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On Generating New $(2+1)$-Dimensional Super Integrable Systems

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Abstract In this paper, we make use of the binormial-residue-representation (BRR) to generate $(2+1)$-dimensional super integrable systems. By using these systems, a new $(2+1)$-dimensional super soliton hierarchy is deduced, which can be reduced to a $(2+1)$-dimensional super nonlinear Schrödinger equation. Especially, two main results are obtained which have important physics applications, one of them is a set of $(2+1)$-dimensional super integrable couplings, the other one is a $(2+1)$-dimensional diffusion equation. Finally, the Hamiltonian structure for the new $(2+1)$-dimensional super hierarchy is produced with the aid of the super trace identity.

Keywords Lie super algebra; $(2+1)$-dimensional super equation; Hamiltonian structure

MR(2010) Subject Classification 35Q51; 37K10

1. Introduction

Seeking new integrable systems has been an important aspect in soliton theory. Different approaches for generating integrable systems have been proposed, such as the results in [1–4]. For the $(1+1)$-dimensional case, Tu has proposed an efficient method for deducing integrable hierarchies of equations and the corresponding Hamiltonian structures [5]. Afterwards, Ma called the method as Tu scheme [6]. From then on, many interesting integrable systems and some corresponding algebraic properties were obtained [7–11]. Actually, the Tu scheme was proposed based on the Lax pair method [12]. Therefore, it is important for us to derive $(2+1)$-dimensional integrable systems by the Lax pair method. For example, Ablowitz et al. discussed the generation on $(1+1)$-dimensional and $(2+1)$-dimensional integrable systems by using the self-dual Yang-Mills hierarchy and its reductions [13], which is actually the Lax pair method. Tu et al. proposed a method for generating $(2+1)$-dimensional hierarchies of evolution equations and their Hamiltonian structures [14], which was called the TAH scheme. However, the TAH scheme did not adopt zero curvature equations to directly derive $(2+1)$-dimensional integrable systems. Therefore, the integrability of the systems obtained could not be ensured. Dorfman et al. proposed a method for generating $(2+1)$-dimensional integrable systems through the construction

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of Hamiltonian operators [15]. Based on this research, Tu et al. presented the BRR method to obtain the $(2+1)$-dimensional AKNS hierarchy [16]. Zhang et al. employed the BRR method to produce a few integrable systems [17].

In this paper, we extend the BRR method to $(2+1)$-dimensional super integrable systems. Firstly, some new (2+1)-dimensional super integrable hierarchies are generated by using the BRR method. Moreover, a new type of (2+1)-dimensional super soliton hierarchy is obtained different from the one in [18] and further reduce it to the $(2+1)$-dimensional super nonlinear Schrödinger equation. In particular, the super integrable hierarchy can be reduced to two different $(2+1)$-dimensional super integrable couplings, which are new findings. Another prominent result is a $(2+1)$-dimensional diffusion equation by reducing the $(2+1)$-dimensional integrable couplings. Furthermore, the super trace identity over the corresponding loop super algebra is used to furnish the super Hamiltonian structure for the $(2+1)$-dimensional super soliton hierarchy.

2. The Lie super algebra $\mathrm{sl}(m/n)$

The definition of the Lie super algebra $\mathrm{sl}(m/n)$ is given by [19]

$$\mathrm{sl}(m/n)=\left(\ X=\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \mathrm{str}\, X = TrA - TrD = 0\ \right), \tag{2.1}$$

where A is $m\times m$ matrix, B is $m\times n$ matrix, C is $n\times m$ matrix and D is $n\times n$ matrix.

The Lie bracket of $\mathrm{sl}(m/n)$ is denoted by

$$s[X,Y]=XY-(-1)^{P(X)P(Y)}YX,\quad \forall\, X,Y\in \mathrm{sl}(m/n), \tag{2.2}$$

with the degration of the element X defined as

$$P(X)=\begin{cases} 0, X=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, & \mathrm{str}\, X=0, \\ 1, X=\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \end{cases} \tag{2.3}$$

The matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ is called the bosonic or the even element, while the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ is called the fermionic or called the odd element. Hu introduced different bases of the Lie super algebra sl(m/n) and presented the super trace identity in 1997 (see [19]), but he did not give its rigorous proof. Recently, Ma gave a systematic proof of the super trace identity and the expression of its constant γ (see [20]). For application, the super Hamiltonian structures of the super AKNS hierarchy and the super Dirac hierarchy have been obtained. Geng and Wu constructed a new super KdV equation by using the Lie super algebra as stated [21]. Zhang and Rui adopted the special Lie super algebra to obtain some super integrable systems [22]. Wei and Xia started from a 3×3 spectral problem to obtain a new six-component super soliton hierarchy and its super Hamiltonian structure [23]. It also has been generalized to a super case for studying super integrable couplings [24–26]. All the results above are $(1+1)$-dimensional super integrable systems, except for the $(2+1)$-dimensional super soliton hierarchy in [22] obtained by the TAH scheme. However, we cannot determine the integrability of this $(2 + 1)$-dimensional soliton hierarchy. In this paper, we introduce a loop super algebra sl(2/1) and deduce a new $(2 + 1)$-dimensional super integrable hierarchy by using the BRR method, which can be reduced to the $(2 + 1)$-dimensional super nonlinear Schrödinger equation. Moreover, two $(2 + 1)$-dimensional super integrable couplings of the $(2 + 1)$-dimensional super nonlinear Schrödinger equation are derived.

