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Some Applications of the (f, g)-Inversion

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Abstract We present three kinds of applications of the (f, g)-inversion due to Ma. By taking explicit functions and sequences in the (f, g)-inversion, we derive identities involving hypergeometric series and harmonic numbers. Then we give several inversion relations involving q-hypergeometric terms. Finally, we combine the (f, g)-inversion and the q-differential operators to derive some q-series identities.

Keywords matrix inversion; (f, g)-inversion; hypergeometric series; basic hypergeometric series; q-differential operator

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1. Introduction and preliminaries

Let $\mathbb{N}$ denote the set of nonnegative integers and $F = (f_{n,k})_{n,k\in\mathbb{N}}$ be an infinite-dimensional lower-triangular matrix, i.e., $f_{n,k} = 0$ for $n < k$. The matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$ is the inverse matrix of F if and only if

$$\sum_{i=k}^{n} f_{n,i} g_{i,k} = \delta_{n,k}, \quad n, k \in \mathbb{N},$$

where δ denotes the usual Kronecker delta. We call the entries $(f_{n,k}, g_{n,k})$ an inversion pair.

The study of inversion pairs began with a series of work by Gould [1], Gould-Hsu [2] and Carlitz [3]. Gessel and Stanton used the inversion pairs to derive a number of hypergeometric summations and transformations [4]. In 1996, Krattenthaler established the operator method and found an inversion pair in a general form [5]. In 2004, Ma established the (f, g)-inversion, which significantly extends Krattenthaler's results. The main result of Ma can be stated as follows.

Theorem 1.1 ([6, Theorem 1.3]) *Let $F = (f_{n,k})_{n,k\in\mathbb{N}}$, $G = (g_{n,k})_{n,k\in\mathbb{N}}$ be two lower triangular matrices with entries given by*

$$f_{n,k} = \frac{\prod_{i=k}^{n-1} f(x_i, y_k)}{\prod_{i=k+1}^{n} g(y_i, y_k)},$$

and

$$g_{n,k} = \frac{f(x_k, y_k) \prod_{i=k+1}^{n} f(x_i, y_n)}{f(x_n, y_n) \prod_{i=k}^{n-1} g(y_i, y_n)},$$

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where $f(x,y)$ and $g(x,y)$ are two arbitrary functions in variables x and y, $\{x_i\}$ and $\{y_i\}$ are two arbitrary sequences. Suppose further $g(x,y)$ is antisymmetric, i.e., $g(x,y)=-g(y,x)$. Then $(f_{n,k},g_{n,k})$ is an inversion pair if and only if for all a,b,c,x,

$$g(a,b)f(x,c)-g(a,c)f(x,b)+g(b,c)f(x,a)=0. \tag{1.1}$$

There is a standard technique for deriving new summation formulas from known ones by using the inversion pairs [7–9]. If $(f_{n,k},g_{n,k})$ is an inversion pair, then we have

$$a(n)=\sum_{k=0}^{n} f_{n,k}b(k) \Longleftrightarrow b(n)=\sum_{k=0}^{n} g_{n,k}a(k). \tag{1.2}$$

If one side in (1.2) is known, then the other produces a new summation formula. Along this approach, we derive some hypergeometric identities and basic hypergeometric identities.

By taking explicit functions $f(x,y),g(x,y)$ and sequences x_i,y_i, we first derive an identity with several parameters in Section 2. As an interesting application, we obtain an identity involving harmonic numbers. Then we consider the q-cases in Section 3 and derive several inversion pairs involving basic hypergeometric terms. Finally, we combine the inversion pair and the q-differential operator to derive some basic hypergeometric identities in Section 4.

We adopt the notation and terminology in [10]. The hypergeometric series and basic hypergeometric series are defined by

$${}_rF_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_r)_kz^k}{(b_1)_k(b_2)_k\cdots(b_s)_kk!},$$

$${}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_s\end{matrix};q,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_kz^k}{(q,b_1,b_2,\ldots,b_s;q)_k}((-1)^kq^{\binom{k}{2}})^{1+s-r},$$

where the rising factorial $(a)_k$ is given by $(a)_k=a(a+1)\cdots(a+k-1)$ and the rising q-factorial $(a;q)_k$ is given by

$$(a;q)_k=\prod_{i=0}^{k-1}(1-aq^i).$$

We also use

$$(a;q)_\infty=\prod_{i=0}^{\infty}(1-aq^i),\quad (a_1,a_2,\ldots,a_r;q)_k=(a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k.$$

2. An inverse relation and its applications

In this section, we derive an identity with several parameters by taking explicit functions $f(x,y),g(x,y)$ and sequences x_i,y_i in Theorem 1.1. As two examples of the applications of the identity, we derive a curious hypergeometric identity and an identity involving harmonic numbers.

Utilizing Theorem 1.1, we obtain the following identity on hypergeometric series.

Theorem 2.1 *Let $c(n)$ be the hypergeometric series*

$$c(n) = {}_rF_s\left[\begin{matrix} a_1,\ldots,a_{r-2},-n,a+n \\ b_1,\ldots,b_s \end{matrix};z\right],$$

where $a_1,\ldots,a_{r-2},b_1,\ldots,b_s$ and a,z are parameters independent of n. We have

$$\frac{(a)_{n+1}(a_1)_n\cdots(a_{r-2})_n z^n}{(b_1)_n\cdots(b_s)_n} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a+2k)(a)_k}{(a+n+1)_k}c(k).$$

Proof Let $g(x,y)=x-y$ and $f(x,y)=x+y$. It is straightforward to check that (1.1) holds. Setting $x_i=a+i$ and $y_i=i$ in Theorem 1.1, we obtain an inversion pair

$$f_{n,k} = \frac{(a+2k)_{n-k}}{(n-k)!} = \frac{(a)_n(a+n)_k}{(a)_{2k}(n-k)!},\quad g_{n,k} = (-1)^{n-k}\frac{a+2k}{a+2n}\frac{(a+n+1)_n}{(a+n+1)_k(n-k)!}. \tag{2.1}$$

