The Application of Differential Characteristic Set Method to Pseudo Differential Operator and Lax Representation

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Abstract Differential characteristic set method is applied to the calculation of pseudo differential operators and Lax representation of nonlinear evolution equations. Firstly, differential characteristic set method and differential division with remainder are used for the calculation of inverse and extraction root of pseudo differential operator, such that the process is simplified since it is unnecessary to solve ordinary differential equation systems and substitute the solutions. Secondly, using differential characteristic set method, the nonlinear partial differential equation systems derived from the generalized Lax equation and Zakharov-Shabat equation, are reduced, and the corresponding nonlinear evolution equation is obtained. The related programs are compiled in Mathematica, a computer-based computer algebra system, and Lax representation of some nonlinear evolution equations can be calculated with the aid of the computer.

Keywords differential characteristic set; differential division with remainder; pseudo differential operator; Lax representation; Zakharov-Shabat equation

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1. Introduction

In integrable systems, constructing Lax representations of nonlinear evolution equations, i.e., finding a pair of differential operators $L$ and $B$, is a key problem, and $L$ and $B$ are called Lax Pair. However, there is no general method to construct Lax representations, so it is significant and interesting to find the Lax representations of nonlinear evolution equations. Using pseudo differential operators (abbr. PDO), we can get a series of Lax representations in a simple way [1].

Originally, PDOs were only used in integrable systems in $1 + 1$ dimensions, and the nonlinear evolution equation is obtained from the generalized Lax equation. According to Sato theory, the approach is extended to higher dimensions, and through the “Dressing operator”, Zakharov-Shabat equation is introduced. Further, PDOs are widely applied to $\tau$-function [2], Darboux transformation [3], Hamilton structure [1] and so on, and developed into a standard technique in the theory of integrable systems.
To construct Lax representations via PDOs, there are many differential polynomials to be solved. For a PDO $L$, such as the calculation of the inverse $L^{-1}$ or the extraction of the $n$th root $L^{\frac{1}{n}}$ of $L$, it is necessary to solve the ordinary differential equation systems firstly, and then $L^{-1}$ and $L^{\frac{1}{n}}$ are obtained by substituting the solutions. From $L^{-1}$ and $L^{\frac{1}{n}}$, the differential operator $B$ is derived. Although the calculation formula of $L^{-1}, L^{\frac{1}{n}}$ and $B$ are straightforward, the calculation of all differential polynomials still requires a lot of work. Nonlinear evolution equation is obtained from the generalized Lax equation or Zakharov-Shabat equation, but it is not direct. From the generalized Lax equation or Zakharov-Shabat equation, a nonlinear partial differential equation system is derived. Via observation, we eliminate the auxiliary fields from the nonlinear partial differential equation system by using compatibility conditions, and obtain the nonlinear evolution equation. Thereby, it is complicated to calculate the construction of Lax representations of nonlinear evolution equation with PDOs.

The main aim of this paper is to apply differential characteristic set method [4-7] to the construction of Lax representations of nonlinear evolution equation with PDOs. For $L^{-1}$ and $L^{\frac{1}{n}}$, using differential characteristic set method, the triangular form differential characteristic set of corresponding ordinary differential equation system is obtained, and then, $L^{-1}, L^{\frac{1}{n}}$ and $B_m$ are calculated by differential division with remainder directly, and it is unnecessary to solve the ordinary differential equations and substitute the solutions into the formula of $L^{-1}, L^{\frac{1}{n}}$ and $B_m$. Thereby, the calculations are simplified. On the other hand, the nonlinear partial differential equation system generated from the generalized Lax equation or Zakharov-Shabat equation is reduced by differential characteristic set method, and for the given suitable order, the first equation of the differential characteristic set is just the desired nonlinear evolution equation.

In the process of calculation, the infinite series are implemented in truncated form, and the truncation order is determined by $B$ or Res.

The programs of differential characteristic set method, PDOs and Lax representations calculation are all provided in MATHEMATICA language. Some Lax representation of nonlinear evolution equations are calculated using the programs.

2. Differential characteristic set method

The differential characteristic set method is based on the order, so the order of differential polynomials should be given first.

2.1. The order of differential polynomials

Let $F$ be a field of characteristic zero. Then $F$ is endowed $n$ operators $D_1, \ldots, D_n$, and the following conditions are satisfied:

1. $D_i(f + g) = D_i(f) + D_i(g)$;
2. $D_i(fg) = gD_i(f) + fD_i(g)$;
3. $D_i(D_j(f)) = D_j(D_i(f))$,

where $f, g \in F$. Then $F$ is called an $n$-element differential field, and $D_1, \ldots, D_n$ are basic
differential operators. $c$ is a constant in $F$ if $D_1(c) = 0$, $i = 1, \ldots, n$.

An arbitrary differential operator $D$ in $F$ is an element of $F$ of the form
\[
D = D_1^{\alpha_1} \cdots D_n^{\alpha_n},
\]
where $\alpha_i$, $i = 1, \ldots, n$ are nonnegative integers, and $D_i^{\alpha_i}$ means that $D_i$ implements $\alpha_i$ times. From (2.1), the mapping
\[
D = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \rightarrow (\alpha_1, \ldots, \alpha_n) = \alpha.
\]
is 1-1. $\alpha \in \mathbb{N}^n$ is called multiple index (abbr. multi-index), meanwhile, $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ is denoted by $D_\alpha$, and $|\alpha| = \sum_{i=1}^n \alpha_i$ is the order of $D_\alpha$. Any finite set of multi-indices is a subset of $\mathbb{N}^n$, and denoted by $\mathbb{N}^n$. In this section, the multi-indices are all $n$-dimensional. For $\forall \alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we have $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$, and
\[
D_\alpha D_\beta = D_\beta D_\alpha = D_{\alpha + \beta} = D_\alpha^{\beta_1} \cdots D_n^{\beta_n}.
\]

For a given $n$-element differential field $F$, $u_1, \ldots, u_m$ are independent indeterminate elements, and can be treated as elements of a certain extended field of $F$. We take the notation
\[
Du = \{D_\sigma u^p \mid \sigma \in \mathbb{N}^n, \ p = 1, 2, \ldots, m\},
\]
where $p$ is the class of $D_\sigma u^p$. $D_\sigma u^p$ is called formal derivative, and denoted by $u_{(p, \sigma)}$ sometimes. $u_{(p, \sigma)}$ is a proper derivative as $|\sigma| > 0$, and $u_{(p, \sigma)}$ is $u^p$ as $\sigma$ is the zero vector.

**Definition 2.1** (diff-graded reverse lex order) $D_\alpha u^p$ is larger than $D_\beta u^q$, and denoted by $D_\alpha u^p > D_\beta u^q$, if
\begin{enumerate}
    \item $(1)$ $p > q$; or
    \item $(2)$ $p = q$, and $|\alpha| < |\beta|$; or
    \item $(3)$ $p = q$, and $|\alpha| = |\beta|$, there exists $i$, s.t., $\alpha_i < \beta_i, \alpha_{i+1} < \beta_{i+1}, \ldots, \alpha_n < \beta_n \ (1 \leq i \leq n)$.
\end{enumerate}

The differential monomial on $F$ is the finite multiplication of elements in $Du$,
\[
c \prod u_{(p, \sigma)}^{k_{(p, \sigma)}},
\]
where $c \in F$, and $k_{(p, \sigma)}$ is a positive integer. Differential polynomial (abbr. d-pol) is a finite sum of differential monomials. $f$ is a d-pol, according to the order in Definition 2.1, the largest derivative occurring in $f$ is called leading derivative, and denoted by \(\text{ld}(f)\), and $f$ can be written as
\[
f = f_d u_{(p, \sigma)}^{d} + f_{d-1} u_{(p, \sigma)}^{d-1} + \cdots + f_0,
\]
where $u_{(p, \sigma)}$ is the leading derivative of $f$, and $f_d \neq 0$. $p$ is called the class of $f$, and denoted by $\text{cl}(f)$; $d$ is called the degree of $f$, and denoted by $\text{deg}(f)$; $f_d$ is called the initial of $f$, and denoted by $\text{Ini}(f)$.
\[
\frac{\partial f}{\partial u_{(p, \sigma)}} = df_{d} u_{(p, \sigma)}^{d-1} + (d-1)f_{d-1} u_{(p, \sigma)}^{d-2} + \cdots
\]
is called the separant of $f$ and denoted by $\text{Sep}(f)$. If $\gamma \in \mathbb{N}^n$ is a nonzero vector,
\[
D_\gamma(f) = \frac{\partial f}{\partial u_{(p, \sigma)}} u_{(p, \gamma + \sigma)} + \cdots
\]
(2.3)
Then \( \text{ld}(D_\gamma(f)) = u_{(p, \gamma + \sigma)} \), and \( \deg(D_\gamma(f)) = 1 \). \( \text{Sep}(f) \) is the initial of \( D_\gamma(f) \) essentially. \( u_{(p, \gamma + \sigma)} \) is the proper derivative of \( \text{ld}(f) \), and is called the principal derivative of \( f \).

\((p, \sigma, d)\) is called the rank of \( f \). \( f \) is called non-trivial if formal derivative occurs in \( f \), or is trivial when the rank is \((0, \sigma^{(0)}, 0)\), where \( \sigma^{(0)} \) is the zero vector. In addition, if \( q \) is not the class of \( f \), \((\sigma, q)\) is called the rank of \( u_{(\sigma, d)} \) in \( f \), where \( \sigma \) is the largest multi-index about \( u^d \), and \( d \) is the highest power of \( D_\sigma u^p \) in \( f \).

All \( d \)-pols make up a ring that is closed to derivative, which is called the differential polynomial ring, and denoted by \( \mathbb{F}\{Du\} \). The order is introduced in \( \mathbb{F}\{Du\} \) as follows.

