Journal of Mathematical Research with Applications
Sept., 2019, Vol. 39, No. 5, pp. 459–468
DOI:10.3770/j.issn:2095-2651.2019.05.003
Http://jmre.dlut.edu.cn

Meet Uniform Continuous Posets

Xuxin MAO[1,*], **Luoshan XU**[2]
1. *College of Science, Nanjing University of Aeronautics and Astronautics, Jiangsu* 210016*, P. R. China;*
2. *Department of Mathematics, Yangzhou University, Jiangsu* 225002*, P. R. China*

Abstract In this paper, as a generalization of uniform continuous posets, the concept of meet uniform continuous posets via uniform Scott sets is introduced. Properties and characterizations of meet uniform continuous posets are presented. The main results are: (1) A uniform complete poset L is meet uniform continuous iff $\uparrow(U\cap\downarrow x)$ is a uniform Scott set for each $x\in L$ and each uniform Scott set U; (2) A uniform complete poset L is meet uniform continuous iff for each $x\in L$ and each uniform subset S, one has $x\wedge\bigvee S=\bigvee\{x\wedge s\mid s\in S\}$. In particular, a complete lattice L is meet uniform continuous iff L is a complete Heyting algebra; (3) A uniform complete poset is meet uniform continuous iff every principal ideal is meet uniform continuous iff all closed intervals are meet uniform continuous iff all principal filters are meet uniform continuous; (4) A uniform complete poset L is meet uniform continuous if L^1 obtained by adjoining a top element 1 to L is a complete Heyting algebra; (5) Finite products and images of uniform continuous projections of meet uniform continuous posets are still meet uniform continuous.

Keywords uniform set; uniform Scott set; complete Heyting algebra; meet uniform continuous poset; principal ideal; uniform continuous projection

MR(2010) Subject Classification 06A06; 06A11; 06B35; 54A10

1. Introduction

In 1972, Dana Scott introduced the notion of continuous lattices in order to provide models for the semantics of programming languages [1]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was introduced and extensively studied [2]. It should be noted that a distinctive feature of the theory of continuous domains is that many of the considerations are closely interlinked with topological ideas. The Scott topology, as an order-theoretical topology, is of fundamental importance in domain theory [2, 3]. Lawson in [3] gave a remarkable characterization that a dcpo L is continuous iff the lattice $\sigma^*(L)$ of all Scott-closed subsets of L is completely distributive. A meet continuous lattice is a complete

Received September 24, 2018; Accepted May 22, 2019
Supported by the National Natural Science Foundation of China (Grant Nos. 11671008; 11101212), the Natural Science Foundation of Jiangsu Province (Grant No. BK20170483) and the Fund of University Speciality Construction of Jiangsu Province (Grant No. PPZY2015B109).
* Corresponding author
E-mail address: xuxinmao@yahoo.com (Xuxin MAO); luoshanxu@hotmail.com (Luoshan XU)

lattice in which the binary meet operation distributes over directed suprema [2]. This algebraic notion has a purely topological characterization that can be generalized to the setting of dcpos by the Scott topology in [2, 4] without involving the meet operations: A dcpo L is called *meet continuous* if for any $x \in L$ and any directed subset D with $\sup D \geqslant x$, one has $x \in cl_\sigma(\downarrow D \cap \downarrow x)$, where $cl_\sigma(\downarrow D \cap \downarrow x)$ is the Scott closure of the set $\downarrow D \cap \downarrow x$.

Based on the concept of uniform sets in the theory of program expansion [5], Bai introduced the concept of uniform continuous posets, a generalization of completely distributive lattices, and showed that uniform continuous posets have many properties in common with continuous lattices (see Theories of uniform continuous partial order sets. Journal of Northwest Normal University, 1996, 32(2)). Later, Ruan and Zhang in [6] introduced the concept of uniform Scott sets and gave more properties of uniform continuous posets.

It is well-known that a dcpo is continuous iff it is meet continuous and quasicontinuous [7–11]. Since uniform continuous posets share some properties with continuous dcpos [6–12], it is natural to consider the meet uniform continuity and the quasi uniform continuity on posets. So, in this paper, in the manner of defining the meet continuity of dcpos, we introduce the concept of meet uniform continuous posets as a generalization of uniform continuous posets via the uniform Scott sets. Properties and characterizations of meet uniform continuous posets are presented.