Let $a(n)=(a)_n c(n)/n!$ and

$$b(k) = \frac{(-1)^k(a)_{2k}(a_1)_k\cdots(a_{r-2})_k z^k}{(b_1)_k\cdots(b_s)_k k!}.$$

We see that

$$a(n) = \sum_{k=0}^{n}\frac{(a)_n(a+n)_k}{(a)_{2k}(n-k)!}b(k).$$

Hence by (1.2), we have

$$\frac{(-1)^n(a)_{2n}(a_1)_n\cdots(a_{r-2})_n z^n}{(b_1)_n\cdots(b_s)_n n!} = \sum_{k=0}^{n}(-1)^{n-k}\frac{a+2k}{a+2n}\frac{(a+n+1)_n}{(a+n+1)_k(n-k)!}\frac{(a)_k c(k)}{k!}.$$

We thus complete the proof by some simplification. □

When the hypergeometric series $c(n)$ has a closed formula, we will derive a summation formula on hypergeometric series.

Example 2.2 The following identity for balanced hypergeometric series of ${}_5F_4$ is the $c=1$ case of [11, Proposition 2.11]:

$${}_5F_4\left[\begin{matrix} -n,n+1,\frac{a}{2},\frac{1+a}{2},1+\frac{2a}{3} \\ \frac{3}{2},1,1+a,\frac{2a}{3} \end{matrix};1\right] = \frac{2-2a+n}{2(1+2n)(1-a)}\frac{(1-a)_n}{n!}.$$

Setting $a=1$ in Theorem 2.1, we thus derive

$$\frac{(n+1)(\frac{a}{2})_n(\frac{1+a}{2})_n(1+\frac{2a}{3})_n}{(\frac{3}{2})_n(1+a)_n(\frac{2a}{3})_n} = \frac{1}{2(1-a)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2-2a+k)(1-a)_k}{(2+n)_k}.$$

Huang showed that a new inversion pair can be obtained from a known one by multiplying suitable factors.

Lemma 2.3 ([12, Lemma 3.6.1]) *Suppose that $F=(f_{n,k})_{n,k\in\mathbb{N}}$ and $G=(g_{n,k})_{n,k\in\mathbb{N}}$ are lower-triangular matrices that are inverses of each other. Let A_n and B_n be two functions of n. Then $F'=(\frac{f_{n,k}A_k}{B_n})_{n,k\in\mathbb{N}}$ and $G'=(\frac{g_{n,k}B_k}{A_n})_{n,k\in\mathbb{N}}$ are also inverses of each other.*

Setting $a=1$ in (2.1) and $A_k=\frac{(2k)!}{k!k!}$, $B_k=1$ in Lemma 2.3, we obtain a new inversion pair:

$$f'_{n,k}=\binom{n}{k}\binom{n+k}{k},\quad g'_{n,k}=(-1)^{n-k}\frac{\binom{2n}{n-k}(2k+1)}{\binom{2n}{n}(n+k+1)}. \tag{2.2}$$

This pair leads to an identity involving harmonic numbers, which seems to be new.

Example 2.4 It is known that [13, Equation (2.37)]

$$(-1)^n\binom{n+p}{p}\binom{n}{p}(2H_n-H_p)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\binom{k}{p}H_k,\quad 0\le p\le k,$$

where H_n denotes the n-th harmonic number $H_n=\sum_{k=1}^{n}\frac{1}{k}$.

Setting $a(n)=(-1)^n\binom{n+p}{p}\binom{n}{p}(2H_n-H_p)$, $b(k)=(-1)^k\binom{k}{p}H_k$ in (1.2). Since (2.2) is an inversion pair, we have

$$(-1)^n\binom{n}{p}H_n=\sum_{k=0}^{n}(-1)^{n-k}\frac{\binom{2n}{n-k}(2k+1)}{\binom{2n}{n}(n+k+1)}(-1)^k\binom{k+p}{p}\binom{k}{p}(2H_k-H_p).$$

After some simplification, we obtain

$$\binom{n}{p}\binom{2n}{n}H_n=\sum_{k=0}^{n}\binom{2n}{n-k}\binom{k+p}{p}\binom{k}{p}\frac{(2k+1)}{(n+k+1)}(2H_k-H_p).$$

In particular, when $p=0$, the above identity becomes

$$\frac{1}{2}\binom{2n}{n}H_n=\sum_{k=0}^{n}\binom{2n}{n-k}\frac{(2k+1)}{(n+k+1)}H_k.$$

3. Inversion relations involving q-hypergeometric terms

In this section, we will give a number of inversion pairs involving basic hypergeometric terms.

Theorem 3.1 *Let $M(a,b)$ be the infinite-dimensional lower-triangular matrix*

$$M(a,b)=\Big(\frac{(\frac{a}{b}q;q)_n(q^{-n},\frac{a}{b}q^{n+1};q)_k}{(q,\frac{a}{b}q^2;q)_k}q^k\Big)_{n,k\in\mathbb{N}}.$$

Then

$$M^{-1}(a,b)=\Big(\frac{(q^{-n};q)_k(1-\frac{a}{b}q^{2k+1})}{(q,\frac{a}{b}q^{n+2};q)_k(1-\frac{a}{b}q)}q^{nk}\Big)_{n,k\in\mathbb{N}}.$$