We brief overview of the BRR method.

(1) The loop super algebra $\tilde{\mathrm{sl}}(2/1)$ is defined by

$$\tilde{\mathrm{sl}}(2/1) = \{X(u, \lambda + \xi) = e_0(\lambda + \xi) + u_1 e_1(\lambda + \xi) + \cdots + u_p e_p(\lambda + \xi)\}, \tag{2.4}$$

where $e_0, e_1, \ldots, e_p$ constitute a set of basis for the Lie super algebra sl(2/1), and $u = (u_1, u_2, \ldots, u_p)$ is the potential function. The ξ means an operator defined by

$$\xi f = f\xi + f_y, \quad \forall f \in \tilde{\mathrm{sl}}(2/1). \tag{2.5}$$

(2) Fix a couple of matrices

$$\begin{aligned} U &= e_0(\lambda + \xi) + u_1 e_1(0) + \cdots + u_p e_p(0), \\ V &= \sum_{m \geq 0} V_{i,m} \lambda^{-m}. \end{aligned} \tag{2.6}$$

Solve the stationary matrix equation

$$V_x = [U, V], \tag{2.7}$$

and obtain a recurrence operator Ψ.

(3) Construct a sequence of matrices $V^{(n)}$ so that

$$V_x^{(n)} - [U, V^{(n)}] \in Ce_1 + \cdots + Ce_p, \tag{2.8}$$

where C stands for a set of complex numbers.

(4) The zero curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{2.9}$$

generates the super integrable hierarchy, especially, the $(2 + 1)$-dimensional super integrable hierarchy.

(5) Rewrite the integrable hierarchy as

$$U_{t_n} = J\Psi^n b_0, \tag{2.10}$$

where J and Ψ are integro-differential operators.

(6) According to the idea given by Magri et al. [27] employ multiple seeds to obtain a hierarchy of the form

$$U_{t_n} = \sum_{i=0}^{n} \binom{n}{i} \Phi^{n-i} J\Psi \partial_y^i, \tag{2.11}$$

where the operator Φ is defined by

$$J\Psi = \Phi J. \tag{2.12}$$

The above procedure is the BRR method. However, it only applies to generating $(1+1)$-dimensional super integrable systems. In the following section, we extend this method to the case of $(2+1)$-dimensional super integrable hierarchies.

3. A new $(2+1)$-dimensional super integrable hierarchy

In this section, we apply the loop algebra $\tilde{\mathrm{sl}(2/1)}$ with operator matrices of the Lie super algebra sl(2/1) to derive a new $(2+1)$-dimensional super integrable hierarchy through employing the BRR method.

We consider the following matrix spectral problem

$$\begin{cases} \varphi_x = U\varphi, \\ \varphi_t = V\varphi, \end{cases} \tag{3.1}$$

where

$$U = \frac{1}{2}\begin{pmatrix} \lambda^{-1}+\xi & u_1+u_2 & u_3 \\ u_1-u_2 & -\lambda^{-1}-\xi & u_4 \\ u_6 & u_5 & 0 \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} V_1 & V_2 & V_3 \\ V_4 & V_5 & V_6 \\ V_7 & V_8 & V_9 \end{pmatrix},$$

with λ being a spectral parameter, λ, u_1, u_2 being bosonic, and u_3, u_4, u_5, u_6 being fermionic in U. The V_1, V_2, V_4, V_5, V_9 are even, and the V_3, V_6, V_7, V_8 are odd.

Starting from the stationary zero curvature equation

$$V_x = [U, V], \tag{3.2}$$

we have

$$\begin{cases} 2V_{1x} = V_{1y} + (u_1+u_2)V_4 + u_3V_7 - V_2(u_1-u_2) - V_3u_6, \\ 2V_{2x} = 2\lambda^{-1}V_2 + 2V_2\xi + V_{2y} + (u_1+u_2)V_5 + u_3V_8 - V_1(u_1+u_2) - V_3u_5, \\ 2V_{3x} = \lambda^{-1}V_3 + V_3\xi + V_{3y} + (u_1+u_2)V_6 + u_3V_9 - V_1u_3 - V_2u_4, \\ 2V_{4x} = -2\lambda^{-1}V_4 - 2V_4\xi - V_{4y} + (u_1-u_2)V_1 + u_4V_7 - V_5(u_1-u_2) - V_6u_6, \\ 2V_{5x} = -V_{5y} + (u_1-u_2)V_2 + u_4V_8 - V_4(u_1+u_2) - V_6u_5, \\ 2V_{6x} = -\lambda^{-1}V_6 - V_6\xi - V_{6y} + (u_1-u_2)V_3 + u_4V_9 - V_4u_3 - V_5u_4, \\ 2V_{7x} = -V_7\lambda^{-1} - V_7\xi + u_6V_1 + u_5V_4 - V_8(u_1-u_2) - V_9u_6, \\ 2V_{8x} = V_8\lambda^{-1} + V_8\xi + u_6V_2 + u_5V_5 - V_7(u_1+u_2) - V_9u_5, \\ 2V_{9x} = u_6V_3 + u_5V_6 - V_7u_3 - V_8u_4. \end{cases} \tag{3.3}$$