The paper is organized as follows. In the preliminary section, we recall some basic notions such as uniform sets, uniform way-below relations and uniform continuous posets. Some basic properties of uniform way-below relations are given. In the third section, the concept of the meet uniform continuous posets via the uniform Scott sets is introduced. Characterizations of meet uniform continuous posets are given by some kind of distributivity, principal ideals, principal filters and closed intervals. It will be established that a uniform complete poset is meet uniform continuous iff every principal ideal is meet uniform continuous iff all closed intervals are meet uniform continuous iff all principal filters are meet uniform continuous. The forth section is devoted to operational properties of meet uniform continuous posets. We will end the paper with some conclusion remarks.

2. Preliminaries

We recall some basic notions and results [2, 6].

Let $(L, \leqslant)$ be a poset. A principal ideal (resp., principal filter) is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$ (resp., $\uparrow x = \{y \in L \mid x \leqslant y\}$). A closed interval $[x, y]$ is a set of the form $\uparrow x \cap \downarrow y$ for $x \leqslant y$. Note that closed intervals are always nonempty. For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists\, x \in A, y \leqslant x\}$, $\uparrow A = \{y \in L \mid \exists\, x \in A, x \leqslant y\}$. A subset A is a lower set (resp., an upper set) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a lower bound (resp., an upper bound) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). A subset A is a consistent set if A has an upper bound. The supremum of A in L is denoted by $\bigvee A$ or $\sup A$. The infimum of A is denoted by $\bigwedge A$ or $\inf A$. If $\sup A$ (resp., $\inf A$) exists, we say that A has a sup (resp., A has an inf).

A nonempty subset D of L is directed if every finite subset of D has an upper bound in D. A poset L is a directed complete partially ordered set (dcpo, for short) if every directed subset of L has a sup. A sup semilattice (resp., a meet semilattice) is a poset in which every nonempty finite subset has a sup (resp., an inf). A poset which is both a sup semilattice and a meet semilattice is a lattice. A poset is a *bounded complete poset* (bc-poset, for short) if every consistent subset has a sup. In particular, a bc-poset has a smallest element. It is easy to prove that a poset is bounded complete if and only if every nonempty subset has an inf. So, every bc-poset is a meet semilattice. A bounded complete dcpo is called a complete semilattice and written bc-dcpo for short. A complete lattice is a poset in which every subset has a sup and an inf.

In a poset L, we say that x approximates y, written $x \ll y$ if whenever D is a directed set that has a supremum $\sup D \geqslant y$, then $x \leqslant d$ for some $d \in D$. The poset L is said to be continuous if every element is the directed supremum of elements that approximate it. A continuous poset which is also a dcpo is called a continuous domain or a domain. A continuous poset which is also a complete lattice is called a continuous lattice.

A subset A of a poset L is Scott closed if $\downarrow A = A$ and for any directed set $D \subseteq A$, $\sup D \in A$ whenever $\sup D$ exists. The complements of the Scott closed sets form a topology, called the Scott topology and denoted by $\sigma(L)$.

Definition 2.1 ([6]) *Let L be a poset and $S \subseteq L$. If for all a, $b \in S$, there exists $c \in L$ such that $a \leqslant c$ and $b \leqslant c$, then S is called a uniform set. A poset L is called uniform complete if every uniform subset of L has a supremum.*

Remark 2.2 (1) All directed sets, all consistent sets and the empty set are uniform sets. So, every uniform complete poset is a bc-dcpo. In particular, every uniform complete poset is a meet semilattice and has a bottom.

(2) If a poset L has a top element, then every subset A of L is a uniform set. But a subset of A may not be uniform in the poset A with the inherited order.

Example 2.3 A bc-dcpo need not be uniform complete. For example, let $X = \{a, b, c\}$ and $\mathcal{P}(X)$ be the powerset of X with the inclusion order. Let $L = \mathcal{P}(X) \backslash \{X\}$. It is clear that L is a bc-dcpo. But the uniform subset $\{\{a\}, \{b\}, \{c\}\}$ of L does not have a supremum. So, L is not uniform complete.