Proof Setting $g(x,y)=f(x,y)=x-y$, $x_i=aq^i$, $y_i=bq^{-(i+1)}$ in Theorem 1.1, we obtain the inversion pair

$$\begin{aligned}f_{n,k}&=\frac{(aq^k-bq^{-(k+1)})(aq^{k+1}-bq^{-(k+1)})\cdots(aq^{n-1}-bq^{-(k+1)})}{(bq^{-(k+2)}-bq^{-(k+1)})(bq^{-(k+3)}-bq^{-(k+1)})\cdots(bq^{-(n+1)}-bq^{-(k+1)})}\\&=\frac{(\frac{a}{b}q;q)_{n+k}}{(\frac{a}{b}q;q)_{2k}}\frac{(q^{-n};q)_k}{(q^{-n};q)_n}=\frac{(\frac{a}{b}q;q)_n(q^{-n},\frac{a}{b}q^{n+1};q)_k}{(\frac{a}{b}q;q)_{2k}(q^{-n};q)_n},\\ g_{n,k}&=\frac{aq^k-bq^{-(k+1)}}{aq^n-bq^{-(n+1)}}\frac{(aq^{k+1}-bq^{-(n+1)})(aq^{k+2}-bq^{-(n+1)})\cdots(aq^n-bq^{-(n+1)})}{(bq^{-(k+1)}-bq^{-(n+1)})(bq^{-(k+2)}-bq^{-(n+1)})\cdots(bq^{-n}-bq^{-(n+1)})}\end{aligned}$$

$$= q^{n-k}\frac{1-\frac{a}{b}q^{2k+1}}{1-\frac{a}{b}q^{2n+1}}\frac{(1-\frac{a}{b}q^{n+k+2})(1-\frac{a}{b}q^{n+k+3})\cdots(1-\frac{a}{b}q^{2n+1})}{(1-q)(1-q^2)\cdots(1-q^{n-k})}$$
$$= q^{n-k}\frac{1-\frac{a}{b}q^{2k+1}}{1-\frac{a}{b}q}\frac{(\frac{a}{b}q;q)_{2n}}{(\frac{a}{b}q^2;q)_n(\frac{a}{b}q^{n+2};q)_k}\frac{(q^{-n};q)_k}{(-1)^k q^{-nk+\binom{k}{2}}}\frac{1}{(q;q)_n}.$$

Taking $A_k = \frac{(\frac{a}{b}q;q)_{2k}q^k}{(q,\frac{a}{b}q^2;q)_k}$ and $B_n = \frac{1}{(q^{-n};q)_n} = \frac{1}{(-1)^n q^{-n-\binom{n}{2}}}\frac{1}{(q;q)_n}$ in Lemma 2.3, we obtain the desired $M(a,b)$ and $M^{-1}(a,b)$. □

Theorem 3.2 *Let $N(a,b)$ be the infinite-dimensional lower-triangular matrix*

$$N(a,b) = (\frac{(\frac{a}{b}q;q)_n}{(q;q)_k}\frac{(q^{-n},\frac{b}{aq};q)_k}{(\frac{b}{a}q^{-n};q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Then

$$N^{-1}(a,b) = (\frac{(q^{-n};q)_k}{(q,\frac{a}{b}q^{2-n};q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Proof Setting $g(x,y)=f(x,y)=x-y$, $x_i = aq^{i+1}$, $y_i = bq^i$ in Theorem 1.1, we get

$$f_{n,k} = \frac{(\frac{a}{b}q;q)_n}{(q;q)_n}\frac{(q^{-n};q)_k}{\frac{a^k}{b^k}(\frac{b}{a}q^{-n};q)_k},$$

$$q_{n,k} = (\frac{aq}{b})^n\frac{(\frac{b}{aq};q)_n}{(q;q)_n}\frac{(q^{-n};q)_k}{(\frac{a}{b}q^{2-n};q)_k}q^k.$$

Taking $A_k = \frac{(\frac{b}{aq};q)_k}{(q;q)_k}\frac{a^k q^k}{b^k}$ and $B_n = \frac{1}{(q;q)_n}$ in Lemma 2.3, we obtain the desired $N(a,b)$ and $N^{-1}(a,b)$. □

Theorem 3.3 *Let $P(a,b)$ be the infinite-dimensional lower-triangular matrix*

$$P(a,b) = (\frac{1-b^2q^{2k}}{1-b^2}\frac{(\frac{b}{a},b^2,q^{-n},abq^n;q)_k}{(q,abq,b^2q^{n+1},\frac{b}{a}q^{1-n};q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Then

$$P^{-1}(a,b) = (\frac{1-abq^{2k}}{1-ab}\frac{(\frac{a}{b},ab,q^{-n},b^2q^n;q)_k}{(q,b^2q,abq^{n+1},\frac{a}{b}q^{1-n};q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Proof Setting $g(x,y)=f(x,y)=(x-y)(1-xy)$, $x_i = aq^i$, $y_i = bq^i$ in Theorem 1.1, we get

$$f_{n,k} = \frac{b^k}{a^k}\frac{(\frac{a}{b},ab;q)_n(b^2q;q)_{2k}}{(q,b^2q;q)_n(ab;q)_{2k}}\frac{(q^{-n},abq^n;q)_k}{(b^2q^{n+1},\frac{b}{a}q^{1-n};q)_k}q^k,$$

$$g_{n,k} = \frac{1-abq^{2k}}{1-ab}\frac{1-b^2q^{2n}}{1-b^2}\frac{a^n}{b^n}\frac{(\frac{b}{a},b^2;q)_n(ab;q)_{2n}}{(q,abq;q)_n(b^2q;q)_{2n}}\frac{(q^{-n},b^2q^n;q)_k}{(abq^{n+1},\frac{a}{b}q^{1-n};q)_k}q^k.$$

Taking $A_k = \frac{a^k}{b^k}\frac{1-b^2q^{2k}}{1-b^2}\frac{(\frac{b}{a},b^2;q)_k(ab;q)_{2k}}{(q,abq;q)_k(b^2q;q)_{2k}}$ and $B_n = \frac{(ab,\frac{a}{b};q)_n}{(b^2q,q;q)_n}$ in Lemma 2.3, we obtain the desired $P(a,b)$ and $P^{-1}(a,b)$. □

Theorem 3.4 *Let $R(a,b)$ be the infinite-dimensional lower-triangular matrix*

$$R(a,b) = (\frac{(\frac{b}{a},q^{-n};q)_k}{(q,\frac{b}{a}q^{1-n};q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Then

$$R^{-1}(a,b)=(\frac{(\frac{a}{b},q^{-n};q)_k}{(q,\frac{a}{b}q^{1-n};q)_k}q^k)_{n,k\in\mathbb{N}}=R(b,a).$$