**Definition 2.2** Let us assume that \( f \) and \( g \) are two non-trivial differential polynomials (abbr. \( d \)-pols), and \((p, \alpha, d_f)\), \((q, \beta, d_g)\) are ranks, respectively. Then \( f \) is larger than \( g \), denoted by \( f > g \), if

1. \( p > q \); or
2. \( p = q \), and \( \alpha > \beta \); or
3. \( p = q \), and \( \alpha = \beta \), and \( d_f > d_g \).

If \((p_f, \sigma_f, d_f) = (p_g, \sigma_g, d_g)\), then \( f \) and \( g \) are incomparable, denoted by \( f \sim g \).

**Lemma 2.3** The order > is a well order on \( \mathbb{F}\{Du\} \). This means that any subset of \( \mathbb{F}\{Du\} \) has smallest elements under >.

From the order for \( \mathbb{F}\{Du\} \), a sequence of \( d \)-pols is called an ascent (a decreasing) sequence if it does not decrease (ascend) according to ranks.

**Lemma 2.4** Every strictly decreasing sequence of \( d \)-pols is finite.

**Definition 2.5** Let \( f \) and \( g \) be \( d \)-plos, and \( g \) be non-trivial. \( f \) is said to be reduced w.r.t. \( g \) if

1. no proper derivative of \( \text{ld}(g) \) occurs in \( f \); and
2. the degree of \( \text{ld}(g) \) in \( f \) is lower than \( \deg(g) \).

Let \( dps \) be a nonempty differential polynomial set (system) (abbr. \( d \)-pol-set), and \( f \) be a \( d \)-pol. \( f \) is said to be reduced w.r.t. \( dps \), if \( f \) is reduced w.r.t. every \( d \)-pol in \( dps \). \( d \)-pol-set \( dqs \) is reduced w.r.t. \( dps \), if every \( d \)-pol in \( dqs \) is reduced w.r.t. \( dps \).

**Definition 2.6** The ascent sequence of \( d \)-pols

\[
f_1 < \cdots < f_k
\]

is differential ascent set (abbr. \( d \)-asc-set) if either

1. \( k = 1 \), and \( f_1 \) is trivial, and is then said to be trivial; or
2. \( k > 1 \), \( f_1, \ldots, f_k \) are all non-trivial, and for \( j < i \), \( f_i \) is reduced w.r.t. all \( f_j \).

Obviously, the number of any \( d \)-asc-set of \( d \)-pol-set is finite.

**Definition 2.7** Assuming that there are two \( d \)-asc-sets

\[
G : g_1 < \cdots < g_k, \text{ and } H : h_1 < \cdots < h_l,
\]

\( G \) is higher than \( H \), and denoted by \( G > H \) if
(1) $H$ is trivial, and $G$ is non-trivial; or
(2) $G$ and $H$ are both non-trivial, and there exists $i$, s.t., $g_1 \sim h_1, \ldots, g_{i-1} \sim h_{i-1}$, and $g_i > h_i$; or
(3) $l < k$, and $g_1 \sim h_1, \ldots, g_l \sim h_l$.

$G$ and $H$ are two $d$-asc-sets. $G$ is equivalent $H$, denoted by $G \sim H$, if $G$ and $H$ are both trivial, or $k = l$ and $g_i \sim h_i$.

**Lemma 2.8** Any strictly decreasing sequence of $d$-asc-set

\[ \text{d-asc-set}_1 > \text{d-asc-set}_2 > \cdots \]

is finite.

Let $dps$ be a set of $d$-pol-set. All of $d$-asc-sets contained in $dps$ form a set, and from Lemma 2.8, the set has the lowest elements. The lowest $d$-asc-sets of $dps$ are called differential basic ascent set (abbr. $d$-bas-asc-set) of $dps$, and any two $d$-bas-asc-sets of $dps$ are equivalent.

**Theorem 2.9** The differential ascent set $G$ is the $d$-bas-asc-set of $d$-pol-set $dps$, iff $dps \setminus G$ does not contain nonzero $d$-pol which reduced w.r.t $G$.

Theorem 2.9 also provides an algorithm to construct the $d$-bas-asc-set of $d$-pol-set. Let $dps$ be a $d$-pol-set. If $dps$ contains trivial nonzero $d$-pol $f$, the $f$ is $d$-bas-asc-set, or a $d$-pol is taken from the lowest polynomials of $dps$, and denoted by $g_1$. If $dps \{g_1\}$ contains the $d$-pols reduced w.r.t. $g_1$, one of the lowest ranks can be taken from $dps \{g_1\}$, denoted by $g_2$. Obviously, $g_1 < g_2$ is a $d$-asc-set. If $dps \setminus \{g_1, g_2\}$ contains the $d$-pols reduced w.r.t. $g_1, g_2$, one of the lowest rank is taken from them, denoted by $g_3$. Then, $g_1 < g_2 < g_3$ is a $d$-asc-set. Continuing, the $d$-asc-set

\[ G : g_1 < g_2 < \cdots < g_k \]

is obtained, and $dps \setminus G$ does not contain the $d$-pol reduced w.r.t $G$. So $G$ is a $d$-bas-asc-set of $dps$.

**Definition 2.10** Let $dps$ and $dqs$ be two differential polynomial sets. $dps$ is larger than $dqs$, denoted by $dps > dqs$, if the $d$-bas-asc-set of $dps$ is larger than $d$-bas-asc-set of $dqs$.

From Lemma 2.8, it is easy to prove the following lemma.

**Lemma 2.11** Any strictly decreasing sequence of $d$-pol-sets in order

\[ dps_1 > dps_2 > \cdots \]

is finite.

Definition 2.7 also provides a way for lowering the order of a $d$-pol-set:

**Theorem 2.12** Let $G$ be a $d$-bas-asc-set of $d$-pol-set $dps$, and $d$-pol-set $dqs$ be reduced w.r.t. $G$. Then $dps \cup dqs < dps$.

### 2.2. Differential division with remainder

Differential division with remainder (also called differential pseudo division with remainder)
The application of d-char-set method to PDO and Lax representation

is the foundation of reduction of d-pol-set, including ordinary differential equation system (abbr. ODEs), and partial differential equation system (abbr. PDEs).

**Theorem 2.13** Let $f$ and $g$ be two d-pols, $g$ be non-trivial. Then the following formula can be obtained in finite steps

$$Jf = \sum_{\alpha} h_{\alpha}D_{\alpha}g + r, \quad (2.4)$$

where $h_{\alpha}$ and $r$ are all d-pols, and $J$ is the power product of Ini($g$) and Sep($g$). $r$ is zero polynomial or reduced w.r.t. $g$. $r$ is called the differential remainder (abbr. d-remainder) of $f$ w.r.t. $g$, denoted by Red($f, g$), and (2.4) is called d-remainder formula.

**Proof** Let $ld(g) = D_{\sigma}u^p$, and $I = \text{Ini}(g)$, $S = \text{Sep}(g)$, and

$$M_{\beta}^p(f) = \{ \beta | D_{\beta}u^p \text{ occurs in } f \},$$

where $D_{\beta}u^p$ is the principal derivative of $g$, i.e., the proper derivative of $D_{\sigma}u^p$. It is discussed in three cases:

1. If $M_{\beta}^p(f) = \emptyset$, and deg($g$) is higher than the degree of $D_{\sigma}u^p$ in $f$, $f$ is reduced w.r.t. $g$;
2. If $M_{\beta}^p(f) \neq \emptyset$, we might as well let $\gamma$ be the largest multi-index, and $\gamma - \sigma \in \mathbb{N}^n$. Thus $D_{\gamma}u^p$ is the leading derivative of $D_{\gamma-\sigma}(g)$, and $S$ is the initial of $D_{\gamma-\sigma}(g)$, i.e., separant of $g$. $f$ can be treated as a polynomial in indeterminate $D_{\gamma}u^p$, according to Euclidean algorithm, in finite steps, the following reduction can be reached

$$S' \cdot f = h_1D_{\gamma-\sigma}(g) + f_1. \quad (2.5)$$

From (2.3), $D_{\gamma}u^p$ does not occur in $f_1$, and the rank of $u^p$ in $f_1$ is lower than in $f$. If $M_{\beta}^p(f_1) = \emptyset$, repeating the above process to $f_1$. Continuing the process, until $M_{\beta}^p(f_k) = \emptyset$, and one can get

$$S^c \cdot f = h_kD_{\gamma-\sigma}(g) + f_k. \quad (2.6)$$

3. If deg($g$) is lower than the degree of $D_{\sigma}u^p$ in $f_k$, $f_k$ can be likewise treated as a polynomial in indeterminate $D_{\sigma}u^p$. Using Euclidean algorithm again, the following formula

$$I^d \cdot f_k = hg + r \quad (2.7)$$

can be derived, and the degree of $D_{\sigma}u^p$ in $r$ is lower than deg($g$).

From (2.6) and (2.7), formula (2.4) can be derived, meanwhile, the algorithm is given out. □

Let $G : g_1 < \cdots < g_{k-1} < g_k$ be a d-asc-set. Successively using d-remainder formula, then

$$r_k = \text{Red}(f, g_k), r_{k-1} = \text{Red}(r_k, g_{k-1}), \ldots, r_1 = \text{Red}(r_2, g_1),$$

and

$$J_1 \cdots J_k f = \sum_{i, \alpha} h_{\alpha}D_{\alpha}(g_i) + r, \quad (2.8)$$

where $J_i$ are the power product of Ini($g_i$) and Sep($g_i$), and $r$ is zero polynomial or reduced w.r.t. $G$. $r$ is called the d-remainder of $f$ w.r.t. $G$, denoted by Red($f, G$).