Definition 2.4 ([6]) *Let L be a uniform complete poset and x, $y \in L$. We say that x is uniform way-below y, written $x \ll_v y$ if for any uniform subset S of L with $\sup S \geqslant y$, there is some $s \in S$ such that $x \leqslant s$. We say that x is uniform compact if $x \ll_v x$. The set of all uniform compact elements is denoted by $UK(L)$. For each $x \in L$, we write $\Downarrow_v x = \{y \in L \mid y \ll_v x\}$ and $\Uparrow_v x = \{y \in L \mid x \ll_v y\}$. A uniform complete poset L is called a uniform continuous poset if for each $x \in L$, $x = \bigvee \Downarrow_v x$.*

Proposition 2.5 ([6]) *Let L be a uniform complete poset and $\perp$ be the bottom. Then for all x, y, u, $z \in L$:*

(1) $x \ll_v y \Longrightarrow x \leqslant y$;

(2) $u \leqslant x \ll_v y \leqslant z \Longrightarrow u \ll_v z$;

(3) If $x \in L \setminus \{\bot\}$, then one has $\bot \ll_v x$, but $\bot \ll_v \bot$ does not hold;

(4) $\Downarrow_v x$ is a uniform set.

Definition 2.6 ([6]) *Let L and M be uniform complete posets. A function $f : L \to M$ is called uniform continuous if f preserves suprema of uniform sets, that is, f is order-preserving and $f(\sup S) = \sup f(S)$ for all uniform subsets S of L.*

Lemma 2.7 *Let L be a uniform complete poset. Then for all $x \in L$ and all uniform subset S, we have $\downarrow x \cap \downarrow S = \downarrow \{x \wedge s \mid s \in S\} = \downarrow (x \wedge \bigvee S) \cap \downarrow S$.*

Proof Straightforward. □

3. Meet uniform continuous posets and characterizations

In this section, via uniform Scott sets in [6], the notion of meet uniform continuous posets is introduced. Properties and characterizations of meet uniform continuous posets are presented.

Definition 3.1 ([6]) *Let L be a uniform complete poset. A subset U of L is called a uniform Scott set if the following two conditions are satisfied:*

(1) $U = \uparrow U$;

(2) For all uniform subset $S \subseteq L$, $\bigvee S \in U$ implies $S \cap U \neq \emptyset$.

The family of all uniform Scott sets of L is denoted by $US(L)$. The family of the complements of all uniform Scott sets of L is denoted by $US^*(L)$, i.e., $US^*(L) = \{F \subseteq L | L \setminus F \in US(L)\}$.

Proposition 3.2 *Let L be a uniform complete poset. Then*

(1) $F \in US^*(L)$ iff $F = \downarrow F$ and $\bigvee S \in F$ for any uniform set $S \subseteq F$;

(2) $L \in US^*(L)$, $\emptyset \notin US^*(L)$;

(3) $\forall x \in L$, $\downarrow x \in US^*(L)$;

(4) For any $\{F_\alpha\}_{\alpha \in \Gamma} \subseteq US^*(L)$, $\bigcap_{\alpha \in \Gamma} F_\alpha \in US^*(L)$ whenever $\bigcap_{\alpha \in \Gamma} F_\alpha \neq \emptyset$;

(5) $\forall x \in L$, $x \in UK(L)$ iff $\uparrow x \in US(L)$.

Proof Straightforward. □

Remark 3.3 By Definition 3.1 and Remark 2.2, uniform Scott sets of a uniform complete poset are Scott open sets of the poset. By Proposition 3.2(1) and Remark 2.2, complements of uniform Scott sets of a complete lattice are all principal ideals. Since unions of two principal ideals need not be principal ideals, clearly an intersection of two uniform Scott sets need not be a uniform Scott set. This reveals that the uniform Scott sets of a complete lattice need not be a topology.

Definition 3.4 *Let L be a uniform complete poset. If for any $x \in L$ and any uniform subset S with $\sup S \geqslant x$, one has $x \in \bigcap \{F \in US^*(L) \mid \downarrow x \cap \downarrow S \subseteq F\}$, then L is called a meet uniform continuous poset.*

Proposition 3.5 *Every uniform continuous poset L is meet uniform continuous.*

Proof Let $x \in L$ and $S \subseteq L$ a uniform set with $\sup S \geqslant x$. It is clear that $\Downarrow_v x \subseteq \downarrow S$ and $\Downarrow_v x \subseteq \downarrow S \cap \downarrow x$. It follows from the uniform continuity of L, Proposition 2.5(4) and Proposition 3.2(1) that $x = \sup \Downarrow_v x \in \bigcap\{F \in US^*(L) \mid \downarrow x \cap \downarrow S \subseteq F\}$. This shows that L is a meet uniform continuous poset. □