Set

$$V_i = \sum_{m\geq 0} V_{i,m}\lambda^m, \quad i = 1,2,\ldots,9, \partial_\pm = 2\partial_x \pm \partial_y, \tag{3.4}$$

it is easy to calculate

$$\begin{cases} \partial_- V_{1,m} = (u_1+u_2)V_{4,m} + u_3V_{7,m} - V_{2,m}(u_1-u_2) - V_{3,m}u_6, \\ 2V_{2,m+1} = 2(V_{2,m})_x - 2V_{2,m}\xi - (V_{2,m})_y - (u_1+u_2)V_{5,m} - u_3V_{8,m} + V_{1,m}(u_1+u_2) + V_{3,m}u_5, \\ V_{3,m+1} = 2(V_{3,m})_x - V_{3,m}\xi - (V_{3,m})_y - (u_1+u_2)V_{6,m} - u_3V_{9,m} + V_{1,m}u_3 + V_{2,m}u_4, \\ 2V_{4,m+1} = -2(V_{4,m})_x - 2V_{4,m}\xi - (V_{4,m})_y + (u_1-u_2)V_{1,m} + u_4V_{7,m} - V_{5,m}(u_1-u_2) - V_{6,m}u_6, \\ \partial_+ V_{5,m} = (u_1-u_2)V_{2,m} + u_4V_{8,m} - V_{4,m}(u_1+u_2) - V_{6,m}u_5, \\ V_{6,m+1} = -2(V_{6,m})_x - V_{6,m}\xi - (V_{6,m})_y + (u_1-u_2)V_{3,m} + u_4V_{9,m} - V_{4,m}u_3 - V_{5,m}u_4, \\ V_{7,m+1} = -2(V_{7,m})_x - V_{7,m}\xi + u_6V_{1,m} + u_5V_{4,m} - V_{8,m}(u_1-u_2) - V_{9,m}u_6, \\ V_{8,m+1} = 2(V_{8,m})_x - V_{8,m}\xi - u_6V_{2,m} - u_5V_{5,m} + V_{7,m}(u_1+u_2) + V_{9,m}u_5, \\ 2(V_{9,m})_x = u_6V_{3,m} + u_5V_{6,m} - V_{7,m}u_3 - V_{8,m}u_4. \end{cases} \tag{3.5}$$

Denoting

$$V_-^{(n)} = \sum_{i=1}^{n} V_{i,m}\lambda^m,\ V_{i,m} = \begin{pmatrix} V_{1,im} & V_{2,im} & V_{3,im} \\ V_{4,im} & V_{5,im} & V_{6,im} \\ V_{7,im} & V_{8,im} & V_{9,im} \end{pmatrix}, \tag{3.6}$$

then we have

$$-V_{-x}^{(n)} + [U, V_-^{(n)}] = \begin{pmatrix} 0 & -\frac{1}{2}V_{2,n+1} & -\frac{1}{4}V_{3,n+1} \\ \frac{1}{2}V_{4,n+1} & 0 & \frac{1}{4}V_{6,n+1} \\ \frac{1}{4}V_{7,n+1} & -\frac{1}{4}V_{8,n+1} & 0 \end{pmatrix}. \tag{3.7}$$

Take $V^{(n)} = V_-^{(n)}$, then (2.9) is satisfied by

$$\begin{aligned} u_{t_n} =& [u_1, u_2, u_3, u_4, u_5, u_6]^T_{t_n} \\ =& [\frac{1}{4}(V_{2,n+1} - V_{4,n+1}), \frac{1}{4}(V_{2,n+1} + V_{4,n+1}), \frac{1}{4}V_{3,n+1}, \\ & -\frac{1}{4}V_{6,n+1}, \frac{1}{4}V_{8,n+1}, -\frac{1}{4}V_{7,n+1}]^T_{t_n} \\ =& \frac{1}{4}\sigma(V_{2,n+1} - V_{4,n+1}, -V_{2,n+1} - V_{4,n+1}, V_{3,n+1}, V_{6,n+1}, V_{8,n+1}, V_{7,n+1})^T_{t_n}, \end{aligned} \tag{3.8}$$

where

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}. \tag{3.9}$$

Next, we write the $(2+1)$-dimensional super integrable hierarchy as a binormial-residue representation by using (2.11). Denote

$$V_{j0}=\sigma\xi^j, j=0,1,2,\ldots, V_{ji}=\begin{pmatrix} V_{1,ji} & V_{2,ji} & V_{3,ji}\\ V_{4,ji} & V_{5,ji} & V_{6,ji}\\ V_{7,ji} & V_{8,ji} & V_{9,ji}\end{pmatrix}. \tag{3.10}$$