Let $H = \{ h_1, \ldots, h_t \}$ be a d-pol-set, and $G : g_1 < \cdots < g_k$ be a d-asc-set. The set $\{ \text{Red}(h_1, G), \ldots, \text{Red}(h_k, G) \} \setminus \{ 0 \}$ is called the d-remainder of $H$ w.r.t. $G$, denoted by Red($H, G$).
Definition 2.14 Let $G$ be the $d$-bas-asc-set of $d$-pol-set $dps$. $G$ is said to be the $d$-bas-set of $dps$ if $\text{Red}(dps, G) = \emptyset$.

2.3. Integrability conditions

Integrability conditions include minimal integrability conditions and supplementary integrability conditions. Minimal integrability conditions are the relations among $d$-pol-set, and can be found out directly. Supplementary integrability conditions are the potential relations among $d$-pol-set, and can be obtained through the completion of the $d$-pol-set.

Minimal integrability conditions are the derivation operation must be compatible, i.e., the property (3) of basic derivation operators: the order of derivation operations can be interchanged. For example, $F_x = f, F_y = g$, then

$$F_{xy} = F_{yx} \Rightarrow f_y = g_x.$$ 

The equation $f_y = g_x$ is the integrable condition, and also called compatible condition. Let $f$ and $g$ be two same class $d$-pols, and the leading derivatives be $D_\alpha u^p$ and $D_\beta u^p$, respectively, i.e.,

$$f = f_d D_\alpha u^p + f_1, \ g = g_d D_\beta u^p + g_1.$$ 

Then

$$S(f, g) = \frac{g_d D_{\sigma - \alpha} f - f_d D_{\sigma - \beta} g}{\text{GCD}(f_d, g_d)} \tag{2.9}$$

is the compatible condition of $f$ and $g$, where $\text{GCD}(f_d, g_d)$ is the greatest common divisor of $f_d$ and $g_d$, and $\sigma = (\sigma_1, \ldots, \sigma_n), \ \sigma_i = \max\{\alpha_i, \beta_i\}$. (2.9) is called $S$-polynomial of $f$ and $g$.

Let $G$ be a $d$-asc-set and all be $S$-polynomials of $G$, denoted by $S_{\text{min}}(G)$. Then $\text{Red}(S_{\text{min}}(G), G)$ is the minimal integrability conditions of $G$, denoted by $I_{\text{min}}(G)$.

Supplementary integrability conditions come from the completion of $d$-pol set, which exercises through the completion of the corresponding multi-indices set of leading derivatives. Let $\Gamma$ be a set of multi-indices and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be two multi-indices. The following notations are used:

- $\gamma = (\gamma_1, \ldots, \gamma_n)$ is the bounded element of $\Gamma$, where $\gamma_i = \max\{\alpha_i | \alpha \in \Gamma\}$.
- $b$ is an operator on $\Gamma$ as $b(\Gamma) = \gamma$, and the $i$-th component of $b(\Gamma)$ is $b_i(\Gamma) = \gamma_i$.
- $\delta^{(i)}$ is the vector whose $i$-th component is 1 and the other components are all zeros.
- The Hadamard’s product of $\alpha$ and $\beta$ is

$$\alpha \circ \beta = (\alpha_1 \beta_1, \ldots, \alpha_n \beta_n) \tag{2.10}$$

- $\alpha + \beta$ is called prolongation (multiple) of $\beta$. All of prolongations of $\alpha$ are denoted by

$$M(\alpha) = \{\alpha + \beta | \beta \in \mathbb{N}^n\} \tag{2.11}$$

The prolongation along $\delta$ direction is

$$M(\alpha, \delta) = \{\alpha + \beta \circ \delta | \beta \in \mathbb{N}^n\}$$

- $M(\Gamma) = \bigcup_{\alpha \in \Gamma} M(\alpha)$ is called the prolongation (multiple) space of $\Gamma$. 

In the differential characteristic set method, completion is indispensable, and there are several completion methods, but it is very difficult to find out the minimal complete set of multi-indices set. A less complete set can be obtained through the following method.

**Definition 2.15** For \( \forall \alpha \in \Gamma, \delta(\alpha) \) is the multiplicative direction of \( \alpha \) defined as
\[
\delta_n(\alpha) = \begin{cases} 
1, & \alpha_n = b_n(\Gamma) \\
0, & \text{otherwise}
\end{cases}
\]
\[
\delta_k(\alpha) = \begin{cases} 
1, & \alpha_k = b_k(\Gamma_{\gamma_k+1, \ldots, \gamma_n}) \\
0, & \text{otherwise}
\end{cases}
\]
where \( \Gamma_{\gamma_k+1, \ldots, \gamma_n} = \{ \beta \in \Gamma | \beta_k = k+1, \ldots, \beta_n = n \} \). \( \delta \) is the multiplicative direction of \( \Gamma \).

**Lemma 2.16** For \( \forall \alpha, \beta \in \Gamma, M(\alpha, \delta(\alpha)) \cap M(\beta, \delta(\beta)) = \emptyset. \)

**Definition 2.17** \( \Gamma \) is called the completion set if \( M(\Gamma) = \bigcup_{\alpha \in \Gamma} M(\alpha, \delta(\alpha)) \).

**Theorem 2.18** \( \Gamma \) is completion set iff
\[
\forall \alpha \in \Gamma, \delta_k(\alpha) = 0 \Rightarrow \alpha + \delta^{(k)} \in \bigcup_{\alpha \in \Gamma} M(\alpha, \delta(\alpha))
\]

**Definition 2.19** \( \Gamma' \) is complete, and called the completion set of \( \Gamma \), if \( \Gamma' \supseteq \Gamma \) and \( M(\Gamma') \supseteq M(\Gamma) \).

For a finite multi-indices set \( \Gamma \), the completion set of \( \Gamma \) is not unique, and the intersection of two completion sets may not be completion set. In order to make up for the detective, a special completion set, closed set will be introduced.

**Definition 2.20** Finite multi-indices set \( \Gamma \) is called closed set if
\[
\alpha \in \Gamma, \delta_k(\alpha) = 0 \Rightarrow \alpha + \delta^{(k)} \in \Gamma,
\]
where \( \delta \) is the multiplicative direction of \( \Gamma \).

From Theorem 2.18, closed set must be complete set.

**Theorem 2.21** The intersection of any two closed sets is also closed set.

**Definition 2.22** The minimal closed set containing \( \Gamma \) is called the closure of \( \Gamma \), denoted by \( \overline{\Gamma} \).

Obviously, the closure of the \( \Gamma \) exists and is unique, and is the intersection of all closed sets containing \( \Gamma \). The closure of \( \Gamma \) can be obtained from the following algorithm.

**Completion Algorithm:**

1. Let \( b_n = b_n(\Gamma) \), and step by step, we determine the following sets:
\[
\Gamma_0 = \{ \alpha \in \Gamma | \alpha_n = 0 \}
\]
\[
\Gamma_1 = \{ \alpha \in \Gamma | \alpha_n = 1 \} \cup \{ \alpha + \delta^{(n)} | \alpha \in \Gamma_0 \}
\]
\[
\vdots
\]
\[
\Gamma_n = \{ \alpha \in \Gamma | \alpha_n = b_n \} \cup \{ \alpha + \delta^{(n)} | \alpha \in \Gamma_{b_n-1} \}
\]

2. Assuming that \( \Gamma_{j_{k+1}, \ldots, j_n} \) are all determined, and \( b_{j_{k+1}, \ldots, j_n} = b_{j_{k+1}, \ldots, j_n}(\Gamma_{j_{k+1}, \ldots, j_n}) \), and
step by step, the following sets are determined
\[ \Gamma_{0j_1\ldots j_n} = \{ \alpha \in \Gamma_{j_1\ldots j_n} | \alpha_k = 0 \} \]
\[ \Gamma_{1j_1\ldots j_n} = \{ \alpha \in \Gamma_{j_1\ldots j_n} | \alpha_k = 1 \} \cup \{ \alpha + \delta(\alpha) | \alpha \in \Gamma_{0j_1\ldots j_n} \} \]
\[ \vdots \]
\[ \Gamma_{lj_1\ldots j_n} = \{ \alpha \in \Gamma_{j_1\ldots j_n} | \alpha_k = l \} \cup \{ \alpha + \delta(\alpha) | \alpha \in \Gamma_{l-1,j_1\ldots j_n} \} \]
where \( l = b_{kj_1\ldots j_n} \).

(3) \( \Gamma = \bigcup_{j_1\ldots j_n} \Gamma_{j_1\ldots j_n} \).

**Theorem 2.23** The result \( \Gamma \) of the above algorithm is the closure of \( \Gamma \).

**Proof** Notice the following facts:
\[ (\Gamma)_{j_k\ldots j_n} = \Gamma_{j_k\ldots j_n}, \quad k = 1, 2, \ldots, n. \]
Let \( \delta \) be the multiplicative direction of \( \Gamma \). For \( \alpha \in \Gamma \), \( \alpha_k < b_k(\Gamma_{\alpha_k+1\ldots\alpha_n}) \) if \( \delta(\alpha_k) = 0 \). From the algorithm, \( \alpha + \delta(\alpha_k) \in \Gamma_{\alpha_k+1\ldots\alpha_n} \subseteq \Gamma \). So, \( \Gamma \) is closed set.