Theorem 3.6 *Let L be a uniform complete poset. Then L is meet uniform continuous iff for all $U \in US(L)$ and all $x \in L$, one has $\uparrow(U \cap \downarrow x) \in US(L)$.*

Proof $\Longrightarrow$. Let $x \in L$ and $U \in US(L)$. Clearly, $\uparrow(U \cap \downarrow x)$ is an upper set. Suppose that S is a uniform subset with $\bigvee S \in \uparrow (U \cap \downarrow x)$. There is $y \in U \cap \downarrow x$ such that $y \leqslant \bigvee S$. It follows from the meet uniform continuity of L that $y \in \bigcap\{F \in US^*(L) \mid \downarrow y \cap \downarrow S \subseteq F\}$. Assume that $\uparrow(U \cap \downarrow x) \cap S = \emptyset$. Then $U \cap \downarrow x \cap \downarrow S = \emptyset$. So, $\downarrow y \cap \downarrow S \subseteq \downarrow x \cap \downarrow S \subseteq L \backslash U$. Since $L \backslash U \in US^*(L)$, we have $y \in L \backslash U$, a contradiction to $y \in U$. This shows that $\uparrow(U \cap \downarrow x) \cap S \neq \emptyset$. By Definition 3.1, $\uparrow(U \cap \downarrow x) \in US(L)$.

$\Longleftarrow$. Let $x \in L$ and $S \subseteq L$ a uniform set with $\sup S \geqslant x$. Assume that there is $F_0 \in US^*(L)$ such that $\downarrow x \cap \downarrow S \subseteq F_0$ but $x \notin F_0$. Then $x \in L \backslash F_0$. So, $x \in \uparrow ((L \backslash F_0) \cap \downarrow x)$ and $\uparrow ((L \backslash F_0) \cap \downarrow x) \in US(L)$. Since $x \leqslant \vee S$, we have $\vee S \in \uparrow ((L \backslash F_0) \cap \downarrow x)$. By Definition 3.1, $\uparrow((L \backslash F_0) \cap \downarrow x) \cap S \neq \emptyset$. So, $(L \backslash F_0) \cap \downarrow x \cap \downarrow S \neq \emptyset$, a contradiction to $\downarrow x \cap \downarrow S \subseteq F_0$. This shows that $x \in \bigcap\{F \in US^*(L) \mid \downarrow x \cap \downarrow S \subseteq F\}$. By Definition 3.4, L is meet uniform continuous. □

By Theorem 3.6, we immediately have

Corollary 3.7 *A uniform complete poset L is meet uniform continuous iff for any $U \in US(L)$ and any lower subset C of L, one has $\uparrow (U \cap C) = \bigcup_{x \in C} \uparrow (U \cap \downarrow x) \in US(L)$.*

Theorem 3.8 *Let L be a uniform complete poset. Then the following statements are equivalent:*

(1) *L is a meet uniform continuous poset;*

(2) *For each $x \in L$ and each uniform subset S, we have $x \wedge \bigvee S = \bigvee\{x \wedge s \mid s \in S\}$;*

(3) *For two uniform subsets S, H, we have $\bigvee S \wedge \bigvee H = \bigvee(\downarrow S \cap \downarrow H)$;*

(4) *For each $x \in L$ and each uniform subset S, $x \leqslant \bigvee S$ implies that $x = \bigvee(\downarrow x \cap \downarrow S)$.*

Proof (1) $\Longrightarrow$ (2). Let L be a meet uniform continuous poset. Then for each $x \in L$ and each uniform subset S, it follows from Remark 2.2 that $x \wedge \bigvee S$ is an upper bound of the set $\{x \wedge s \mid s \in S\}$. Hence, the set $\{x \wedge s \mid s \in S\}$ is a uniform set of L. Let $y = \bigvee\{x \wedge s \mid s \in S\}$. By Lemma 2.7, we have

$$y = \bigvee\{x \wedge s \mid s \in S\} = \bigvee \downarrow\{x \wedge s \mid s \in S\} = \bigvee(\downarrow x \cap \downarrow S) = \bigvee(\downarrow(x \wedge \bigvee S) \cap \downarrow S).$$

This shows that $\downarrow (x \wedge \bigvee S) \cap \downarrow S \subseteq \downarrow y \in US^*(L)$. It follows from the meet uniform continuity of L and $x \wedge \bigvee S \leqslant \bigvee S$ that $x \wedge \bigvee S \in \bigcap\{F \in US^*(L) \mid \downarrow (x \wedge \bigvee S) \cap \downarrow S \subseteq F\}$. So, we have $x \wedge \bigvee S \in \downarrow y$. This shows that $x \wedge \bigvee S = y = \bigvee\{x \wedge s \mid s \in S\}$.