Case 1 Choose the initial data

$$V_{1,00}=1, V_{5,00}=-1, V_{2,00}=V_{3,00}=V_{4,00}=V_{6,00}=V_{7,00}=V_{8,00}=V_{9,00}=0, \tag{3.11}$$

we can compute from (3.5) that

$$\begin{aligned}
V_{2,01}=&u_1+u_2, V_{3,01}=u_3, V_{4,01}=u_1-u_2, V_{6,01}=u_4, V_{7,01}=u_6, V_{8,01}=u_5,\\
\partial_- V_{1,01}=&(u_1+u_2)(u_1-u_2)+u_3u_6-(u_1+u_2)(u_1-u_2)-u_3u_6=0,\\
&\Rightarrow V_{1,01}=0,\\
\partial_+ V_{5,01}=&(u_1-u_2)(u_1+u_2)+u_4u_5-(u_1-u_2)(u_1+u_2)-u_4u_5=0,\\
&\Rightarrow V_{5,01}=0,\\
(V_{9,01})_x=&\frac{1}{2}(u_6u_3+u_5u_4-u_6u_3-u_5u_4)=0,\\
&\Rightarrow V_{9,01}=0.
\end{aligned} \tag{3.12}$$

Substituting the above results into (3.5), one gets

$$\begin{aligned}
V_{2,02}=&(u_1+u_2)_x-(u_1+u_2)\xi-\frac{1}{2}(u_1+u_2)_y, V_{3,02}=2u_{3x}-u_3\xi-u_{3y},\\
V_{4,02}=&-(u_1+u_2)_x-(u_1-u_2)\xi-\frac{1}{2}(u_1-u_2)_y, V_{6,02}=-2u_{4x}-u_4\xi-u_{4y},\\
V_{7,02}=&-2u_{6x}-u_6\xi, V_{8,02}=2u_{5x}-u_5\xi,\\
\partial_- V_{1,02}=&(u_1+u_2)V_{4,02}+u_3V_{7,02}-V_{2,02}(u_1-u_2)-V_{3,02}\ u_6,\\
&\Rightarrow V_{1,02}=-\frac{1}{2}(u_1+u_2)(u_1-u_2)-u_3u_6,\\
\partial_+ V_{5,02}=&(u_1-u_2)V_{2,02}+u_4V_{8,02}-V_{4,02}(u_1+u_2)-V_{6,02}\ u_5,\\
&\Rightarrow V_{5,02}=\frac{1}{2}(u_1-u_2)(u_1+u_2)+u_4u_5,\\
(V_{9,02})_x=&\frac{1}{2}(u_6V_{3,02}+u_5V_{6,02}-V_{7,02}u_3-V_{8,02}u_4),\\
&\Rightarrow V_{9,02}=u_6u_3-u_5u_4.
\end{aligned} \tag{3.13}$$

Inserting (3.13) into (3.5) gives

$$\begin{aligned}
V_{2,03}=&(u_1+u_2)_{xx}-(u_1+u_2)_{xy}+[(u_1+u_2)_y-2(u_1+u_2)_x]\xi+\frac{1}{4}(u_1+u_2)_{yy}+\\
&(u_1+u_2)\xi^2-\frac{1}{2}(u_1-u_2)(u_1+u_2)^2-\frac{1}{2}(u_1+u_2)u_4u_5-\frac{1}{2}u_3u_6(u_1+u_2)-\\
&\frac{1}{2}(u_3u_{5y}+u_{3y}u_5+2u_3u_{5x}-2u_{3x}u_5),\\
V_{3,03}=&4u_{3xx}-2(2u_{3x}-u_{3y})\xi-4u_{3xy}+u_{3yy}+u_3\xi^2+2(u_1+u_2)u_{4x}+(u_1+u_2)_xu_4-\\
&\frac{1}{2}(u_1+u_2)_yu_4-\frac{1}{2}(u_1+u_2)(u_1-u_2)u_3+u_3u_5u_4,
\end{aligned}$$

$$
\begin{aligned}
V_{4,03} =&(u_1-u_2)_{xx}+(u_1-u_2)_{xy}+\frac{1}{4}(u_1-u_2)_{yy}+[2(u_1-u_2)_x+(u_1-u_2)_y]\xi+\\
&(u_1-u_2)\xi^2-\frac{1}{2}(u_1+u_2)(u_1-u_2)^2-\frac{1}{2}(u_1-u_2)u_3u_6-\frac{1}{2}u_4u_5(u_1-u_2)+\\
&\frac{1}{2}u_{4y}u_6+\frac{1}{2}u_4u_{6y}-u_4u_{6x}+u_{4x}u_6,\\
V_{6,03} =&4u_{4xx}+4u_{4xy}+u_{4yy}+(4u_{4x}+2u_{4y})\xi+u_4\xi^2+2(u_1-u_2)u_{3x}+(u_1-u_2)_xu_3+\\
&\frac{1}{2}(u_1-u_2)_yu_3-\frac{1}{2}(u_1+u_2)(u_1-u_2)u_4+u_4u_6u_3,\\
V_{7,03} =&4u_{6xx}+4u_{6x}\xi+u_6\xi^2-u_5(u_1-u_2)_x-2u_{5x}(u_1-u_2)+\frac{1}{2}u_5(u_1-u_2)_y-\\
&\frac{1}{2}u_6(u_1+u_2)(u_1-u_2)+u_5u_4u_6,\\
V_{8,03} =&4u_{5xx}-4u_{5x}\xi+u_5\xi^2-\frac{1}{2}u_6(u_1+u_2)_y-u_6(u_1+u_2)_x-2u_{6x}(u_1+u_2)-\\
&\frac{1}{2}u_5(u_1+u_2)(u_1-u_2)+u_6u_3u_5.
\end{aligned}
\tag{3.14}
$$