Assuming that \( \Gamma' \) is another closed set, and \( \Gamma' \supseteq \Gamma \). Obviously, \( b_n(\Gamma') \geq b_n(\Gamma) \), and \( \Gamma_0 \subseteq \Gamma'_0 \).
\[ \{ \alpha \in \Gamma | 1 \leq \alpha_1 \} \subseteq \Gamma_1' \cdot \Gamma' \text{ is closed set, it means } \Gamma_1' \subseteq \Gamma' \cdot \text{Similarly, it shows that } \Gamma_j' \subseteq \Gamma_j' \cdot \text{step by step. Further, using mathematical induction, } \Gamma_{j_k\ldots j_n} \subseteq \Gamma_{j_k'\ldots j_n} \text{ can be proved. These relations indicate } \Gamma \subseteq \Gamma'. \]
Therefore, \( \Gamma \) is the closure of \( \subseteq \Gamma \).

\( \Gamma \) may not be the minimal completion set of \( \Gamma \), but it is easy to calculate the multiplicative direction, and so is \( \Gamma \). In addition, based on \( \Gamma \), a completion set which is smaller than \( \Gamma \) in size can be obtained from the following algorithm.

**Contraction Algorithm:**

For every \( \Gamma_{j_2\ldots j_n} \), set \( \Gamma_{j_2\ldots j_n} \setminus \{ \alpha \} \), if the following conditions are satisfied:

1. \( \Gamma_{j_2\ldots j_n} \) has two elements at least;
2. \( \alpha = (b_{1j_2\ldots j_n}) \notin \Gamma \).

Repeating the above steps until at least one condition is not satisfied. \( \Gamma^* = \bigcup_{j_2\ldots j_n} \Gamma_{j_2\ldots j_n} \) is also a completion set of \( \Gamma \), called modified completion set. Generally, the number of elements in \( \Gamma^* \) is less than \( \Gamma \).

The main aim of completion of multi-indices is to complete the corresponding \( d \)-pol set, and supplementary integrability conditions can be offered. In the following algorithm, supplementary integrability conditions of \( d \)-pol-set set \( G \) are determined.

**Supplementary integrability conditions Algorithm:**

1. \( G \) is separated into a disjoint union of \( G_1, \ldots, G_m \), where \( G_p = \{ g \in G | \text{cl}(g) = p \} \), then \( G = \bigcup_{p=1}^m G_p \) is a classification of \( G \).
2. Let \( G_p = \{ g_1, \ldots, g_k \} \), and the corresponding multi-indices set be \( \Gamma_p = \{ \alpha^1, \ldots, \alpha^k \} \).

\( \Gamma_p \) is completed through the above Completion Algorithm and Contraction Algorithm, while \( \overline{\Gamma}^i \) is the all prolongations of \( \alpha^i \), then \( \overline{\Gamma}_p = \bigcup_{i=1}^k \overline{\Gamma}^i \) is the completion set of \( \Gamma_p \), and \( \overline{\Gamma}^i \cap \overline{\Gamma}^j = \emptyset \) if \( i \neq j \), meanwhile it forms the \( d \)-pol-set \( \overline{\Gamma}_i = \{ D_{\beta - \alpha}| \beta \in \overline{\Gamma}^i \} \), and \( \overline{\Gamma}_p = \bigcup_{i=1}^k \overline{\Gamma}_i \).
(3) \( \{ S(f, g) | \forall f \in \mathcal{F}, g \in \mathcal{G}, i \neq j \} \) is the collection of \( S \)-polynomials of \( \mathcal{G} \), and \( S(\mathcal{G}) \) is the collection of \( S \)-polynomials of \( \mathcal{G} \).

(4) \( I_{\text{supp}} = \text{Red}(S(\mathcal{G}), G) \) is the supplementary conditions of \( G \).

**Definition 2.24** A non-trivial \( d \)-asc-set \( G \) is said to be passive, if \( I_{\text{min}}(G) \) and \( I_{\text{supp}}(G) \) are both empty.

### 2.4. Differential characteristic set

Based on the above preparing, the main result is given.

**Theorem 2.25** Let \( \text{dps} \) be a finite \( d \)-pol system. Then there exists a passive \( d \)-bas-set \( \text{dcs} \) of \( \text{dps} \), called differential characteristic set (abbr. \( d \)-char-set) of \( \text{dps} \), s.t.,

\[
d \text{-Zero}(\text{dcs}/J) \subset d \text{-Zero}(\text{dps}) \subset d \text{-Zero}(\text{dcs}),
\]

\[
d \text{-Zero}(\text{dps}) = d \text{-Zero}(\text{dcs}/J) + \sum_i d \text{-Zero}(\text{dps}_i^1) + \sum_i d \text{-Zero}(\text{dps}_i^\prime),
\]

where \( d \text{-Zero}(\text{dps}) \) are all Zero points of \( \text{dps} \) in field \( \mathbb{F} \). \( J \) is the product of initials and separants of \( \text{dcs} \). \( d \text{-Zero}(\text{dcs}/J) \) are all Zero points of \( \text{dps} \) which are not Zero points of \( J \). \( \text{dps}_i^1 \) and \( \text{dps}_i^\prime \) are the enlarged \( d \)-pol-set obtained from the initials and separants of \( \text{dcs} \).

**Proof** Let \( \text{dps}_0 = \text{dps} \). A \( d \)-bas-asc-set \( \text{dbs}_0 \) of \( \text{dps}_0 \) is selected, and let \( r s_0 = \text{Red}((\text{dps}_0 \setminus \text{dbs}_0), \text{dbs}_0) \). If \( r s_0 = \emptyset \), \( \text{dbs}_0 \) is the \( d \)-bas-set of \( \text{dps} \), otherwise, let \( \text{dps}_0 = \text{dps}_0 \cup r s_0 \). Selecting \( \text{dbs}_0 \) of \( \text{dps}_0 \), and calculating \( r s_0 = \text{Red}((\text{dps}_0 \setminus \text{dbs}_0), \text{dbs}_0) \) again. Repeating the above procedure until \( r s_0 = \emptyset \).

Let \( I_{\text{min}}^0 = I_{\text{min}}(\text{dbs}_0) \) be the minimal integrability conditions of \( \text{dbs}_0 \). If \( I_{\text{min}}^0 \neq \emptyset \), let \( \text{dps}_0 = \text{dps}_0 \cup I_{\text{min}}^0 \), and selecting \( \text{dbs}_0 \) of \( \text{dps}_0 \) and calculating \( I_{\text{min}}(\text{dbs}_0) \) of \( \text{dbs}_0 \) again, and repeating the above procedure until \( I_{\text{min}}^0 = \emptyset \), otherwise, calculating supplementary integrability conditions \( I_{\text{supp}}^0 = I_{\text{supp}}(\text{dbs}_0) \) of \( \text{dbs}_0 \).

If \( I_{\text{supp}}^0 = \emptyset \), \( \text{dbs}_0 = \text{dcs} \) is the \( d \)-char-set of \( \text{dps} \), otherwise, let \( \text{dps}_1 = \text{dps}_0 \cup I_{\text{supp}}^0 \). Selecting the \( d \)-bas-asc-set \( \text{dbs}_1 \) of \( \text{dps}_1 \), and calculating minimal integrability conditions \( I_{\text{min}}^1 \) and supplementary integrability conditions \( I_{\text{supp}}^1 \) of \( \text{dbs}_1 \). If \( I_{\text{supp}}^1 = \emptyset \), \( \text{dbs}_1 = \text{dcs} \), otherwise, let \( \text{dps}_2 = \text{dps}_1 \cup I_{\text{supp}}^1 \). The series \( d \)-pol-sets: \( \text{dps}_i, \text{dbs}_i, \text{rs}_i, I_{\text{min}}^i, I_{\text{supp}}^i \) are gained. Let \( \text{RIS}_i = r s_i \cup I_{\text{min}}^i \cup I_{\text{supp}}^i \). Then, there exist following relations:

\[
\text{dps}_i = \text{dps}_{i-1} \cup \text{RIS}_{i-1}
\]

\[
\text{dbs}_0 > \text{dbs}_1 > \cdots > \text{dbs}_{i-1} > \text{dbs}_i > \cdots
\]

From Lemma 2.8, the series in (2.17) should be ended at a certain stage \( k \), that is in finite steps \( \text{dbs}_k = \text{dcs} \) is a passive \( d \)-bas-set, and \( \text{RIS}_k = \emptyset \).

According to the above-mentioned construction process of \( d \)-char-set, we see that

\[
d \text{-Zero}(\text{dps}) = d \text{-Zero}(\text{dps}_0) = \cdots = d \text{-Zero}(\text{dps}_k)
\]
From the $d$-remainder formula, and the emptiness of $RIS_k$, we have
\[
\text{d-Zero}(dps) = \text{d-Zero}(dps_k/J) + \sum d\text{-Zero}(dps'_{ki}) + \sum d\text{-Zero}(dps''_{ki}), \tag{2.19}
\]
where each $dps'_{ki}$ and $dps''_{ki}$ is the enlarged $d$-plo-set derived from $dps_k$ by adjoining to its initials and separants of $dcs$. For each $i$, it is clear from (2.18) that we have
\[
\text{d-Zero}(dps'_{ki}) = \text{d-Zero}(dps_k), \quad \text{d-Zero}(dps''_{ki}) = \text{d-Zero}(dps_k). \tag{2.20}
\]

The formula (2.15) can be obtained from (2.18)–(2.20). The formula (2.14) is also obtained from the construction immediately.

In the above deducing procedure, $d$-char-set $dcs$ can be formed in the following scheme:
\[
dps = \dps_0 \subset \dps_1 \subset \cdots \subset \dps_k \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\
dbs_0 > dbs_1 > \cdots > dbs_k = dcs \quad \text{(W)} \\
\downarrow \quad \downarrow \quad \cdots \quad \downarrow \\
RIS_0 \uparrow RIS_1 \uparrow \cdots \uparrow RIS_k = \emptyset
\]
(W) is called Wu-Ritt process. The $d$-pol-sets $dps'_{i}$ and $dps''_{i}$ in (2.15) can be further split into the sum of $d$-char-set as in (2.15). The procedure needs to be terminated in finite steps since all $dps'_{i}$ and $dps''_{i}$ are lower than $dps$. So the following theorem can be obtained finally.

