leq 1} (\nabla \cdot (n_k \nabla c_k), \omega)_{(H^1(\Omega)^*, H^1(\Omega))})^2 + C(\sup_{\|\omega\|_{H^1(\Omega)} \leq 1} (rn_k, \omega)_{(H^1(\Omega)^*, H^1(\Omega))})^2 + \\
&\qquad C(\sup_{\|\omega\|_{H^1(\Omega)} \leq 1} (\mu \tilde{n}_k^2, \omega)_{(H^1(\Omega)^*, H^1(\Omega))})^2 \\
&\quad \leq C\Big(\int_\Omega u \cdot \nabla n_k \omega \mathrm{d}x\Big)^2 + C\Big(\int_\Omega \triangle n_k \omega \mathrm{d}x\Big)^2 + C\Big(\int_\Omega \nabla \cdot (n_k \nabla c_k) \omega \mathrm{d}x\Big)^2 + \\
&\qquad C\Big(\int_\Omega rn_k \omega \mathrm{d}x\Big)^2 + C\Big(\int_\Omega \mu \tilde{n}_k^2 \omega \mathrm{d}x\Big)^2 \\
&\quad \leq C\Big(\int_\Omega (u \cdot \nabla \omega) n_k \mathrm{d}x\Big)^2 + C\Big(\int_\Omega \nabla n_k \cdot \nabla \omega \mathrm{d}x\Big)^2 + C\Big(\int_\Omega (n_k \nabla c_k) \cdot \nabla \omega \mathrm{d}x\Big)^2 + \\
&\qquad C\Big(\int_\Omega rn_k \omega \mathrm{d}x\Big)^2 + C\Big(\int_\Omega \mu \tilde{n}_k^2 \omega \mathrm{d}x\Big)^2 \\
&\quad \leq C\|\nabla \omega\|_{L^2(\Omega)}^2 \|un_k\|_{L^2(\Omega)}^2 + C\|\nabla n_k\|_{L^2(\Omega)}^2 \|\nabla \omega\|_{L^2(\Omega)}^2 + C\|n_k \nabla c_k\|_{L^2(\Omega)}^2 \|\nabla \omega\|_{L^2(\Omega)}^2 + \\
&\qquad C\|n_k\|_{L^2(\Omega)}^2 \|\omega\|_{L^2(\Omega)}^2 + C\|\tilde{n}_k^2\|_{L^2(\Omega)}^2 \|\omega\|_{L^2(\Omega)}^2 \\
&\quad \leq C\|n_k\|_{L^4(\Omega)}^2 \|u\|_{L^4(\Omega)}^2 + C\|\nabla n_k\|_{L^2(\Omega)}^2 + C\|n_k\|_{L^3(\Omega)}^2 \|\nabla c_k\|_{L^6(\Omega)}^2 + C\|\tilde{n}_k^2\|_{L^2(\Omega)}^2 + C \\
&\quad \leq C\|u\|_{L^4(\Omega)}^2 + C\|\nabla n_k\|_{L^2(\Omega)}^2 + C\|\nabla c_k\|_{L^6(\Omega)}^2 + C\|\tilde{n}_k^2\|_{L^2(\Omega)}^2 + C \\
&\quad \leq C(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2) + C\|\nabla n_k\|_{L^2(\Omega)}^2 + C(\|\triangle c_k\|_{L^2(\Omega)}^2 + \|c_k\|_{L^2(\Omega)}^2) + \\
&\qquad C\|\tilde{n}_k^2\|_{L^2(\Omega)}^2 + C \\
&\quad \leq C\|\nabla u\|_{L^2(\Omega)}^2 + C\|\nabla n_k\|_{L^2(\Omega)}^2 + C\|\triangle c_k\|_{L^2(\Omega)}^2 + C\|\tilde{n}_k\|_{L^4(\Omega)}^4 + C.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\int_0^T \|n_{kt}\|_{H^1(\Omega)^*}^2 \mathrm{d}\tau \\
&\quad \leq C\int_0^T (\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla n_k\|_{L^2(\Omega)}^2 + \|\triangle c_k\|_{L^2(\Omega)}^2 + \|\tilde{n}_k\|_{L^4(\Omega)}^4) \mathrm{d}\tau + CT \leq C.
\end{aligned} \tag{3.30}$$

Applying the Aubin-Lions compactness lemma, there exists a subsequence $\{n_k\}_{k\in N}$ such that

$$n_k \to n_* \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.31}$$

From (3.31), we observe that

$$n_k \to n_* \text{ a.e in } (0,T) \times \Omega.$$

This implies that

$$n_k^2 \to n_*^2 \text{ a.e in } (0,T) \times \Omega. \tag{3.32}$$

By the part (5) of Lemma 3.8 and (3.32), we have

$$n_k^2 \to n_*^2 \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

Hence, we obtain

$$n_k \to n_* \text{ in the norm } |\cdot|_Y.$$

On the other hand, Lemma 3.8 (4) allows us to conclude that

$$\nabla c_k \to \nabla c \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

We have

$$n_k \nabla c_k \to n_* \nabla c \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

Finally we take the limit in (3.24) as $k \to \infty$ to conclude that n_* corresponds to the weak solution of problem (3.25) and we conclude that $n = n_*$ by the uniqueness of solution. □

Lemma 3.11 ([10]) *The set $\Gamma(B_Y(0,R))$ is relatively compact in $B_Y(0,R)$.*

Using Lemmas 3.9–3.11 and the Schauder's fixed theorem, we can obtain Theorem 3.7.

Proof of Theorem 2.3 By the Moser-type iteration [19, Lemma A.1], we can use the Lemma 3.4 to obtain $\|n\|_{L^\infty(\Omega)} \leq C$. Moreover, we can prove $\|c\|_{W^{1,\infty}(\Omega)} \leq C$ and $\|u\|_{W^{1,\infty}(\Omega)} \leq C$ by using the standard parabolic regularity arguments. Then the local-in-time solution can be extended to the global-in-time solution. Thus we obtain Theorem 2.3. □

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