Case 2 Choose the initial data

$$
V_{1,10}=\xi, V_{5,10}=-\xi, V_{2,10}=V_{3,10}=V_{4,10}=V_{6,10}=V_{7,10}=V_{8,10}=V_{9,10}=0. \tag{3.15}
$$

In terms of (3.5), we get that

$$
\begin{aligned}
&V_{2,11} =(u_1+u_2)\xi+\frac{1}{2}(u_1+u_2)_y, V_{3,11}=u_3\xi+u_{3y},\\
&V_{4,11} =(u_1-u_2)\xi+\frac{1}{2}(u_1-u_2)_y, V_{6,11}=u_4\xi+u_{4y}, V_{7,11}=u_6\xi, V_{8,11}=u_5\xi,\\
&\partial_-V_{1,11}=(u_1+u_2)V_{4,11}+u_3V_{7,11}-V_{2,11}(u_1-u_2)-V_{3,11}u_6,\\
&\qquad\Rightarrow V_{1,11}=\partial_-^{-1}[-\frac{1}{2}(u_1+u_2)(u_1-u_2)-u_3u_6]_y,\\
&\partial_+V_{5,11}=(u_1-u_2)V_{2,11}+u_4V_{8,11}-V_{4,11}(u_1+u_2)-V_{6,11}u_5,\\
&\qquad\Rightarrow V_{5,11}=\partial_+^{-1}[-\frac{1}{2}(u_1-u_2)(u_1+u_2)-u_4u_5]_y,\\
&(V_{9,11})_x=\frac{1}{2}(u_6V_{3,11}+u_5V_{6,11}-V_{7,11}u_3-V_{8,11}u_4),\\
&\qquad\Rightarrow V_{9,11}=0.
\end{aligned}
\tag{3.16}
$$

From (3.5) and (3.16), we can obtain

$$
\begin{aligned}
V_{2,12} =&(u_1+u_2)_x\xi+\frac{1}{2}(u_1+u_2)_{xy}-(u_1+u_2)\xi^2-(u_1+u_2)_y\xi-\frac{1}{4}(u_1+u_2)_{yy}-\\
&\frac{1}{2}(u_1+u_2)V_{5,11}+\frac{1}{2}V_{1,11}(u_1+u_2)+\frac{1}{2}u_{3y}u_5+\frac{1}{2}u_3u_{5y},\\
V_{3,12} =&2u_{3x}\xi+2u_{3xy}-2u_{3y}\xi-u_3\xi^2-u_{3yy}+V_{1,11}u_3+\frac{1}{2}(u_1+u_2)_yu_4,\\
V_{4,12} =&-(u_1-u_2)_x\xi-\frac{1}{2}(u_1-u_2)_{xy}-(u_1-u_2)\xi^2-(u_1-u_2)_y\xi-\frac{1}{4}(u_1-u_2)_{yy}+\\
&\frac{1}{2}(u_1-u_2)V_{1,11}-\frac{1}{2}V_{5,11}(u_1-u_2)-\frac{1}{2}u_4u_{6y}-\frac{1}{2}u_{4y}u_6,\\
V_{6,12} =&-2u_{4x}\xi-2u_{4xy}-u_4\xi^2-2u_{4y}\xi-u_{4yy}+(u_1-u_2)u_3\xi-\frac{1}{2}(u_1-u_2)_yu_3-V_{5,11}u_4,
\end{aligned}
$$

$$V_{7,12} = -2u_{6x}\xi - u_6\xi^2 + u_6V_{1,11} - \frac{1}{2}u_5(u_1-u_2)_y,$$
$$V_{8,12} = 2u_{5x}\xi - u_5\xi^2 + \frac{1}{2}u_6(u_1+u_2)_y - u_5V_{5,11}. \tag{3.17}$$

Case 3 Choose the initial data

$$V_{1,20} = \xi^2, V_{5,20} = -\xi^2, V_{2,20} = V_{3,20} = V_{4,20} = V_{6,20} = V_{7,20} = V_{8,20} = V_{9,20} = 0. \tag{3.18}$$

Substituting (3.18) into (3.5) produces

$$\begin{aligned} V_{2,21} &= (u_1+u_2)\xi^2 + (u_1+u_2)_y\xi + \frac{1}{2}(u_1+u_2)_{yy}, \\ V_{3,21} &= u_3\xi^2 + 2u_{3y}\xi + u_{3yy}, \\ V_{4,21} &= (u_1-u_2)\xi^2 + (u_1-u_2)_y\xi + \frac{1}{2}(u_1-u_2)_{yy}, \\ V_{6,21} &= u_4\xi^2 + 2u_{4y}\xi + u_{4yy}, V_{7,21} = u_6\xi^2, V_{8,21} = u_5\xi^2. \end{aligned} \tag{3.19}$$

Now, we make use of (2.11) to rewrite (3.8) into a binormial-residue representation. Firstly, we consider some reduced cases by using (3.11)–(3.19).