**Theorem 2.26** (zero decomposition theorem) There exists an algorithmic procedure which permits us to give for any finite $d$-plo-set $dps$ a decomposition of the form:
\[
d\text{-Zero}(dps) = \sum d\text{-Zero}(dcs_i/J_i),
\]
where $dcs_i$ is passive $d$-char-set, and $J_i$ is the multiplication of initials and separants of $dcs_i$.

The $d$-char-set is in triangular or trapezoid form. It is important for many problems which contain $d$-pols. For ODEs, there are no integrability conditions, then $d$-bas-set is $d$-char-set.

3. Pseudo differential operators

In the subsequent sections, $D$ denotes $\frac{\partial}{\partial x}$, and prime denotes the differential in $x$. For any function $f(x)$, $Df$ is regarded as an operator as follows:
\[
Df = f' + fD, \tag{3.1}
\]
and for function $g(x)$,
\[
Df(g) = (f' + fD)g = f'g + fg'. \tag{3.2}
\]
So,
\[
D^n f = \sum_{i=0}^{n} \frac{n(n-1)\cdots(n-i+1)}{i!} f^{(i)} D^{n-i}. \tag{3.3}
\]

$D^{-1}$ is the inverse of $D$ and defined as
\[
DD^{-1} = D^{-1} D = 1. \tag{3.4}
\]
The application of $d$-char-set method to PDO and Lax representation

$D^{-1}$ can be understood as a formal integration symbol, and for a function $f(x)$,

$$D^{-1} f = f D^{-1} - f' D^{-2} + f'' D^{-3} + \cdots,$$

(3.5)

and

$$D^{-n} f = \sum_{i=0}^{\infty} \frac{(-n)(-n-1)\cdots(-n-i+1)}{i!} f^{(i)} D^{-n-i}, \quad n > 0. $$

(3.6)

From (3.3) and (3.6), the Leibniz rule is employed as

$$D^n f = \sum_{i=0}^{\infty} \frac{n(n-1)\cdots(n-i+1)}{i!} f^{(i)} D^{n-i}, \quad n \in \mathbb{Z}. $$

(3.7)

A PDO of order $n$ is defined as an infinite series

$$U = \sum_{i=-\infty}^{n} u_i D^i, $$

(3.8)

where coefficients $u_i$ are differential polynomials. According to the operations of series and (3.7), one can define the operations between PDO and PDO, between differential operators and PDO, and between PDO and functions. The multiplication of PDOs $U$ and $V$ can be treated as a composition of operators, i.e., for any function $f$,

$$(UV)f = U(Vf) = (U \circ V)f.$$ (3.9)

The multiplication is non-commutative, but for arbitrary integers $m$ and $n$,

$$D^m \circ D^n = D^n D^m = D^{n+m} = D^m \circ D^n.$$ (3.10)

Let $V = \sum_{j=-\infty}^{m} v_j D^j$. Then the multiplication is

$$UV = \sum_{k=0}^{\infty} \left[ \sum_{i=0}^{k} \sum_{j=0}^{k-i} e_{n-j}^{i-j} u_j v_i^{(k-i-j)} \right] D^{m+n-k}. $$

(3.11)

Based on multiplication, the power, the inverse, and the extraction of a root of PDOs can be defined, and composed of non-commutative algebra [1], and play a very important role in integrable systems. The concrete calculation of the inverse and the extraction of a root of PDOs is presented utilizing differential characteristic set method in Section 4.

Given a PDO, its positive part and negative part are shown as follows

$$(U)_+ = \sum_{i=0}^{n} u_i D^i, $$

(3.12)

$$(U)_- = \sum_{i=-\infty}^{-1} u_i D^i, $$

(3.13)

and its residue is defined as the coefficient of $D^{-1}$,

$$\text{Res}(U) = u_{-1}. $$

(3.14)

$(U)_+$ is a normal linear differential operator. From residue, the properties of first integrals, conservation laws, etc., of a nonlinear evolution equation can be deduced.
4. Lax representation

In integrable systems, according to Lax scheme [1, 8], if evolution equation
\[
\frac{\partial u}{\partial t} = K(u),
\]  
(4.1)
can be represented as the compatibility conditions of equations
\[
L \psi = \lambda \psi, \quad \frac{\partial \psi}{\partial t} = B \psi, \quad \frac{\partial \lambda}{\partial t} = 0,
\]
(4.2a, 4.2b, 4.2c)
then the equation (4.1) is integrable, where \( L \) and \( B \) are the differential operators with coefficients being differential polynomials about \( u \). Differentiating (4.2a) w.r.t. \( t \), and substituting into (4.2b), the famous Lax equation
\[
\frac{\partial L}{\partial t} = [B, L],
\]
(4.3)
is yielded. (4.3) is called the Lax representation and \( L \) and \( B \) are called Lax pair of the evolution equation (4.1). Using the inverse scattering method, the explicit solution can be derived.

For a given equation, however, unfortunately, there exists no general method to construct its Lax pair. Starting with a linear differential operator \( L \), PDOs provides a simple way to construct a whole hierarchy of integrable evolution equations. Firstly, \( 1+1 \) dimensional systems are presented.

Let \( L \) be a PDO of order \( n \)
\[
L = D^n + \sum_{i=0}^{n-2} u_i(x, t)D^i,
\]
(4.4)
where \( t = (t_1, t_2, \ldots) \). Then, the unique \( n \)-th root and the inverse of \( L \) exist in following forms
\[
L^{\frac{1}{n}} = D + \sum_{i=0}^{\infty} v_i(x, t)D^{-i}, \quad \quad L^{-1} = D^{-n} + \sum_{i=0}^{\infty} v_i(x, t)D^{-n-i}.
\]
(4.5, 4.6)
\( L^{\frac{1}{n}} \) and \( L^{-1} \) commute with \( L \). Linear differential operators \( B_m = (L^{\frac{1}{n}})_+ \), \( m = 1, 2, \ldots \) are the partners of \( L \) in Lax pair. From Lax equations
\[
\frac{\partial L}{\partial t_m} = [B_m, L],
\]
(4.7)
a hierarchy is defined, and called Gelfand-Dickey hierarchy. (4.7) is called the generalized Lax equation. Furthermore,
\[
\frac{\partial L_k^{\frac{1}{n}}}{\partial t_m} = [B_m, L_k^{\frac{1}{n}}], \quad k = 1, 2, \ldots
\]
(4.8)
can be derived from (4.7). Then,
\[
J_k = \int \text{Res} L_k^{\frac{1}{n}} dx, \quad k = 1, 2, \ldots
\]
(4.9)
are all first integrals of the \( n \)-th hierarchy, and
\[
\frac{\partial J_k}{\partial t_m} = \int \text{Res}[B_m, L^k] \, dx = 0, \quad k = 1, 2, \ldots.
\]

Using Sato theory [2, 9, 10], this method can be extended to more than 1 + 1 dimensional systems. Let
\[
W = 1 + w_1 D^{-1} + w_2 D^{-2} + w_3 D^{-3} + \cdots
\]
where the coefficients \( w_i, i = 1, 2, \ldots \) are the function of \( x, t = (t_1, t_2, \ldots) \), which are the fields of the theory satisfying nonlinear evolution equations. \( W \) is called “dressing” operator [1, 11, 12]. From the multiplication of PDOs, the inverse of \( W \) exists and is also a PDO, denoted by \( W^{-1} \), i.e.,
\[
WW^{-1} = W^{-1}W = 1.
\]

Based on Cramer Rule and the theory of ordinary differential equation, one can yield the famous Sato equation
\[
\frac{\partial W}{\partial t_n} = B_n W - WB_n,
\]
\[
B_n = (WD^nW^{-1})_+.
\]

The operator \( L \) is defined by
\[
L = WD^{-1},
\]
and
\[
\frac{\partial L}{\partial t_n} = \frac{\partial W}{\partial t_n}DW^{-1} + WD\frac{\partial W^{-1}}{\partial t_n}.
\]

Since \( WW^{-1} = 1 \),
\[
0 = \frac{\partial 1}{\partial t_n} = \frac{\partial (WW^{-1})}{\partial t_n} = \frac{\partial W}{\partial t_n}W^{-1} + W\frac{\partial W^{-1}}{\partial t_n},
\]
we have
\[
\frac{\partial W^{-1}}{\partial t_n} = -W^{-1}\frac{\partial W}{\partial t_n}W^{-1}.
\]

Substituting (4.17) into (4.13a), the generalized Lax equation can be derived
\[
\frac{\partial L}{\partial t_n} = [B_n, L].
\]

For an arbitrary positive integer \( m \),
\[
L^m = (WDW^{-1})(WDW^{-1}) \cdots (WDW^{-1}) = WD^mW^{-1},
\]
therefore, a general form of the generalized Lax equation can be obtained by repeating the above analysis with \( L^m \) instead of \( L \)
\[
\frac{\partial L^m}{\partial t_n} = [B_n, L^m],
\]
and the following equation holds too
\[
\frac{\partial L^n}{\partial t_m} = [B_m, L^n].
\]
The Zakharov-Shabat equation can be derived from (4.20) and (4.21) directly

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m].$$

(4.22)

PDO $L$ and linear differential operator $B$ have more general forms

$$L = u_n D^n + u_{n-1} D^{n-1} + u_1 D + u_0 + u_{-1} D^{-1} + \cdots, \quad B = (L^\infty)^\geq k,$$

(4.23)

and are called “nonstandard cases” [9].