(2) $\Longrightarrow$ (3). For two uniform subsets S, H, it follows from (2) that $\bigvee S \wedge \bigvee H = \bigvee\{\bigvee S \wedge h \mid h \in H\} = \bigvee\{s \wedge h \mid s \in S, h \in H\} = \bigvee(\downarrow S \cap \downarrow H)$.

(3) $\Longrightarrow$ (4). For each $x \in L$ and each uniform subset S with $x \leqslant \bigvee S$, it follows from (3) that $x = x \wedge \bigvee S = \bigvee(\downarrow x) \wedge \bigvee S = \bigvee(\downarrow x \cap \downarrow S)$.

(4) $\Longrightarrow$ (1). For any $x \in L$ and any uniform subset S with $\sup S \geqslant x$, we need to show that $x \in \bigcap\{F \in US^*(L) \mid \downarrow x \cap \downarrow S \subseteq F\}$. Assume that there is $F_0 \in US^*(L)$ such that $\downarrow x \cap \downarrow S \subseteq F_0$ but $x \notin F_0$. It follows from $\sup S \geqslant x$ and (4) that $x = \bigvee(\downarrow x \cap \downarrow S) \in L \backslash F_0$. By Definition 3.1, we have $\downarrow x \cap \downarrow S \cap (L \backslash F_0) \neq \emptyset$, a contradiction to $\downarrow x \cap \downarrow S \subseteq F_0$. This shows that $x \in \bigcap\{F \in US^*(L) \mid \downarrow x \cap \downarrow S \subseteq F\}$. So, L is meet uniform continuous. $\square$

Corollary 3.9 *If L is a meet uniform continuous poset, then L and L^1 are both meet continuous dcpos, where $L^1 = L \cup \{1\}$ is obtained from L by adjoining a top element $1 \notin L$ such that $x \leqslant 1$ for all $x \in L$.*

Proof Follows from Remark 2.2, Theorem 3.8, Theorem 3.1 in [8], Definition O-4.1 and Remark III-2.2 in [2]. $\square$

Corollary 3.10 *A complete lattice L is meet uniform continuous if and only if L is a complete Heyting algebra (cHa, for short). Particularly, every meet uniform continuous complete lattice is a meet continuous lattice.*

Proof Apply Remark 2.2 and Theorem 3.8. $\square$

Remark 3.11 It is easy to see that every finite lattice is a continuous lattice. But finite lattices need not be distributive. For example, both the diamond lattice and the pentagon lattice are non-distributive. By Corollary 3.10, neither the diamond lattice nor the pentagon lattice is meet uniform continuous, showing that continuous lattices need not be meet uniform continuous.

Example 3.12 Let $L = (0,1) \times (0,1) \cup \{(0,0),\ (1,1)\}$ be equipped with the usual order. Then L is a distributive complete lattice. But L is not meet continuous [2, Counterexample O-4.5(1)]. By Corollary 3.10, L is not meet uniform continuous.

4. Operational properties of meet uniform continuous posets

In this section, operational properties of meet uniform continuous posets are concerned. We consider the meet uniform continuity of subposets, extensions by adjoining a top element, finite products and images under uniform continuous projections of meet uniform continuous posets.