Taking $n=1$, we have

$$\begin{cases} u_{1t_1} = \frac{1}{2}u_{1x}, \\ u_{2t_1} = \frac{1}{2}u_{2x}, \\ u_{3t_1} = \frac{1}{2}u_{3x}, \\ u_{4t_1} = \frac{1}{2}u_{4x}, \\ u_{5t_1} = \frac{1}{2}u_{5x}, \\ u_{6t_1} = \frac{1}{2}u_{6x}. \end{cases} \tag{3.20}$$

Setting $n=2$, we obtain

$$\begin{cases} u_{1t_2} = \frac{1}{4}(2u_{2xx} + \frac{1}{2}u_{2yy} - u_3u_{5x} - u_{3x}u_5 + u_4u_{6x} - u_{4x}u_6 + \frac{1}{2}u_{3y}u_5 + \frac{1}{2}u_3u_{5y} + \\ \qquad \frac{1}{2}u_4u_{6y} + \frac{1}{2}u_{4y}u_6 - u_1^2u_2 + u_2^3 - u_2u_3u_6 - u_2u_4u_5 + 2u_2V_{1,11} - 2u_2V_{5,11}), \\ u_{2t_2} = \frac{1}{4}(2u_{1xx} + \frac{1}{2}u_{1yy} - u_3u_{5x} - u_{3x}u_5 - u_4u_{6x} + u_{4x}u_6 + \frac{1}{2}u_{3y}u_5 + \frac{1}{2}u_3u_{5y} - \\ \qquad \frac{1}{2}u_4u_{6y} - \frac{1}{2}u_{4y}u_6 + u_1u_2^2 - u_1^3 - u_1u_3u_6 - u_1u_4u_5 + 2u_1V_{1,11} - 2u_1V_{5,11}), \\ u_{3t_2} = \frac{1}{4}(4u_{3xx} + \frac{1}{2}u_{1y}u_4 + \frac{1}{2}u_{2y}u_4 + u_{1x}u_4 + u_{2x}u_4 + 2u_1u_{4x} + 2u_2u_{4x} - \\ \qquad \frac{1}{2}u_1^2u_3 + \frac{1}{2}u_2^2u_3 + u_3u_5u_4 + 2V_{1,11}u_3), \\ u_{4t_2} = -\frac{1}{4}(4u_{4xx} - \frac{1}{2}u_{1y}u_3 + \frac{1}{2}u_{2y}u_3 + u_{1x}u_3 - u_{2x}u_3 + 2u_1u_{3x} - 2u_2u_{3x} - \\ \qquad \frac{1}{2}u_1^2u_4 + \frac{1}{2}u_2^2u_4 + u_4u_6u_3 - 2V_{5,11}u_4), \\ u_{5t_2} = \frac{1}{4}(4u_{5xx} + \frac{1}{2}u_6u_{1y} + \frac{1}{2}u_6u_{2y} - u_6u_{1x} - u_6u_{2x} - 2u_{6x}u_1 - 2u_{6x}u_2 - \frac{1}{2}u_5u_1^2 + \\ \qquad \frac{1}{2}u_5u_2^2 + u_6u_3u_5 - 2u_5V_{5,11}), \\ u_{6t_2} = -\frac{1}{4}(4u_{6xx} - \frac{1}{2}u_5u_{1y} + \frac{1}{2}u_5u_{2y} - u_5u_{1x} + u_5u_{2x} - 2u_{5x}u_1 + 2u_{5x}u_2 - \\ \qquad \frac{1}{2}u_6u_1^2 + \frac{1}{2}u_6u_2^2 + u_5u_4u_6 + 2u_6V_{1,11}). \end{cases} \tag{3.21}$$

In particular, set $u_3 = u_4 = u_5 = u_6 = 0$, $t_2 = t$, then Eq. (3.21) reduces to generalized $(2+1)$-dimensional nonlinear Schrödinger equation

$$\begin{cases} u_{1t} = \frac{1}{4}[2u_{2xx} + \frac{1}{2}u_{2yy} - u_1^2u_2 + u_2^3 - u_2\partial_-^{-1}(u_1^2-u_2^2)_y + u_2\partial_+^{-1}(u_1^2-u_2^2)_y], \\ u_{2t} = \frac{1}{4}[2u_{1xx} + \frac{1}{2}u_{1yy} + u_1u_2^2 - u_1^3 - u_1\partial_-^{-1}(u_1^2-u_2^2)_y + u_1\partial_+^{-1}(u_1^2-u_2^2)_y]. \end{cases} \tag{3.22}$$

