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Wave Breaking in the Periodic Integrable Hunter-Saxton Equation with a Dispersive Term

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Abstract Considered here is the periodic Cauchy problem for an integrable Hunter-Saxton equation with a dispersive term. Firstly, we derive a precise blow-up criterion of strong solutions to the equation. Secondly, sufficient conditions guaranteeing the development of breaking waves in finite time are demonstrated by applying some conservative quantities and the method of characteristics, respectively. Finally, the exact blow-up rate is determined.

Keywords Hunter-Saxton equation; integrable equation; blow-up; wave-breaking

MR(2010) Subject Classification 35G25; 35L05; 35B30

1. Introduction

The nonlinear partial differential equations of second order of the general form [1]

$$u_{xt} = u + c_0u^2 + c_1uu_x + c_2uu_{xx} + c_3u_x^2 + d_0u^3 + d_1u^2u_x + d_2u^2u_{xx} + d_3uu_x^2 \tag{1.1}$$

are of interest and physically, they are always found in the description of the short-wave behavior of nonlinear systems. Recently, Hone, Novikov and Wang show that when having an infinite hierarchy of local higher symmetries, the form (1.1) contains many interesting equations, especially some valuable integrable ones up to scaling transform.

For instance, in case that $c_j = 0, j = 0, 1, 2, 3$ and $d_0 = d_1 = 0, d_3 = 2d_2$, the form (1.1) becomes the short-pulse equation

$$u_{xt} = u + (u^3)_{xx},$$

which was first proposed as an equation for pseudo-spherical surfaces with an associated inverse scattering problem [2, 3]. Later, Schäfer and Wayne in [4] derived it as a model of ultra-short optical pulses in nonlinear media. Mathematical properties of the short-pulse equation were studied recently in details, including the construction of the Lax pair, recursion operator, bi-Hamiltonian structure and the non-existence of smooth traveling wave solutions [4–6], the local and global well-posednesses in energy space [4, 7], and the blow-up phenomena both on the line and in the periodic domain [8].

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The second interesting member of the form (1.1) is the Ostrovsky-Hunter equation

$$u_{xt} = u + (u^2)_{xx}, \tag{1.2}$$

which was derived by Vakhnenko [9] as a model for the propagation of short-wave perturbations in a relaxing medium. It models small-amplitude long waves in rotating fluids of finite depth, under the assumption of no-high frequency dispersion [10]. Eq. (1.2) has some different names, such as the Vakhnenko equation [11], the short-wave equation [12], and the reduced Ostrovsky equation [13]. Local existence of solutions of the Ostrovsky-Hunter equation in $H^s(\mathbb{R})$ for $s > 3/2$ was obtained in [14]. Then sufficient conditions for the wave breaking of the Ostrovsky-Hunter equation on an infinite line and in a periodic domain were both given in [15]. They also specified the blow-up rate of the wave breaking based on the method of characteristics. Grimshaw and Pelinovsky [16] proved global existence of small-norm solutions in $H^3(\mathbb{R})$ by using a new transformation of equation (1.2) to the integrable Tzitzéica equation.

The third interesting member of the form (1.1) is the equation

$$u_{xt} = u + 2uu_{xx} + u_x^2, \tag{1.3}$$

which is one of the integrable generalized short-pulse equation and its integrability has been studied by Hone, Novikov and Wang in [1]. It possesses an infinite hierarchy of local higher symmetries and the first higher symmetry is

$$u_\tau = \frac{u_{3x}}{(1+4u_{2x})^{3/2}}.$$

The symmetries of equation (1.3) can be generated by a recursion operator

$$\Re = HD_x = (\frac{1}{1+4u_{xx}}D_x + D_x\frac{1}{1+4u_{xx}} - 8u_\tau D_x^{-1}u_\tau)D_x,$$

where D_x is a symplectic operator satisfying $D_x u_\tau = -\frac{1}{4}\delta_u\sqrt{1+4u_{xx}}$, and the operator H and D_x^{-1} form a compatible Hamiltonian pair. Furthermore, Eq. (1.3) admits the following lax pair

$$\Phi_x = \begin{pmatrix} 0 & 1+4u_{xx} \\ -\lambda & 0 \end{pmatrix}\Phi, \quad \Phi_t = \begin{pmatrix} u_x & -\frac{1}{4\lambda} + 2u + 8uu_{xx} \\ \frac{1}{4} - 2\lambda u & -u_x \end{pmatrix}\Phi.$$

Hone, Novikov and Wang [1] showed that Eq. (1.3) can be considered as a short-wave limit of the Camassa-Holm equation. Taking its x derivative, we can get

$$m_t = 2um_x + 4u_x m, \quad m = 1 + 4u_{xx},$$

which shares the same form with the Camassa-Holm (CH) equation (in the case $m = u - u_{xx}$). The Camassa-Holm (CH) equation was proposed as a model describing the uni-directional propagation of the shallow water waves over a flat bottom, where $u(t,x)$ represents the free surface of shallow water in nondimensional variables or wave speed [17]. The CH equation has a number of remarkable properties, including complete integrability, wave-breaking, etc. (see [18–26] and the references therein).