Proof Setting $g(x,y)=f(x,y)=(x-y)(1-x)(1-y)$, $x_i=aq^i$, $y_i=bq^i$ in Theorem 1.1, we get

$$f_{n,k}=\frac{b^k}{a^k}\frac{(\frac{a}{b},a;q)_n}{(q,bq;q)_n}\frac{(q^{-n},bq;q)_k}{(\frac{b}{a}q^{1-n},a;q)_k}q^k,$$

$$g_{n,k}=\frac{a^n}{b^n}\frac{(\frac{b}{a},a;q)_n}{(q,bq;q)_n}\frac{(q^{-n},bq;q)_k}{(\frac{a}{b}q^{1-n},a;q)_k}q^k.$$

Taking $A_k=\frac{a^k}{b^k}\frac{(\frac{b}{a},a;q)_k}{(q,bq;q)_k}$ and $B_n=\frac{(\frac{a}{b},a;q)_n}{(q,bq;q)_n}$ in Lemma 2.3, we obtain the desired $R(a,b)$ and $R^{-1}(a,b)$. $\Box$

Theorem 3.5 *Let $S(a,b)$ be the infinite-dimensional lower-triangular matrix*

$$S(a,b)=(\frac{(a;q)_n}{(bq^{-n};q)_n}\frac{(\frac{a}{b}q^n,bq^{-k},q^{-n};q)_k}{(q,a,\frac{a}{b}q;q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Then

$$S^{-1}(a,b)=(\frac{1-\frac{a}{b}q^{2k}}{1-\frac{a}{b}}\frac{(a;q)_n}{(bq^{-n};q)_n}\frac{(\frac{a}{b},bq^{-k},q^{-n};q)_k}{(q,a,\frac{a}{b}q^{n+1};q)_k}q^{nk})_{n,k\in\mathbb{N}}.$$

Proof Setting $g(x,y)=f(x,y)=(x-y)(1-x)(1-y)$, $x_i=aq^i$, $y_i=bq^{-i}$ in Theorem 1.1, we get

$$f_{n,k}=\frac{(\frac{a}{b},a;q)_n}{(q,bq^{-n};q)_n(\frac{a}{b};q)_{2k}}\frac{(q^{-n},bq^{-k},\frac{a}{b}q^n;q)_k}{(a;q)_k}(-1)^nq^{\binom{n}{2}+n},$$

$$g_{n,k}=q^{n-k}\frac{1-\frac{a}{b}q^{2k}}{1-\frac{a}{b}}\frac{(a;q)_n(\frac{a}{b};q)_{2n}}{(q,\frac{a}{b}q,bq^{-n};q)_n}\frac{(q^{-n},bq^{-k};q)_k}{(\frac{a}{b}q^{n+1},a;q)_k}(-1)^kq^{nk-\binom{k}{2}}.$$

Taking $A_k=\frac{(\frac{a}{b};q)_{2k}}{(\frac{a}{b}q,q;q)_k}q^k$ and $B_n=\frac{(\frac{a}{b};q)_n}{(q;q)_n}(-1)^nq^{\binom{n}{2}+n}$ in Lemma 2.3, we obtain the desired $S(a,b)$ and $S^{-1}(a,b)$. $\Box$

Now we give two examples to illustrate the applications of the inverse relations.

Example 3.6 It is known that the following transformation formula for very-well-poised ${}_8\phi_7$ series holds [10, Equation (III.18)]:

$${}_8\phi_7\left[\begin{matrix}a,a^{\frac12}q,-a^{\frac12}q,b,c,d,e,q^{-n}\\ a^{\frac12},-a^{\frac12},\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e},aq^{n+1}\end{matrix};q,\frac{a^2q^{n+2}}{bcde}\right]=\frac{(aq,\frac{aq}{de};q)_n}{(\frac{aq}{d},\frac{aq}{e};q)_n}{}_4\phi_3\left[\begin{matrix}\frac{aq}{bc},d,e,q^{-n}\\ \frac{aq}{b},\frac{aq}{c},\frac{de}{aq^n}\end{matrix};q,q\right].$$

Taking $a\to\frac{a}{q}$ and $b\to1$ in Theorem 3.1 gives

$$M(\frac{a}{q},1)=(f_{n,k})_{n,k\in\mathbb{N}}=(\frac{(a;q)_n(q^{-n},aq^n;q)_k}{(q,aq;q)_k}q^k)_{n,k\in\mathbb{Z}},$$

$$M^{-1}(\frac{a}{q},1)=(g_{n,k})_{n,k\in\mathbb{N}}=(\frac{(q^{-n};q)_k(1-aq^{2k})}{(q,aq^{n+1};q)_k(1-a)}q^{nk})_{n,k\in\mathbb{Z}}.$$

Now setting

$$a(k)=\frac{(a,b,c,d,e;q)_k}{(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e};q)_k}(\frac{a^2q^2}{bcde})^k$$

and

$$b(n)=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(1-aq^{2k})q^{nk}}{(q,aq^{n+1};q)_k(1-a)}\frac{(a,b,c,d,e;q)_k}{(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e};q)_k}(\frac{a^2q^2}{bcde})^k, \tag{3.1}$$

we see that

$$b(n)=\sum_{k=0}^{n}g_{n,k}a(k).$$

By (1.2), we derive that

$$\frac{(a,b,c,d,e;q)_n}{(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e};q)_n}(\frac{a^2q^2}{bcde})^n=\sum_{k=0}^{n}\frac{(a;q)_n(q^{-n},aq^n;q)_kq^k}{(q,aq;q)_k}\frac{(aq,\frac{aq}{de};q)_k}{(\frac{aq}{d},\frac{aq}{e};q)_k}{}_4\phi_3\left[\begin{matrix}\frac{aq}{bc},d,e,q^{-k}\\ \frac{aq}{b},\frac{aq}{c},\frac{de}{aq^k}\end{matrix};q,q\right].$$