Let $u_5 = u_6 = 0$. Then Eq. (3.21) becomes

$$\begin{cases} u_{1t_2} = \frac{1}{4}[2u_{2xx} + \frac{1}{2}u_{2yy} - u_1^2 u_2 + u_2^3 - u_2\partial_-^{-1}(u_1^2 - u_2^2)_y + u_2\partial_+^{-1}(u_1^2 - u_2^2)_y], \\ u_{2t_2} = \frac{1}{4}[2u_{1xx} + \frac{1}{2}u_{1yy} + u_1 u_2^2 - u_1^3 - u_1\partial_-^{-1}(u_1^2 - u_2^2)_y + u_1\partial_+^{-1}(u_1^2 - u_2^2)_y], \\ u_{3t_2} = \frac{1}{4}[4u_{3xx} + \frac{1}{2}u_{1y}u_4 + \frac{1}{2}u_{2y}u_4 + u_{1x}u_4 + u_{2x}u_4 + 2u_1 u_{4x} + 2u_2 u_{4x} - \\ \qquad \frac{1}{2}u_1^2 u_3 + \frac{1}{2}u_2^2 u_3 - \partial_-^{-1}(u_1^2 - u_2^2)_y u_3], \\ u_{4t_2} = -\frac{1}{4}[4u_{4xx} - \frac{1}{2}u_{1y}u_3 + \frac{1}{2}u_{2y}u_3 + u_{1x}u_3 - u_{2x}u_3 + 2u_1 u_{3x} - 2u_2 u_{3x} - \\ \qquad \frac{1}{2}u_1^2 u_4 + \frac{1}{2}u_2^2 u_4 + \partial_+^{-1}(u_1^2 - u_2^2)_y u_4], \end{cases} \tag{3.23}$$

which is a $(2+1)$-dimensional super integrable coupling of (3.22).

Again let $u_3 = u_4 = 0$. Then Eq. (3.21) reduces to

$$\begin{cases} u_{1t_2} = \frac{1}{4}[2u_{2xx} + \frac{1}{2}u_{2yy} - u_1^2 u_2 + u_2^3 - u_2\partial_-^{-1}(u_1^2 - u_2^2)_y + u_2\partial_+^{-1}(u_1^2 - u_2^2)_y], \\ u_{2t_2} = \frac{1}{4}[2u_{1xx} + \frac{1}{2}u_{1yy} + u_1 u_2^2 - u_1^3 - u_1\partial_-^{-1}(u_1^2 - u_2^2)_y + u_1\partial_+^{-1}(u_1^2 - u_2^2)_y], \\ u_{5t_2} = \frac{1}{4}[4u_{5xx} + \frac{1}{2}u_6 u_{1y} + \frac{1}{2}u_6 u_{2y} - u_6 u_{1x} - u_6 u_{2x} - 2u_{6x}u_1 - 2u_{6x}u_2 - \\ \qquad \frac{1}{2}u_5 u_1^2 + \frac{1}{2}u_5 u_2^2 + u_5\partial_+^{-1}(u_1^2 - u_2^2)_y], \\ u_{6t_2} = -\frac{1}{4}[4u_{6xx} - \frac{1}{2}u_5 u_{1y} + \frac{1}{2}u_5 u_{2y} - u_5 u_{1x} + u_5 u_{2x} - 2u_{5x}u_1 + 2u_{5x}u_2 - \\ \qquad \frac{1}{2}u_6 u_1^2 + \frac{1}{2}u_6 u_2^2 - u_6\partial_-^{-1}(u_1^2 - u_2^2)_y], \end{cases} \tag{3.24}$$

which is another $(2+1)$-dimensional super integrable coupling of (3.22), the super integrable coupling (3.23) is different from the super integrable coupling (3.24), these results are not found in previous research on integrable systems.

When $u_1 = u_2$, Eq. (3.22) becomes the $(2+1)$-dimensional diffusion equation

$$u_{1t} = \frac{1}{2}u_{1xx} + \frac{1}{8}u_{1yy}. \tag{3.25}$$

The $(2+1)$-dimensional super integrable hierarchy (3.8) can be rewritten into a general binormial representation

$$\begin{cases} u_{1t_n} = \frac{1}{4}\sum_{i=0}^{n}\binom{n}{i}(V_{2,n+1-i} - V_{4,n+1-i}), \\ u_{2t_n} = \frac{1}{4}\sum_{i=0}^{n}\binom{n}{i}(V_{2,n+1-i} + V_{4,n+1-i}), \\ u_{3t_n} = \frac{1}{4}\sum_{i=0}^{n}\binom{n}{i}V_{3,n+1-i}, \\ u_{4t_n} = -\frac{1}{4}\sum_{i=0}^{n}\binom{n}{i}V_{6,n+1-i}, \\ u_{5t_n} = \frac{1}{4}\sum_{i=0}^{n}\binom{n}{i}V_{8,n+1-i}, \\ u_{6t_n} = -\frac{1}{4}\sum_{i=0}^{n}\binom{n}{i}V_{7,n+1-i}. \end{cases} \tag{3.26}$$