Lemma 4.1 *Let L be a uniform complete poset and $A \subseteq L$ a lower set. Then $x \wedge_A t = x \wedge_L t$ for all $x,\ t \in A$, where $x \wedge_A t$ and $x \wedge_L t$ denote the meet of x and t in the poset A and L, respectively. In particular, every lower set of a uniform complete poset is a meet semilattice.*

Proof Straightforward. $\square$

Lemma 4.2 *Let L be a uniform complete poset and $A \in US^*(L)$. Then $\bigvee S = \bigvee_A S$ for all uniform subset S in A, where $\bigvee_A S$ denotes the supremum of S in the poset A with the inherited order. In particular, the poset A is uniform complete.*

Proof Apply Proposition 3.2(1). □

Proposition 4.3 *Let L be a meet uniform continuous poset and $A \in US^*(L)$. Then A in the inherited order is also a meet uniform continuous poset. Particularly, every principal ideal of L is meet uniform continuous, and a cHa.*

Proof For any $x \in A$ and any uniform subset S in A, it follows from the meet uniform continuity of L, Lemmas 4.1, 4.2 and Theorem 3.8 that $x \wedge_A (\bigvee_A S) = x \wedge_L (\bigvee_L S) = \bigvee_L \{x \wedge_L s \mid s \in S\} = \bigvee_A \{x \wedge_A s \mid s \in S\}$. By Theorem 3.8, A in the inherited order is also a meet uniform continuous poset. □

Theorem 4.4 *Let L be a uniform complete poset. Then L is a meet uniform continuous poset if and only if every principal ideal is a meet uniform continuous poset.*

Proof $\Longrightarrow$. Follows from Proposition 4.3.

$\Longleftarrow$. Assume that each principal ideal of L is meet uniform continuous. For any $x \in L$ and any uniform subset S of L with $\sup_L S := h \geqslant x$, let $A = \downarrow h$. By Proposition 3.2(3) and Lemma 4.2, we have $\sup_L S = \sup_A S$. It follows from $x \leqslant \sup_A S$, the meet uniform continuity of A, Theorem 3.8 and Lemma 4.2 that $x = \bigvee_A (\downarrow x \cap \downarrow S) = \bigvee_L (\downarrow x \cap \downarrow S)$. By Theorem 3.8, L is a meet uniform continuous poset. □

Corollary 4.5 *Let L be a uniform complete poset. Then every closed interval of L is meet uniform continuous iff each principal filer $\uparrow x$ is meet uniform continuous.*

Proof Apply Theorem 4.4 to the principal filters of L. □

Theorem 4.6 *Let L be a uniform complete poset. Then the following statements are equivalent:*

(1) L is a meet uniform continuous poset;

(2) Every principal ideal of L is meet uniform continuous;

(3) Every closed interval of L is meet uniform continuous;

(4) Every principal filer of L is meet uniform continuous.

Proof (1) $\Longleftrightarrow$ (2). Follows from Theorem 4.4.

(3) $\Longleftrightarrow$ (4). Follows from Corollary 4.5.

(1) $\Longrightarrow$ (3). Assume that L is a meet uniform continuous poset. Let $[a, b]$ be a closed interval of L. It follows from Proposition 4.3 that the principal ideal $\downarrow b$ is a cHa. Since $[a, b]$ is a closed interval of $\downarrow b$, $[a, b]$ is also a cHa and thus meet uniform continuous.

(3) $\Longrightarrow$ (2). By Remark 2.2(1), L has a bottom $\perp$. Then every principal ideal $\downarrow x$ of L is the closed interval $[\perp, x]$ and hence meet uniform continuous. □

Proposition 4.7 *Let L be a complete Heyting algebra and $1 \in L$ be the top element. If the poset $L \backslash \{1\}$ is uniform complete, then $L \backslash \{1\}$ is meet uniform continuous.*

Proof For each $x \in L\backslash\{1\}$ and each uniform subset S in $L\backslash\{1\}$, it follows from the uniform completeness of $L\backslash\{1\}$ that the supremum of S in the poset $L\backslash\{1\}$ exists and is denoted by $\bigvee_{L\backslash\{1\}} S \neq 1$. So, $\bigvee_{L\backslash\{1\}} S = \bigvee_L S$. By Lemma 4.1, $x \wedge_{L\backslash\{1\}} t = x \wedge_L t$ for any $t \in L\backslash\{1\}$. Thus, $x \wedge_{L\backslash\{1\}} (\bigvee_{L\backslash\{1\}} S) = x \wedge_L (\bigvee_L S) = \bigvee_L \{x \wedge_L s \mid s \in S\} = \bigvee_{L\backslash\{1\}} \{x \wedge_{L\backslash\{1\}} s \mid s \in S\}$. It follows from Theorem 3.8 that the poset $L\backslash\{1\}$ is meet uniform continuous. $\square$

Example 4.8 Let $X = \{a, b, c\}$ and $\mathcal{P}(X)$ be the powerset of X with the inclusion order. It is clear that $\mathcal{P}(X)$ is a cHa. Let $L = \mathcal{P}(X)\backslash\{X\}$. By Example 2.3, L is a bc-dcpo but L is not uniform complete. So, L is not meet uniform continuous. This shows that the uniform completeness condition of $L\backslash\{1\}$ in Proposition 4.7 cannot be replaced with the condition of bounded completeness.