After some simplification, we obtain

$$\frac{(b,c,d,e;q)_n}{(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{e};q)_n}(\frac{a^2q^2}{bcde})^n=\sum_{k=0}^{n}\frac{(q^{-n},aq^n,\frac{aq}{de};q)_kq^k}{(q,\frac{aq}{d},\frac{aq}{e};q)_k}{}_4\phi_3\left[\begin{matrix}\frac{aq}{bc},d,e,q^{-k}\\ \frac{aq}{b},\frac{aq}{c},\frac{de}{aq^k}\end{matrix};q,q\right].$$

Example 3.7 It is known that the following summation formula for very-well-poised ${}_{10}\phi_9$ series holds [10, Exercise 2.12]:

$${}_{10}\phi_9\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,\sqrt{b},-\sqrt{b},\sqrt{bq},-\sqrt{bq},\frac{a}{b},\frac{a^2}{b}q^{n+1},q^{-n}\\ \sqrt{a},-\sqrt{a},\frac{aq}{\sqrt{b}},-\frac{aq}{\sqrt{b}},\frac{aq}{\sqrt{bq}},-\frac{aq}{\sqrt{bq}},bq,\frac{b}{a}q^{-n},aq^{n+1}\end{matrix};q,q\right]=\frac{(aq,\frac{a^2q}{b^2};q)_n}{(\frac{aq}{b},\frac{a^2q}{b};q)_n}.$$

Taking $a\to\frac{a}{\sqrt{b}}$ and $b\to\sqrt{b}$ in Theorem 3.3 results in

$$P(\frac{a}{\sqrt{b}},\sqrt{b})=(\frac{1-bq^{2k}}{1-b}\frac{(\frac{b}{a},b,q^{-n},aq^n;q)_k}{(q,aq,bq^{n+1},\frac{b}{a}q^{1-n};q)_k}q^k)_{n,k\in\mathbb{N}},$$

$$P^{-1}(\frac{a}{\sqrt{b}},\sqrt{b})=(\frac{1-aq^{2k}}{1-a}\frac{(\frac{a}{b},a,q^{-n},bq^n;q)_k}{(q,bq,aq^{n+1},\frac{a}{b}q^{1-n};q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Setting $b\to\frac{a^2q}{b}$ in the above inversion, we get

$$Q(a,b)=(f_{n,k})_{n,k\in\mathbb{N}}=(\frac{1-\frac{a^2q}{b}q^{2k}}{1-\frac{a^2q}{b}}\frac{(\frac{aq}{b},\frac{a^2q}{b},q^{-n},aq^n;q)_k}{(q,aq,\frac{a^2q^{n+2}}{b},\frac{aq^{2-n}}{b};q)_k}q^k)_{n,k\in\mathbb{N}},$$

$$Q^{-1}(a,b)=(g_{n,k})_{n,k\in\mathbb{N}}=(\frac{1-aq^{2k}}{1-a}\frac{(\frac{b}{aq},a,q^{-n},\frac{a^2}{b}q^{n+1};q)_k}{(q,\frac{a^2q^2}{b},aq^{n+1},\frac{b}{a}q^{-n};q)_k}q^k)_{n,k\in\mathbb{N}}.$$

Set

$$a(k)=\frac{(\frac{a^2q^2}{b},\sqrt{b},-\sqrt{b},\sqrt{bq},-\sqrt{bq},\frac{a}{b};q)_k}{(\frac{b}{aq},\frac{aq}{\sqrt{b}},-\frac{aq}{\sqrt{b}},\frac{aq}{\sqrt{bq}},-\frac{aq}{\sqrt{bq}},bq;q)_k}$$

and

$$b(n)=\sum_{k=0}^{n}\frac{1-aq^{2k}}{1-a}\frac{(\frac{b}{aq},a,q^{-n},\frac{a^2}{b}q^{n+1};q)_k}{(q,\frac{a^2q^2}{b},aq^{n+1},\frac{b}{a}q^{-n};q)_k}q^k\frac{(\frac{a^2q^2}{b},\sqrt{b},-\sqrt{b},\sqrt{bq},-\sqrt{bq},\frac{a}{b};q)_k}{(\frac{b}{aq},\frac{aq}{\sqrt{b}},-\frac{aq}{\sqrt{b}},\frac{aq}{\sqrt{bq}},-\frac{aq}{\sqrt{bq}},bq;q)_k},$$

we see that

$$b(n)=\sum_{k=0}^{n}g_{n,k}a(k),$$

and thus

$$\frac{(\frac{a^2q^2}{b},\sqrt{b},-\sqrt{b},\sqrt{bq},-\sqrt{bq},\frac{a}{b};q)_n}{(\frac{b}{aq},\frac{aq}{\sqrt{b}},-\frac{aq}{\sqrt{b}},\frac{aq}{\sqrt{bq}},-\frac{aq}{\sqrt{bq}},bq;q)_n}=\sum_{k=0}^{n}\frac{1-\frac{a^2q}{b}q^{2k}}{1-\frac{a^2q}{b}}\frac{(\frac{aq}{b},\frac{a^2q}{b},q^{-n},aq^n;q)_k}{(q,aq,\frac{a^2q^{n+2}}{b},\frac{aq^{2-n}}{b};q)_k}q^k\frac{(aq,\frac{a^2q}{b^2};q)_k}{(\frac{aq}{b},\frac{a^2q}{b};q)_k}.$$