5. Differential characteristic set method for Lax representation

In the construction of Lax representations, differential characteristic set method is mainly applied in two aspects: calculating the inverse or the extraction of a root of PDOs, and reducing the nonlinear PDEs which are yielded from the generalized Lax equation or Zakharov-Shabat equation.

For simplicity, we start with the detailed calculation procedure of the inverse of dress operator $W$. Let

$$W = 1 + w_1 D^{-1} + w_2 D^{-2} + w_3 D^{-3} + \cdots,$$

(5.1)

and the inverse of $W$ be

$$W^{-1} = 1 + v_1 D^{-1} + v_2 D^{-2} + v_3 D^{-3} + \cdots.$$  

(5.2)

$WW^{-1} \equiv 1$ gives

$$1 \equiv (1 + w_1 D^{-1} + w_2 D^{-2} + w_3 D^{-3} + \cdots) \circ (1 + v_1 D^{-1} + v_2 D^{-2} + v_3 D^{-3} + \cdots)$$

$$= 1 + (w_1 + v_1) D^{-1} + (w_3 + w_2 v_1 + w_1 v_2 + v_3 - w_1 v'_1) D^{-2} + \cdots.$$  

(5.3)

From (5.3), the ODEs on $v_i$, $i = 1, 2, \ldots$, denoted by dos, is given

$$w_1 + v_1 = 0,$$

$$w_2 + w_1 v_1 + v_2 = 0,$$

$$w_3 + w_2 v_1 + w_1 v_2 + v_3 - w_1 v'_1 = 0,$$

$$w_4 + w_3 v_1 + w_2 v_2 + w_1 v_3 + v_4 - 2 w_2 v'_1 - w_1 v'_2 = 0,$$

$$\cdots,$$

(5.4)

where $w_i, i = 1, 2, \ldots$ can be treated as known variables. Solving dos and substituting $v_i$ into (5.2), $W^{-1}$ can be obtained. However, the number of equations in dos is infinite, so $W^{-1}$ is not obtained completely. From the last section, $W$ and $W^{-1}$ are used to construct the differential operator $B_m = (WD^m W^{-1})_+$. For a given $m$, $B_m$ is confirmed, and has finite terms. So after a certain term, $(WD^m W^{-1})_+$ is not changed. Therefore, this term can be treated as the order of truncation of $W$ and $W^{-1}$, thus dos has finite equations. Actually, in the calculation process, the minimum order of $W$ and $W^{-1}$ decrease step by step from $m$, until

$$(WD^m W^{-1})_+ = [(1 + w_1 D^{-1} + w_2 D^{-2} + \cdots) \circ D^m \circ (1 + v_1 D^{-1} + v_2 D^{-2} + \cdots)]_+$$

(5.5)

does not change, i.e., and truncation order is the maximum order such that $(WDW^{-1})_+$ does not change.
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The process to solve $dos$ is quite tedious and complicated, though $dos$ is composed of linear ordinary differential equations. Since the equations in $dos$ are all d-pols, differential characteristic set method is used to solve $dos$. Let $v_1 < v_2 < \cdots$. According to (W), the d-char-set $dcs$ of $dos$ can be obtained. Since $dos$ is a system of linear ordinary differential equations, integrability conditions do not exist, and the d-bas-set $dbs$ of $dos$ is $dcs$. We see that $dcs$ is a system of linear ordinary differential equations. In fact, since $dos_0 = dos$ is a linear system, the d-bas-asc-set $dbs_0$ of $dos_0$ is also a linear system, then $v_i$ and its derivatives do not occur in the initials and separatants of $dbs_0$. From formula (2.4), the degree of $v_i$ is 1 in $dcs$, then $v_i$ does not occur in $Red(W^{-1}, dcs)$. Therefore, $Red(W^{-1}, dcs)$ is the desired result, denoted by $W^{-1}$, i.e.,

$$W^{-1} = Red(W^{-1}, dcs).$$

In the process of $W^{-1}$ calculation, $D$ does not participate as a parameter variable.

With the same argument, $B_m$ can be calculated through

$$B_m = Red((WD^nW^{-1})_+, dcs),$$

where $dcs$ is also the d-char-set of (5.4).

Analogously, differential characteristic set method can be applied to the calculations of extraction root of PDOs. Let

$$L^\frac{1}{n} = D + v_1D^{-1} + v_2D^{-2} + \cdots$$

be the $n$-th root of $L$ in (4.4). For a given $m$, the differential operator $B_m = (L^\frac{1}{n})_+$ is determined, so the truncation order of $L^\frac{1}{n}$ is the maximum order s.t.,

$$(L^\frac{1}{n})_+ = [(D + v_1D^{-1} + v_2D^{-2} + \cdots)^m]_+$$

does not change. From

$$L = (L^\frac{1}{n})^n,$$

an ODEs $dos$ on $v_i$, $i = 1, 2, \ldots$ can be given. Utilizing differential characteristic set method, the d-char-set $dcs$ of $dos$ is obtained, then $Red(L^\frac{1}{n}, dcs)$ is the $n$th root of $L$, denoted by $L^\frac{1}{n}$, i.e.,

$$L^\frac{1}{n} = Red(L^\frac{1}{n}, dcs)$$

and $B_m = Red((L^\frac{1}{n})_+, dcs)$ is the desired differential operator.
In the above calculations, using characteristic set method and differential division with remainder, it is not necessary to solve the ODEs and substitute the solutions into \( W^{-1}, L^\pm \) and \( B_m \), so the computational procedure is simplified, and the desired result can be obtained rapidly with the help of computer.

The nonlinear evolution equation is yielded from the generalized Lax equation or Zakharov-Shabat equation. However, in general, it is embarrassed that the desired result cannot be obtained directly. From generalized Lax equation or Zakharov-Shabat equation, nonlinear PDEs are derived. The PDEs are composed of \( d \)-pols and contain some auxiliary fields, even overdetermined for some cases. To get the evolution equation, one needs to eliminate the auxiliary fields and superfluous equations from the PDEs. The usual way is to make use of compatible conditions via observing, so the process not only is complicated, but also requires certain skills, and the workload of calculation is heavy as the number of equations is large.

Since it is a powerful technique to deal with PDEs, differential characteristic set method aims for reducing the nonlinear PDEs derived from the generalized Lax equation or Zakharov-Shabat equation. Based on the given order, constructing the \( d \)-char-set of the PDEs, superfluous equations are eliminated. Since the \( d \)-char-set is in triangular form, choosing a suitable order, including dependent variables order and independent variables order, the first \( d \)-pol of the results is the desired nonlinear evolution equation. Differential characteristic set method provides a general algorithm to handle the PDEs derived from the generalized Lax equation or Zakharov-Shabat equation, and the whole procedure can be performed in computer.

In the computer algebra system MATHEMATICA, programs of differential characteristic set method, PDOs and Lax representation have been compiled. Considering differential characteristic set method based on order, in the program designing, the order is given by macro definition and called by parameter mode, which makes it convenient to choose order according to the requirements.

6. Examples

In this section, some illustrative examples are presented to show that differential characteristic set method is used for Lax representation of nonlinear evolution equations.

Example 6.1 Boussinesq Equation.

Given differential operator

\[
L = D^3 + uD + v,
\]

where \( u = u(x, t), \ v = v(x, t) \) and linear differential operator \( B_2 = (L^\pm)_+ \). The maximum order \( D^{-1} \) is obtained when \( B_2 \) does not change, so assuming the 3-rd root of \( L \) is

\[
L^\frac{1}{3} = D + w_1 D^{-1},
\]

and

\[
B_2 = (L^\frac{1}{3})_+ = D^2 + 2w_1.
\]
From
\[ L = (L^\frac{1}{3})^3, \]  
(6.4)
we have
\[ 3w_1 - u = 0. \]  
(6.5)
The solution of (6.5) is
\[ w_1 = \frac{1}{3}u. \]
Then
\[ L^\frac{1}{3} = D + \frac{1}{3}uD^{-1}, \]  
(6.6)
and
\[ B_2 = (L^\frac{1}{3})_+ = D^2 + \frac{2}{3}u. \]  
(6.7)
Substituting \( L \) and \( B_2 \) into the generalized Lax equations (4.3), we have
\[ u_t - 2v_x + u_{xx} = 0, \]
\[ v_t + \frac{2}{3}uu_x - v_{xx} + \frac{2}{3}u_{xxx} = 0. \]  
(6.8)
Under the order of dependent variables \( u < v \), and the order of independents variables \( x < t \), the \( d \)-char-set of (6.8) is
\[ 4u_x^2 + 3u_{tt} + 4uu_{xx} + u_{xxxx} = 0, \]  
(6.9a)
\[ u_t - 2v_x + u_{xx} = 0, \]  
(6.9b)
\[ 6v_t + 4uu_x - 3u_{xt} + u_{xx} = 0, \]  
(6.9c)
(6.9a) is Boussinesq Equation, and can be written as
\[ u_{tt} = -\frac{1}{3}u^{(4)} - \frac{4}{3}(uu')'. \]  
(6.10)
\textbf{Example 6.2} Kaup-Kupershmidt Equation.