Corollary 4.9 *Let L be a complete Heyting algebra with the top element $1 \ll_v 1$. Then $L\backslash\{1\}$ is meet uniform continuous.*

Proof It follows from $1 \ll_v 1$ and Proposition 3.2(5) that the set $\{1\}$ is a uniform Scott set. By Corollary 3.10, Lemma 4.2 and Proposition 4.7, $L\backslash\{1\}$ is meet uniform continuous. $\square$

Corollary 4.10 *Let L be a uniform complete poset. If $L^1 = L \cup \{1\}$ is a cHa, then L is meet uniform continuous.*

Proof Follows from Proposition 4.7. $\square$

Example 4.11 Let $L = \{\perp, a, b, c\}$. The order on L is defined as follows: $\perp \leqslant a$, $\perp \leqslant b$, $\perp \leqslant c$. It is easy to see that L is a meet uniform continuous poset. But $L^1 = L \cup \{1\}$ is the diamond lattice. By Remark 3.11, L^1 is not meet uniform continuous. Particularly, L^1 is not a cHa.

Proposition 4.12 *Let L and M be meet uniform continuous posets. Then the product $L \times M$ is meet uniform continuous.*

Proof Let $S \subseteq L \times M$ be a uniform set. It is easy to see that $p(S)$ is a uniform set of L and $q(S)$ is a uniform set of M, where $p : L \times M \to L$ and $q : L \times M \to M$ are the projection maps. It follows from the uniform completeness of L and M that $\sup_{L\times M} S = (\sup_L p(S), \sup_M q(S))$. This shows that the product $L \times M$ is uniform complete.

Let $(a, m) \in L \times M$. It is directly to show that $\downarrow_{L\times M} (a, m) = \downarrow_L a \times \downarrow_M m$. It follows from Proposition 4.3 and the meet uniform continuities of L and M that the principal ideal $\downarrow_L a$ of L and the principal ideal $\downarrow_M m$ of M are both cHa. Hence, the principal ideal $\downarrow_{L\times M} (a, m)$ of $L \times M$ is also a cHa. By Theorem 4.4, the product $L \times M$ is meet uniform continuous. $\square$

Let L be a uniform complete poset. A function $p : L \to L$ is called a *uniform continuous projection* if $p^2 = p$ and p is uniform continuous.

Lemma 4.13 *For a uniform complete poset L and a uniform continuous projection $p : L \to L$,*

(1) For all uniform subset S of $p(L)$, we have $\bigvee_L S = \bigvee_{p(L)} S$. In particular, the poset $p(L)$ is uniform complete;

(2) For all x, $t \in p(L)$, we have $p(x \wedge_L t) = p(x \wedge_{p(L)} t)$.

Proof (1) Let S be a uniform subset of $p(L)$. Then S is also a uniform subset of L. Since $p : L \to L$ is a uniform continuous projection, we have $p(\bigvee_L S) = \bigvee_L p(S) = \bigvee_L S$. This shows that $\bigvee_L S = \bigvee_{p(L)} S$.

(2) Let $x, t \in p(L)$. By (1) and Remark 2.2(1), L and $p(L)$ are both meet semilattices. Since p is order-preserving and $p^2 = p$, we have $p(x \wedge_L t) \leqslant p(x) = x$ and $p(x \wedge_L t) \leqslant p(t) = t$. This shows that $p(x \wedge_L t) \leqslant x \wedge_{p(L)} t \leqslant x \wedge_L t$. So, $p(x \wedge_L t) = p^2(x \wedge_L t) \leqslant p(x \wedge_{p(L)} t) \leqslant p(x \wedge_L t)$. Hence, $p(x \wedge_L t) = p(x \wedge_{p(L)} t)$. □

Theorem 4.14 *Let L be a meet uniform continuous poset and $p : L \to L$ a uniform continuous projection. Then $p(L)$ in the inherited order is meet uniform continuous.*