4. Hamiltonian structure

In this section, we will establish the super Hamiltonian structure of the new super integrable hierarchy (3.8) by super trace identity [19, 20]

$$\frac{\delta}{\delta u}\int \mathrm{Str}(V\frac{\partial U}{\partial \lambda})\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{Str}(\frac{\partial U}{\partial u}V), \tag{4.1}$$

where the constant γ is determined by

$$\gamma = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln|\mathrm{Str}(VV)|. \tag{4.2}$$

According to super trace identity on Lie super algebras, a direct calculation reads

$$\mathrm{Str}(V\frac{\partial U}{\partial\lambda})=\frac{1}{4}(-\lambda^{-2}V_1+\lambda^{-2}V_5),\ \mathrm{Str}(V\frac{\partial U}{\partial u_1})=\frac{1}{4}(V_4+V_2),\ \mathrm{Str}(V\frac{\partial U}{\partial u_2})=\frac{1}{4}(V_4-V_2),$$
$$\mathrm{Str}(V\frac{\partial U}{\partial u_3})=\frac{1}{4}V_7,\ \mathrm{Str}(V\frac{\partial U}{\partial u_4})=\frac{1}{4}V_8,\ \mathrm{Str}(V\frac{\partial U}{\partial u_5})=\frac{1}{4}V_6,\ \mathrm{Str}(V\frac{\partial U}{\partial u_6})=\frac{1}{4}V_3. \tag{4.3}$$

Substitute the above formula into the super trace identity (4.1) yields

$$\frac{\delta}{\delta u}\int(-\lambda^{-2}V_1+\lambda^{-2}V_5)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(V_4+V_2,V_4-V_2,V_7,V_8,V_6,V_3)^T. \tag{4.4}$$

Comparing the coefficients of λ^{n+2} on both sides of (4.4) gives rise to

$$\begin{aligned}&\frac{\delta}{\delta u}\int\frac{V_{1,n+2}-V_{5,n+2}}{n+1}\mathrm{d}x\\&\quad=(V_{2,n+1}+V_{4,n+1},-V_{2,n+1}+V_{4,n+1},V_{7,n+1},V_{8,n+1},V_{6,n+1},V_{3,n+1})^T.\end{aligned} \tag{4.5}$$

By employing the computing formula (4.2) on the constant γ, we obtain $\gamma=0$. So, we conclude that

$$\begin{aligned}\frac{\delta H_n}{\delta u}&=(V_{2,n+1}+V_{4,n+1},-V_{2,n+1}+V_{4,n+1},V_{7,n+1},V_{8,n+1},V_{6,n+1},V_{3,n+1})^T,\\ H_n&=\int\frac{V_{1,n+2}-V_{5,n+2}}{n+1}\mathrm{d}x,\quad n\geqslant 0.\end{aligned} \tag{4.6}$$

Therefore, the new super integrable hierarchy (3.8) possesses the following super Hamiltonian structure

$$u_{t_n}=\begin{pmatrix}0&-\frac{1}{4}&0&0&0&0\\ \frac{1}{4}&0&0&0&0&0\\ 0&0&0&&0&\frac{1}{4}\\ 0&0&0&0&-\frac{1}{4}&0\\ 0&0&0&\frac{1}{4}&0&0\\ 0&0&-\frac{1}{4}&0&0&0\end{pmatrix}\begin{pmatrix}V_{2,n+1}+V_{4,n+1}\\ -V_{2,n+1}+V_{4,n+1}\\ V_{7,n+1}\\ V_{8,n+1}\\ V_{6,n+1}\\ V_{3,n+1}\end{pmatrix}=J\frac{\delta H_n}{\delta u},\quad n\geq 0. \tag{4.7}$$

5. Remarks and conclusions

By using the binormial-residue-representation, a new $(2+1)$-dimensional super integrable system is obtained, from which a new $(2+1)$-dimensional super soliton hierarchy is deduced, including the $(2+1)$-dimensional super nonlinear Schrödinger equation. In addition, two main results are obtained. One of them is a set of $(2+1)$-dimensional super integrable couplings and another one is a $(2+1)$-dimensional diffusion equation. Furthermore, the super Hamiltonian structure for the new $(2+1)$-dimensional super hierarchy can be presented by taking use of the super trace identity.

It is important to investigate the symmetry analyses of (3.22), (3.23), and (3.24) according to [28–30]. However, since there exist operators $\partial_{\pm}^{-1}$ in these partial differential equations, it may be difficult to discuss their Lie group properties. Can we follow the ways for generating symmetries of the KP equation to discuss the symmetries of (3.22), (3.23) and (3.24)? Recently, lumps solutions, and interaction solutions show a special kind of integrability of exact solutions to integrable systems [31–34]. As its reduction, we gain the nonlinear integrable equations. How to obtain the solutions of reduced equations is a very important and difficult work, and we will plan to study and discuss these problems in the near future.

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