Given differential operator
\[ L = D^3 + 2uD + u', \]  
(6.11)
where \( u = u(x,t_1,t_2,t_3,t_4,t_5) \) and linear differential operator \( B_5 = (L^\frac{1}{3})_+ \). The maximum order \( D^{-4} \) is obtained when \( B_5 \) does not change, so let the 3-rd root of \( L \) be
\[ L^\frac{1}{3} = D + w_1D^{-1} + w_2D^{-2} + w_3D^{-3} + w_4D^{-4}, \]  
(6.12)
and
\[ (L^\frac{1}{3})_+ = D^5 + 5w_1D^3 + (5w_2 + 10w_1')D^2 + (10w_1'' + 5w_3 + 10w_2' + 10w_1')D \]
\[ + 20w_1w_2 + 5w_4 + 20w_1w_1' + 10w_3' + 10w_2'' + 5w_1'''. \]  
(6.13)
From
\[ L = (L^\frac{1}{3})^5, \]  
(6.14)
the following ODEs is derived
\[ 3w_1 - 2u = 0, \]
\[ 3w_2 - u' + 3w_1' = 0, \]
\[ 3w_2^2 + 3w_3 + 3w_2' + w_2'' = 0, \]
\[ 6w_1w_2 + 3w_4 + 3w_3' + w_2'' = 0. \]  \( (6.15) \)

Under the order \( u < w_1 < w_2 < w_3 < w_4 \), the d-char-set \( dcs \) of (6.15) is
\[ 3w_1 - 2u = 0, \]
\[ 3w_2 + u' = 0, \]
\[ 4u^2 + 9w_3 - u'' = 0, \]
\[ 3w_4 - 4uu' = 0. \]  \( (6.16) \)

According to formula (2.8),
\[ B_5 = \text{Red}((L^5), dcs) = D^5 + \frac{10}{3} uD^3 + 5u'D^2 + \left(\frac{20}{9}u^2 + \frac{35}{9}u''\right)D + \frac{20}{9}uu' + \frac{10}{9}u''. \]  \( (6.17) \)

From the generalized Lax equations (4.3), the following overdetermined PDEs is derived,
\[ 40u_xu^2 + 20u_{xxx}u + 2u_{t_5} + 50u_xu_{xx} + 2u_{xxxx} = 0, \]
\[ 20u_{xx}u^2 + 40u_x^2u + 10u_{xxxx}u + 25u_x^2 + u_{xt_5} + 35u_xu_{xxx} + u_{xxxxx} = 0. \]  \( (6.18) \)

There are no integrability conditions and dependent order for (6.18) that contains only one dependent variable \( u \). Under the order of independent variables \( x < t_5 \), the d-char-set of (6.18) is
\[ u_{t_5} + 20u^2u_x + 10uu_{xxx} + 25u_xu_{xx} + u_{xxxx} = 0. \]  \( (6.19) \)

(6.19) is Kaup-Kupershmidt Equation. The 3-rd root of \( L \) can be obtained via \( d \)-remainder of (6.12) w.r.t. (6.16), if it is used in subsequent calculations.

The above examples are both 1 + 1 dimensional systems, as a 2 + 1 dimensional example, KP equation can be considered firstly.

**Example 6.3** KP Equation.

KP equation is a 2 + 1 dimensional system, and is the simplest non-trivial equation of KP hierarchy. The KP hierarchy is generated by the generalized Lax equation (4.18) or Zakharov-Shabat equation (4.22). Now, utilizing differential characteristic set method, the way to get KP is shown concretely. Let PDO
\[ W = 1 + w_1D^{-1} + w_2D^{-2} + w_3D^{-3} + \cdots, \]  \( (6.20) \)

and
\[ W^{-1} = 1 + v_1D^{-1} + v_2D^{-2} + v_3D^{-3} + \cdots. \]  \( (6.21) \)

According to the Sato theory, \( L = WDW^{-1} \), and \( B_m = (L^m)_+ = (WD^mW^{-1})_+ \). As evaluating
The application of d-char-set method to PDO and Lax representation

$m = 2$ and $m = 3$

\[(WD^2W^{-1})_+ = D^2 + (v_1 + w_1)D + v_2 + v_1w_1 + w_2 + 2v_1',\]

\[(WD^3W^{-1})_+ = D^3 + (v_1 + w_1)D^2 + (v_2 + v_1w_1 + w_2 + 3v_1')D^+\]

\[v_3 + v_2w_1 + v_1w_2 + w_3 + 2w_1v_1' + 3v_2' + 3v_1''\]

(6.22)

where the maximum order $D^{-3}$ is obtained when $B_3$ does not change, then the truncation order of $W$ and $W^{-1}$ are both $D^{-3}$, meanwhile $B_2$ does not change too, and

\[
L = D + v_1 + w_1 + (v_2 + v_1w_1 + w_2 + v_1')D^{-1} + (v_3 + v_2w_1 + v_1w_2 + w_3 + v_1')D^{-2} +
\]

\[(v_3w_1 - v_1''w_1 + v_2w_2 + v_1w_3 - w_2v_1' + v_1')D^{-3}.
\]

(6.23)

From (5.3), i.e., $WW^{-1} = 1$, the following ODEs is yielded

\[
w_1 + v_1 = 0,
\]

\[
w_2 + w_1v_1 + v_2 = 0,
\]

\[
w_3 + w_2v_1 + w_1v_2 + v_3 - v_1'w_1 = 0.
\]

(6.24)

Under the order $w_1 < w_2 < w_3 < v_1 < v_2 < v_3$, the d-char-set $dcs_1$ of (6.24) is

\[
v_1 + w_1 = 0,
\]

\[
v_2 - w_1^2 + w_2 = 0,
\]

\[
v_3 + w_1^3 - 2w_2w_1 + w_1'w_1 + w_3 = 0.
\]

(6.25)

Then from (2.8)

\[
B_2 = \text{Red}((WD^2W^{-1})_+, dcs_1) = D^2 - 2w',
\]

(6.26a)

\[
B_3 = \text{Red}((WD^3W^{-1})_+, dcs_1) = D^3 - 3w_1'D + 3w_1w_1' - 3w_2' - 3w_1'';
\]

(6.26b)

\[
L = \text{Red}(L, dcs_1) = D - w_1'D^{-1} + [w_1w_1' - w_2']D^{-2} -
\]

\[
[w_1'w_2^2 - (w_1w_2)' + (w_1')^2 + w_3']D^{-3}.
\]

(6.27)

Let the $L$ in (6.27) have the form

\[
L = D + u_1D^{-1} + u_2D^{-2} + u_3D^{-3}.
\]

(6.28)

The KP equation is derived in terms of $u$'s instead of $w$'s. Comparing (6.27) with (6.28), the relation between $u$'s and $w$'s is

\[
u_1 = -w_1',
\]

\[
u_2 = w_1w_1' - w_2',
\]

\[
u_3 = -w_1'w_1'^2 + w_2'w_1 - (w_1')^2 + w_2w_1' - w_3'.
\]

(6.29)

Under the order $u_1 < u_2 < u_3 < w_1 < w_2 < w_3$, d-char-set $dcs_2$ of (6.29) is

\[
u_1 + u_1' = 0,
\]

\[
u_2 + u_1w_1 + u_2' = 0,
\]

\[
u_1'^2 + w_2u_1 + u_3 + w_2w_1 + u_3' = 0.
\]

(6.30)
Using differential division with remainder, the \( d \)-remainder of (6.26a) and (6.26b) w.r.t. \( dcs_2 \) are \( B_2 \) and \( B_3 \), respectively, then \( B_2 \) and \( B_3 \) are written in terms of \( u \)'s, i.e.,

\[
B_2 = \text{Red}(B_2, dcs_2) = D^2 + 2u_1', \\
B_3 = \text{Red}(B_3, dcs_2) = D^3 + 3u_1'D + 3u_2 + 3u_2'.
\]  

(6.31)

Let \( u_1 = \frac{u}{2}, u_2 = v, t_1 = y, t_2 = t \). Zakharov-Shabat equation (4.22) leads to

\[
-\frac{3}{2}u_y - 6v_x - \frac{3}{2}u_{xx} = 0, \\
u_t - 3v_y + \frac{3}{2}u_{ux} - \frac{3}{2}u_{xy} - 3u_{xx} - \frac{1}{2}u_{xxxx} = 0.
\]  

(6.32)

Under the order \( u < v, x < y < t \), the \( d \)-char-set of (6.32) is

\[
6u_x^2 + 3u_{yy} + 4u_{xt} + 6uu_{xx} + u_{xxxx} = 0, \\
-\frac{3}{2}u_y - 6v_x - \frac{3}{2}u_{xx} = 0, \\
4u_t - 12v_y + 6uu_x - 3u_{xy} + u_{xxx} = 0.
\]  

(6.33)

The first equation in (6.33) is KP equation, and it can be written as

\[
3u_{yy} = (4u_t - 6uu_x - u_{xxx})_x.
\]

or

\[
u_t = \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}D^{-1}u_{yy}.
\]  

(6.34)

KP equation can also be yielded by the generalized Lax equation with \( L \) and \( B_2, B_3 \), but it requires more complicated calculations.

In the generalized Lax equation (4.7), if \( L \) is an infinite series, \( L \) also adopts a truncation form. The coefficient of \( D^{-1} \), i.e., Res plays an important part in integrable systems and PDOs, and has the property (4.10). Thus, the truncation order of \( L \) in (4.7) reaches the maximum when the coefficient of \( D^{-1} \) of Possion bracket \([B_m, L]\) does not change. The next two examples are the cases.

**Example 6.4 mKP Equation.**

Let

\[
L = D + u_0 + u_1D^{-1} + u_2D^{-2} + u_3D^{-3} + \cdots,
\]  

(6.35)

and

\[
B_m = (L^m)_{\geq 1}, \quad m = 1, 2, 3, \ldots.
\]  

(6.36)

\( L \) and \( B_m \) are nonstandard cases. mKP hierarchy can be derived from the generalized Lax equation (4.7) or Zakharov-Shabat equation (4.22).