After some simplification, we obtain

$${}_5\phi_4\left[\begin{matrix}\frac{a^2q}{b^2},\frac{aq\sqrt{q}}{\sqrt{b}},-\frac{aq\sqrt{q}}{\sqrt{b}},q^{-n},aq^n\\ \frac{a\sqrt{q}}{\sqrt{b}},-\frac{a\sqrt{q}}{\sqrt{b}},\frac{a^2q^{n+2}}{b},\frac{aq^{2-n}}{b}\end{matrix};q,q\right]=\frac{(\frac{a^2q^2}{b},\sqrt{b},-\sqrt{b},\sqrt{bq},-\sqrt{bq},\frac{a}{b};q)_n}{(\frac{b}{aq},\frac{aq}{\sqrt{b}},-\frac{aq}{\sqrt{b}},\frac{aq}{\sqrt{bq}},-\frac{aq}{\sqrt{bq}},bq;q)_n}.$$

In particular, when $b=q$, the above identity becomes

$${}_5\phi_4\left[\begin{matrix}\frac{a^2}{q},aq,-aq,q^{-n},aq^n\\ a,-a,a^2q^{n+1},aq^{1-n}\end{matrix};q,q\right]=\frac{(a^2q,\sqrt{q},-\sqrt{q},q,-q,\frac{a}{q};q)_n}{(\frac{1}{a},a\sqrt{q},-a\sqrt{q},a,-a,q^2;q)_n}.$$

4. q-Differential operator

In this section, we combine the inversion formula and the q-differential operator to derive some identities.

The q-differential operator D_q and the q-shifted operator η, acting on the variable x, are defined by

$$D_q\{f(x)\}=\frac{f(x)-f(xq)}{x},\quad \eta\{f(x)\}=f(xq),$$

respectively. Moreover,

$$\eta^{-1}\{f(x)\}=f(xq^{-1}),\quad \theta=\eta^{-1}D_q.$$

Let a be a parameter and T be an operator. The operator aT and T^n are given by

$$(aT)\{f(x)\}=a\cdot T\{f(x)\}$$

and

$$T^0\{f(x)\}=f(x),\quad T^n\{f(x)\}=T\{T^{n-1}\{f(x)\}\},\ n=1,2,\ldots.$$

We also use the notation

$$\Phi={}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,T\right]$$

to denote the operator

$$\Phi\{f(x)\}=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}\Big((-1)^kq^{\binom{k}{2}}\Big)^{1+s-r}T^k\{f(x)\}.$$

In this following context, all operators only act on the variable d.

Zhang and Wang [14, Lemma 2.3] derived the following formulas

$$D_q^n\{\frac{(du;q)_\infty}{(dv;q)_\infty}\}=\frac{v^n(u/v;q)_n(duq^n;q)_\infty}{(dv;q)_\infty},$$

$$\theta^n\{\frac{(du;q)_\infty}{(dv;q)_\infty}\}=\frac{v^nq^{-\binom{n}{2}}(u/v;q)_n(du;q)_\infty}{(dvq^{-n};q)_\infty}.\tag{4.1}$$

Lemma 4.1 ([15, Theorem 2.3]) *We have*

$$ {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,bD_q\right]\{\frac{(du;q)_\infty}{(dv;q)_\infty}\}=\frac{(du;q)_\infty}{(dv;q)_\infty}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1,\ldots,a_r,u/v \\ b_1,\ldots,b_s,du \end{matrix};q,bv\right], \tag{4.2}$$

where b is independent of d.

We generalize Theorem 2.16 of [15] as follows.

Lemma 4.2 *We have*

$$ {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-b\theta\right]\{\frac{(du;q)_\infty}{(dv;q)_\infty}\}=\frac{(du;q)_\infty}{(dv;q)_\infty}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1,\ldots,a_r,u/v \\ b_1,\ldots,b_s,\frac{q}{dv} \end{matrix};q,\frac{bq}{d}\right],$$

where b is independent of d.

Proof By (4.1), we have

$$\begin{aligned} & {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-b\theta\right]\{\frac{(du;q)_\infty}{(dv;q)_\infty}\} \\ &=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}(-b)^k\theta^k\{\frac{(du;q)_\infty}{(dv;q)_\infty}\} \\ &=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}(-b)^k\frac{v^kq^{-\binom{k}{2}}(u/v;q)_k(du;q)_\infty}{(dvq^{-k};q)_\infty} \\ &=\frac{(du;q)_\infty}{(dv;q)_\infty}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1,\ldots,a_r,u/v \\ b_1,\ldots,b_s,\frac{q}{dv} \end{matrix};q,\frac{bq}{d}\right]. \quad \square \end{aligned}$$

Theorem 4.3 *We have*

$$\begin{aligned} & \frac{1-\frac{aq}{b}}{1-\frac{aq^{2n+1}}{b}}\frac{(a_1,\ldots,a_{r-1},u/v;q)_n}{(\frac{aq}{b},b_1,\ldots,b_{s-1},du;q)_n}z^n\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r} \\ &=\sum_{k=0}^{n}\frac{(q^{-n},\frac{a}{b}q^{n+1};q)_kq^k}{(q,\frac{a}{b}q^2;q)_k}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1,\ldots,a_{r-1},q^{-k},u/v \\ b_1,\ldots,b_{s-1},\frac{a}{b}q^{k+2},du \end{matrix};q,zq^k\right], \end{aligned}$$

where $z,a,b,a_1,\ldots,a_{r-1}$ and $b_1,\ldots,b_{s-1}$ are independent of n and d.