Proof Let $x \in p(L)$ and S a uniform subset of $p(L)$. By the meet uniform continuity of L, Theorem 3.8 and Lemma 4.13, $p(L)$ is uniform complete and $x \wedge_{p(L)} (\bigvee_{p(L)} S) = x \wedge_L (\bigvee_L S) = \bigvee_L \{x \wedge_L s \mid s \in S\} = \bigvee_{p(L)} \{x \wedge_{p(L)} s \mid s \in S\}$. By Theorem 3.8, $p(L)$ in the inherited order is also a meet uniform continuous poset. □

5. Concluding remarks

The Scott topology, as an intrinsic topology, is of fundamental importance in domain theory. The concept of meet continuous dcpos was successfully defined [4] by the Scott topology. To give more characterizations of uniform continuous posets, we introduce the concept of meet uniform continuous posets via the uniform Scott sets which are some special Scott open sets. Though the uniform Scott sets of a poset is not a topology, they indeed form a semi-topology in some sense and play roles in characterizing (meet) uniform continuous posets similar to that of Scott open sets in characterizing (meet) continuous posets. We see that a uniform complete poset L is meet uniform continuous if L^1 obtained by adjoining a top element 1 to L is a complete Heyting algebra and that finite products and images of uniform continuous projections of meet uniform continuous posets are still meet uniform continuous. We also gave characterizations of meet uniform continuous posets by some kind of distributivity, principal ideals, principal filters and closed intervals. It is proved (Theorem 3.8) that a uniform complete poset L is meet uniform continuous iff for each $x \in L$ and each uniform subset S, one has $x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$. It is also established (Theorem 4.6) that a uniform complete poset is meet uniform continuous iff every principal ideal is meet uniform continuous iff all closed intervals are meet uniform continuous iff all principal filters are meet uniform continuous.

It is expected in the future that the ideas and methods in this paper can be used to study countable uniform sets and other related topics [11–15].

Acknowledgements We thank the referees for their valuable time and comments.

References

[1] D. SCOTT. *Continuous Lattices*. Springer-Verlag, Berlin, 1972.

[2] G. GIERZ, K. HOFMANN, K. KEIMEL, et al. *Continuous Lattices and Domains*. Cambridge University Press, Cambridge, 2003.

[3] J. LAWSON. *The duality of continuous posets*. Houston J. Math., 1979, **5**: 357–394.

[4] Hui KOU, Yingming LIU, Maokang LUO. *On meet-continuous dcpos*. Kluwer Academic Publishers, 2003.

[5] Hong ZHU. *A theory of program expansion*. Sci. China, Ser. A, 1988, (8): 887–896. (in Chinese)

[6] Xiaojun RUAN, Xiaozhi ZHANG. *Some proerties of uniform continuous posets*. J. Nanchang Univ. (Nat. Sci. Edition), 2008, **32**(2): 119–120. (in Chinese)

[7] G. GIERZ, J. LAWSON. *Generalized continuous and hyper-continuous lattices*. Rocky Mountain J. Math., 1981, **11**: 271–296.

[8] Xiangmei DAI, Luoshan XU. *Heredity and invariance of meet continuous dcpos*. Mohu Xitong yu Shuxue, 2010, **24**(3): 76–81.

[9] Luoshan XU. *Continuity of posets via Scott topology and sobrification*. Topology Appl., 2006, **153**(11): 1886–1894.

[10] Xuxin MAO, Luoshan XU. *Quasicontinuity of posets via Scott topology and sobrification*. Order, 2006, **23**(4): 359–369.

[11] Xuxin MAO, Luoshan XU. *Meet continuity properties of posets*. Theoret. Comput. Sci., 2009, **410**(42): 4234–4240.

[12] Xiaojun RUAN, Xiaoquan XU. *Uniform minimal sets and their applications to uniform continuous posets*. Mohu Xitong yu Shuxue, 2014, **28**(4): 80–83.

[13] Zhongqiang Yang. *Normally supercompact spaces and completely distributive posets*. Topology Appl., 2001, **109**(2): 257–265.

[14] Bin ZHAO, Yihui ZHOU. *The category of supercontinuous poset*. J. Math. Anal. Appl., 2006, **320**: 632–641.

[15] Wenfeng ZHANG, Xiaoquan XU. *Meet Precontinuous Posets*. Elsevier Sci. B. V., Amsterdam, 2014.