In (4.7), the truncation order of \( L \) is obtained when the \( D^{-1} \) entry of \([B_m, L]\) does not change. As \( m = 2, 3 \), let \( u_0 = u, u_1 = v, u_2 = w, u_3 = r \) and \( t_1 = y, t_2 = t \)

\[
B_2 = D^2 + 2uD, \\
B_3 = D^3 + 3uD^2 + (3u^2 + 3v + 3u')D.
\]  

(6.37)
In [14] and [15], graded Lie algebra is exhibited for KP hierarchy and mKP hierarchy. The results of the application of d-char-set method to PDO and Lax representation are as follows:

\begin{align}
    u_y - 2uu_x - 2v_x - u_{xx} &= 0, \quad (6.38a) \\
v_y - 2vu_x - 2w_x - 2w_x - v_{xx} &= 0, \quad (6.38b) \\
w_y - 4wu_x - 2wu_x - w_{xx} &= 0, \quad (6.38c)
\end{align}

and

\begin{align}
u_t - 6(wu_x u + wu_x + v + v_x) - 3(u_x u + u_x u_x + u_x v_x + v_x v_x) - v_{xx} &= 0, \quad (6.39a) \\
u_t - 12wu_x u - 9u_x - 6(r_x v_r + w_x + w_{xx}) - 3(w_x u_x^2 + v_x u_x + v_{xx}) + v_{xx} &= 0, \quad (6.39b) \\
u_t - 3ru_x u^2 - 18ru_x u - 3ru_x u - 9u_x r_x - 9u_x = 0. \quad (6.39d)
\end{align}

(6.38a), (6.38b), (6.39a), (6.39b) compose new PDEs of mKP. Under the order $u < v < w < r$ and $x < y < t$, the d-char-set of mKP is

\begin{align}
12(u_x u_x u_x^3 - u_w u_x u_x^2) + 6(6u_x u_x - u_y u_x + u_y u_x) u_t - 12u_x u_x^2 - 3u_y u_x + 4u_x u_x + (3u_y - 4u_x) u_x - u_x u_x - u_x u_x u_x = 0, \\
24(u_x u_x u_x - 12u_x u_x u_x^2 - 12u_y u_x + 8u_x + 24u_y u_x + 6u_y u_x) - 2u_x u_x - (3u_y + 4u_x u_x + 4u_x u_x) u_x + \\
(3u_y + 4u_x u_x + 2u_x u_x + u_x u_x u_x) - u_x u_x u_x = 0, \\
6u_y u_x - 12u_x^2 + 6u_y u_x + 3u_y u_x - 12u_x + 12u_x u_x + u_x u_x = 0, \\
6(u_y + u_x) u_x^2 - 12u_x^2 + (3u_y - 4u_x + 6u_x u_x + u_x u_x) u_x + 6u_x^2 + (2u_x - 6u_y - 6u_x - 3u_y + u_x u_x) u_x = 0, \\
36u_x u_x u_x u_x - 18(8u_x u_x u_x + 3u_x^3 - 3u_y u_x) u_x - (54u_y + 72u_x + 24u_x) u_x u_x = 0, \\
36(3u_y u_x + u_y u_x - 2u_x u_x u_x + 6u_x^2 u_x + 3u_x u_x u_x + 2u_x u_x u_x u_x) + 54u_x u_x^3 + 6u_x u_x + 12u_x - 6u_y u_x + 6r_x - u_x u_x u_x - (12u_x u_x^2 + 9u_y u_x u_x) - 3u_y u_x = 0, \\
12u_x^2 u_x u_y + (6u_x^2 - 3u_y u_x - u_x u_x) u_x + u_x u_x u_x u_x = 0. \quad (6.40)
\end{align}

The first equation in (6.40) is mKP equation. In [9,13], the form of mKP is

\begin{align}
u_t = \frac{1}{4} u_x x + \frac{3}{2} u_x u_x - \frac{3}{2} u_x D^{-1} u_y + \frac{3}{4} D^{-1} u_y = 0. \quad (6.41)
\end{align}

Using $D^2 = 1$, (6.41) can be transformed into the first equation (6.40). mKP equation can also be yielded from Zakharov-Shabat equation (4.22) with $m = 2, n = 3$.

The applications of PDO to KP hierarchy [14] and mKP hierarchy [15] have been discussed. In [14] and [15], graded Lie algebra is exhibited for KP hierarchy and mKP hierarchy. The results.
are the same as (6.34) and (6.41), respectively in form, except for slightly different coefficients. Furthermore, in [14], if
\[ A_m = P(0, \ldots, 0, \frac{1}{3}(-\sqrt{3})^{m-1}) , \]
as \( m = 3 \), the result is exactly (6.34).

**Example 6.5** 2 + 1 dimensional Dym equation.

2 + 1 dimensional Dym hierarchy is also a nonstandard case, and is associated with the following PDO
\[ L = uD + u_0 + u_1 D^{-1} + u_2 D^{-2} + u_3 D^{-3} + u_4 D^{-4} + \cdots . \]  
(6.42)
The differential operators \( B_m \) are taken as
\[ B_m = (L^m)_{\geq 2} . \]  
(6.43)
As \( m = 2, 3 \), let \( u_0 = v, u_1 = w, u_2 = r \) and \( t_1 = y, t_2 = t \). To calculate 2 + 1 dimensional Dym equation, the truncation order of \( L \) is obtained when \( B_3 \) does not change, and
\[ B_2 = (L^2)_{\geq 2} = u^2 D^2, \]
\[ B_3 = (L^3)_{\geq 2} = u^3 D^3 + 3u^2(v + u_x)D^2 . \]  
(6.44)
The following equation system is obtained from Zakharov-Shabat equation
\[
\begin{align*}
(6r_x + 3w_{xx} + 12ru_x + 6(wv)_x + 3uw_x - 3wux_x)u^3 &- (3wu_x^2 + 3w_y - 6vwu_x + 3v_{xy} + u_{xy})u^2 + 2wv - 3v^2u_y + (u_t - 3w)u_x = 0, \\
3v_{xy} + u_{xy} + [2r_t - 6wuy - 6v_{xy} - 3(vu_x)_y - 6u_yv_x + u_{xt} - 2(u_xu_y)_x]u &- u_yu_x^2 + 2wv - 3v^2u_y + (u_t - 3w)u_x = 0, \\
(3r_{xx} + w_{xxx})u^4 &- 6(w - v_x)w_xu^3 + (6v + 15u_x)r_xu^3 + (6r - 2w)u_{xx}u^3 + ![3v_{xx} + (6v + 3u_x)w_{xx} - 2wu_{xxx}]u^3 + [12ru_x^2 - 3ry - 6w^2u_x + 12vuw_x + (6v + 9u_x)uw_x + (6v - 2u_x^2 + 12vu_x)w_x - 3wxy - 8uxu_{xx} - v_{xx}u_x]u^2 + \\
(6v + 9u_x)uw_x + (6v - 2u_x^2 + 12vu_x)w_x - 3wxy - 8uxu_{xx} - v_{xx}u_x &u^2 + 3v_{xy} + u_{xy} - 3u_2v_y - u_xu_yv_x - 2u_yv_x-u_x^2u_yv_x + 2wv + 2wv - 6uwu_x + 3v^2y + \\
u_wu_x - 3u_yv_x - u_yu_xv_x & = 0. 
\end{align*}
\]  
(6.45)
Under the order \( u < v < w < r \) and \( x < y < t \), one can calculate \( d \)-char-set of (6.45). The \( d \)-char-set of (6.45) is so complicated that it is not presented wholly, and only the first equation, i.e., 2 + 1 dimensional Dym equation, is presented
\[
\begin{align*}
(u_{xy}u_{xxxx} - u_{xyy}u_{xxx}) & u^7 + (2u_{xy}u_{xy}u_{xxx} - 2u_xu_{yy}u_{xxx} + 6u_xu_{xy}u_{xxx} + uy_{xxx}u_{xxxx} + \\
u_yu_{xxx} + u_{xxxx})u^6 + 6u_x^2(u_{xy}u_{xxx} - u_yu_{xxxx})u^5 + (-6u_yu_{xxx}u_x^3 - 4u_{xy}u_{xxx} + 4u_{xt}u_{xyy})u^4 + \\
(3u_{xy}u_{yy} - 4uyu_{xx} + 4u_{xy}u_{xy} + 4uyu_{xxx} - 3u_{xyy}u_{xxx} - 4u_{xt}u_{xyy})u^3 + \\
(6u_yu_{xy}^2 - 3uyu_{xyy} + 3uyu_{xyy}u_{xxx})u^2 - 12u_y^2uyu_{xy} + 6uy^2u_x^2 & = 0. 
\end{align*}
\]  
(6.46)
Equation (6.46) can also be derived from the generalized Lax equation. Dym equation has been
discussed in [9, 16]. The results of \( B_2 \) and \( B_3 \) are all the same as (6.44), and it can be verified that (6.46) is identical with 2 + 1 dimensional Dym equation

\[
4u_t = u^3 u_{xxx} - 3 \frac{1}{u} (u^2 D^{-1} \frac{1}{u})_y,
\]

which is presented in [9].

In (6.41) and (6.47), \( D^{-1} = \int f(x) \, dx \) is the integration, and can be transformed into derivative form through \( DD^{-1} = 1 \). So using differential characteristic set method, the Lax representation of nonlinear evolution equations with integration can be obtained too. Although the form of the final result is more complicated, differential characteristic set method is a general and effective way for PDEs, especially for nonlinear PDEs.

7. Conclusion and discussion

Differential characteristic set method can be applied to Lax representation of nonlinear evolution equations. It not only decreases the steps and the burden of calculation, then simplifies the calculations, but also provides a complete procedure for PDOs and the construction of the Lax representation of nonlinear evolution on computer algebra system. Further, based on the procedure, differential characteristic set method can be applied to the research of bi-Hamilton structure, tan-functions, Darboux transformations, and other sides of the integrable systems.

Utilizing differential characteristic set method reduces the nonlinear partial equation systems derived from the generalized Lax equation or Zakharov-Shabat equation. There is a shortage that the results are complicated sometimes. How to simplify the results should be revisited.

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References