Proof By (1.2) and Theorem 3.1, we have

$$a(n)=\sum_{k=0}^{n}\frac{(\frac{a}{b}q;q)_n(q^{-n},\frac{a}{b}q^{n+1};q)_k}{(q,\frac{a}{b}q^2;q)_k}q^kb(k)\Leftrightarrow b(n)=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(1-\frac{a}{b}q^{2k+1})}{(q,\frac{a}{b}q^{n+2};q)_k(1-\frac{a}{b}q)}q^{nk}a(k). \tag{4.3}$$

Let

$$\begin{aligned} b(n)&=\sum_{k=0}^{n}\frac{(a_1,\ldots,a_{r-1},q^{-n};q)_k}{(q,b_1,\ldots,b_{s-1},\frac{a}{b}q^{n+2};q)_k}\frac{z^kq^{nk}}{v^k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}D_q^k \\ &=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(1-\frac{a}{b}q^{2k+1})}{(q,\frac{a}{b}q^{n+2};q)_k(1-\frac{a}{b}q)}q^{nk}\frac{1-\frac{aq}{b}}{1-\frac{aq^{2k+1}}{b}}\frac{(a_1,\ldots,a_{r-1};q)_k}{(b_1,\ldots,b_{s-1};q)_k}\frac{z^k}{v^k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}D_q^k. \end{aligned}$$

Taking $a(k)=\frac{1-\frac{aq}{b}}{1-\frac{aq^{2k+1}}{b}}\frac{(a_1,\ldots,a_{r-1};q)_k}{(b_1,\ldots,b_{s-1};q)_k}\frac{z^k}{v^k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}D_q^k$ in (4.3), we derive

$$\begin{aligned}&\frac{1-\frac{aq}{b}}{1-\frac{aq^{2n+1}}{b}}\frac{(a_1,\ldots,a_{r-1};q)_n}{(\frac{a}{b}q,b_1,\ldots,b_{s-1};q)_n}\frac{z^n}{v^n}\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}D_q^n\\&=\sum_{k=0}^n\frac{(q^{-n},\frac{a}{b}q^{n+1};q)_k}{(q,\frac{a}{b}q^2;q)_k}q^k{}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_{r-1},q^{-k}\\ b_1,\ldots,b_{s-1},\frac{a}{b}q^{k+2}\end{matrix};q,\frac{zq^k}{v}D_q\right].\end{aligned}$$

Acting on $\frac{(du;q)_\infty}{(dv;q)_\infty}$, by (4.1) and (4.2), we get

$$\begin{aligned}&\frac{1-\frac{aq}{b}}{1-\frac{aq^{2n+1}}{b}}\frac{(a_1,\ldots,a_{r-1};q)_n}{(\frac{a}{b}q,b_1,\ldots,b_{s-1};q)_n}\frac{z^n}{v^n}\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}\frac{v^n(u/v;q)_n(duq^n;q)_\infty}{(dv;q)_\infty}\\&=\sum_{k=0}^n\frac{(q^{-n},\frac{a}{b}q^{n+1};q)_k}{(q,\frac{a}{b}q^2;q)_k}q^k\frac{(du;q)_\infty}{(dv;q)_\infty}{}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_{r-1},q^{-k},u/v\\ b_1,\ldots,b_{s-1},\frac{a}{b}q^{k+2},du\end{matrix};q,zq^k\right].\end{aligned}$$

After some simplification, we complete the proof. □

Theorem 4.4 *We have*

$$\begin{aligned}&\frac{(a_1,\ldots,a_{r-1},u/v;q)_n}{(\frac{aq}{b},b_1,\ldots,b_{s-1},du;q)_n}z^n\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}\\&=\sum_{k=0}^n\frac{(q^{-n},\frac{b}{aq};q)_k}{(q,\frac{b}{a}q^{-n};q)_k}q^k{}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_{r-1},q^{-k},u/v\\ b_1,\ldots,b_{s-1},\frac{a}{b}q^{2-k},du\end{matrix};q,zq\right],\end{aligned}$$

where $z,a,b,a_1,\ldots,a_{r-1}$ *and* $b_1,\ldots,b_{s-1}$ *are independent of* n *and* d.

Proof By (1.2) and Theorem 3.2 we have

$$a(n)=\sum_{k=0}^n\frac{(\frac{a}{b}q;q)_n}{(q;q)_k}\frac{(q^{-n},\frac{b}{aq};q)_k}{(\frac{b}{a}q^{-n};q)_k}q^kb(k)\Leftrightarrow b(n)=\sum_{k=0}^n\frac{(q^{-n};q)_k}{(q,\frac{a}{b}q^{2-n};q)_k}q^ka(k).\tag{4.4}$$

Let

$$b(n)=\sum_{k=0}^n\frac{(q^{-n};q)_k}{(q,\frac{a}{b}q^{2-n};q)_k}q^k\frac{(a_1,\ldots,a_{r-1};q)_k}{(b_1,\ldots,b_{s-1};q)_k}\frac{z^k}{v^k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}D_q^k.$$

Taking $a(k)=\frac{(a_1,\ldots,a_{r-1};q)_k}{(b_1,\ldots,b_{s-1};q)_k}\frac{z^k}{v^k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}D_q^k$ in (4.4), we derive

$$\begin{aligned}&\frac{(a_1,\ldots,a_{r-1};q)_n}{(\frac{aq}{b},b_1,\ldots,b_{s-1};q)_n}\frac{z^n}{v^n}\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}D_q^n\\&=\sum_{k=0}^n\frac{(q^{-n},\frac{b}{aq};q)_k}{(q,\frac{b}{a}q^{-n};q)_k}q^k{}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_{r-1},q^{-k}\\ b_1,\ldots,b_{s-1},\frac{a}{b}q^{2-k}\end{matrix};q,\frac{zq}{v}D_q\right].\end{aligned}$$

Acting on $\frac{(du;q)_\infty}{(dv;q)_\infty}$, we complete the proof after some simplification. □

In a similar way, we derive the following theorem by (1.2) and Theorem 3.3.

Theorem 4.5 *We have*

$$\begin{aligned}&\frac{1-b^2}{1-b^2q^{2n}}\frac{(a_1,\ldots,a_{r-2},abq,u/v;q)_n}{(b_1,\ldots,b_{s-2},\frac{b}{a},b^2,du;q)_n}z^n\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}\\&=\sum_{k=0}^n\frac{1-abq^{2k}}{1-ab}\frac{(\frac{a}{b},ab,q^{-n},b^2q^n;q)_k}{(q,b^2q,abq^{n+1},\frac{a}{b}q^{1-n};q)_k}q^k{}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_{r-2},q^{-k},abq^k,u/v\\ b_1,\ldots,b_{s-2},b^2q^{k+1},\frac{b}{a}q^{1-k},du\end{matrix};q,zq\right],\end{aligned}$$

where $z, a, b, a_1, \ldots, a_{r-2}$ and $b_1, \ldots, b_{s-2}$ are independent of n and d.

For the operator θ, we derive some identities by (1.2) and Theorems 3.1–3.3.

Theorem 4.6 *We have*

$$\frac{1-\frac{aq}{b}}{1-\frac{aq^{2n+1}}{b}}\frac{(a_1,\ldots,a_{r-1},u/v;q)_n}{(\frac{aq}{b},b_1,\ldots,b_{s-1},\frac{q}{dv};q)_n}(\frac{zq}{d})^n\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r}$$
$$=\sum_{k=0}^{n}\frac{(q^{-n},\frac{a}{b}q^{n+1};q)_k q^k}{(q,\frac{a}{b}q^2;q)_k}{}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_{r-1},q^{-k},u/v\\ b_1,\ldots,b_{s-1},\frac{a}{b}q^{k+2},\frac{q}{dv}\end{matrix};q,\frac{zq^{k+1}}{d}\right],$$

where $z, a, b, a_1, \ldots, a_{r-1}$ and $b_1, \ldots, b_{s-1}$ are independent of n and d.

Theorem 4.7 *We have*

$$\frac{(a_1,\ldots,a_{r-1},u/v;q)_n}{(\frac{aq}{b},b_1,\ldots,b_{s-1},\frac{q}{dv};q)_n}(\frac{zq}{d})^n\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r}$$
$$=\sum_{k=0}^{n}\frac{(q^{-n},\frac{b}{aq};q)_k}{(q,\frac{b}{a}q^{-n};q)_k}q^k{}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_{r-1},q^{-k},u/v\\ b_1,\ldots,b_{s-1},\frac{a}{b}q^{2-k},\frac{q}{dv}\end{matrix};q,\frac{zq^2}{d}\right],$$

where $z, a, b, a_1, \ldots, a_{r-1}$ and $b_1, \ldots, b_{s-1}$ are independent of n and d.

Theorem 4.8 *We have*

$$\frac{1-b^2}{1-b^2q^{2n}}\frac{(a_1,\ldots,a_{r-2},abq,u/v;q)_n}{(b_1,\ldots,b_{s-2},\frac{b}{a},b^2,\frac{q}{dv};q)_n}(\frac{zq}{d})^n\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r}$$
$$=\sum_{k=0}^{n}\frac{1-abq^{2k}}{1-ab}\frac{(\frac{a}{b},ab,q^{-n},b^2q^n;q)_k}{(q,b^2q,abq^{n+1},\frac{a}{b}q^{1-n};q)_k}q^k{}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1,\ldots,a_{r-2},q^{-k},abq^k,u/v\\ b_1,\ldots,b_{s-2},b^2q^{k+1},\frac{b}{a}q^{1-k},\frac{q}{dv}\end{matrix};q,\frac{zq^2}{d}\right],$$

where $z, a, b, a_1, \ldots, a_{r-2}$ and $b_1, \ldots, b_{s-2}$ are independent of n and d.

We conclude with two examples of the applications of these theorems.

Example 4.9 Let $a_1 = a$, $a_2 = a^{\frac{1}{2}}q$, $a_3 = -a^{\frac{1}{2}}q$, $a_4 = b$, $a_5 = c$, $b_1 = a^{\frac{1}{2}}$, $b_2 = -a^{\frac{1}{2}}$, $b_3 = \frac{aq}{b}$, $b_4 = \frac{aq}{c}$. Taking $\frac{aq}{b} \to a$ in Theorem 4.3, we have

$$\frac{1-a}{1-aq^{2n}}\frac{(a,a^{\frac{1}{2}}q,-a^{\frac{1}{2}}q,b,c,u/v;q)_n}{(a,a^{\frac{1}{2}},-a^{\frac{1}{2}},\frac{aq}{b},\frac{aq}{c},du;q)_n}z^n$$
$$=\sum_{k=0}^{n}\frac{(q^{-n},aq^n;q)_k q^k}{(q,aq;q)_k}{}_7\phi_6\left[\begin{matrix}a,a^{\frac{1}{2}}q,-a^{\frac{1}{2}}q,b,c,q^{-k},u/v\\ a^{\frac{1}{2}},-a^{\frac{1}{2}},\frac{aq}{b},\frac{aq}{c},aq^{k+1},du\end{matrix};q,zq^k\right].$$

After some simplification, we get

$$\frac{(b,c,u/v;q)_n}{(\frac{aq}{b},\frac{aq}{c},du;q)_n}z^n=\sum_{k=0}^{n}\frac{(q^{-n},aq^n;q)_k q^k}{(q,aq;q)_k}{}_7\phi_6\left[\begin{matrix}a,a^{\frac{1}{2}}q,-a^{\frac{1}{2}}q,b,c,q^{-k},u/v\\ a^{\frac{1}{2}},-a^{\frac{1}{2}},\frac{aq}{b},\frac{aq}{c},aq^{k+1},du\end{matrix};q,zq^k\right].$$

Example 4.10 Let $a_1 = a$, $a_2 = a^{\frac{1}{2}}q$, $a_3 = b$, $b_1 = a^{\frac{1}{2}}$, $b_2 = \frac{aq}{b}$. Taking $\frac{aq}{b} \to b^2$ in Theorem 4.7, we have

$$\frac{(a,a^{1/2}q,b,u/v;q)_n}{(b^2,a^{1/2},\frac{aq}{b},\frac{q}{dv};q)_n}(\frac{zq}{d})^n=\sum_{k=0}^{n}\frac{(q^{-n},\frac{1}{b^2};q)_k}{(q,\frac{1}{b^2}q^{1-n};q)_k}q^k{}_5\phi_4\left[\begin{matrix}a,a^{1/2}q,b,q^{-k},u/v\\ a^{1/2},\frac{aq}{b},b^2q^{1-k},\frac{q}{dv}\end{matrix};q,\frac{zq^2}{d}\right].$$

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