Eq. (1.3) can also be considered as the Hunter-Saxton (HS) equation with a dispersion term u (see [27]). In fact, dropping the dispersion term u from right-hand side of (1.3) and replacing

u with $\frac{1}{2}u$, we can get the following HS equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2,$$

which was derived by Hunter and Saxton [28] as an asymptotic model of liquid crystals. It was shown that the x derivative of the HS equation corresponds to geodesic flow on an infinite-dimensional homogeneous space with constant positive curvature [29]. The HS equation is completely integrable [30, 31] and has a bi-Hamiltonian structure [28, 32]. The initial value problems for the HS equation on the line and the circle were studied in [28, 33]. Global solutions of the HS equation was investigated in [34].

Recently, Li and Yin [27] studied the periodic Cauchy problem of Eq. (1.3). A precise blow-up criterion and a blow-up result of strong solutions to the Eq. (1.3) were established. They also present its travelling wave solutions by using the travelling wave solutions of the sinh-Gordon equation and a period stretch between these two equations.

By setting $u \to -u$, Eq. (1.3) becomes

$$u_{xt} = u - 2uu_{xx} - u_x^2. \tag{1.4}$$

The goal of the present paper is to consider the behavior of solutions to the Cauchy problem of Eq. (1.4), i.e.,

$$\begin{cases} u_{xt} = u - 2uu_{xx} - u_x^2, \quad t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), \quad t \geq 0, \ x \in \mathbb{R}. \end{cases} \tag{1.5}$$

It is worth mentioning that here we take a different method from that used in [27]. The first step is to establish a wave-breaking criterion. The theory of transport equations implies that the solution u will not blow up as long as the slope of the velocity, i.e., u_x remains bounded, while the solution blows up in finite time when the slope u_x is unbounded from blow. Then we try to find conditions of the initial data which can guarantee the wave breaking in finite time by the conservative quantities. Furthermore, applying the method of characteristics, a sufficient condition for the wave breaking in the Cauchy problem (1.5) that is different from the sufficient conditions of Theorem 2.3 is established. The blow-up rate at which the waves break is also specified consequently.

The remainder of the paper is organized as follows. Section 2 gives a sufficient condition for wave breaking. The blow-up rate of the wave breaking is studied in Section 3 and the other wave breaking condition is established based on the method of characteristics.

Notation Throughout this paper, we identity all spaces of periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e., $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. The norm of the Sobolev space $H^s(\mathbb{S})$, $s \in \mathbb{R}$, is denoted by $\|\cdot\|_{H^s}$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity.

2. Wave breaking phenomena

In this section, we establish a blow up criterion and also derive a sufficient condition for the breaking of waves for the initial-value problem (1.5). Firstly, we introduce the following lemma which is useful in the subsequent sections.

Lemma 2.1 ([21]) *Suppose that $v \in C^1([0,T); H^2(\mathbb{S}))$ for some $T > 0$, then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{S}$ with*

$$m(t) := \inf_{x\in\mathbb{S}} (v_x(t,x)) = v_x(t,\xi(t)).$$

The function $I(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{dm(t)}{dt} = v_{tx}(t,\xi(t)), \text{ a.e., on } (0,T).$$

Applying the Kato method for the Cauchy problem for abstract quasi-linear equation of evolution [35], we can obtain the following local well-posedness for the system (1.5).

Theorem 2.2 *Let $u_0 \in H^s$ with $s > 3/2$. Suppose that u_0 satisfies the following condition*

$$\int_{\mathbb{S}} (u_0 + u_{0,x}^2)\mathrm{d}x = 0.$$

Then there exists a time $T = T(u_0) > 0$ such that the Cauchy problem (1.5) has a unique strong solution u, such that

$$u = u(\cdot, u_0) \in C([0,T]; H^s \cap C^1([0,T]; H^{s-1})$$

with the following two conservation laws:

$$\int_{\mathbb{S}} u_x^2(t,x)\mathrm{d}x = \int_{\mathbb{S}} u_{0,x}^2(x)\mathrm{d}x, \quad \int_{\mathbb{S}} u(t,x)\mathrm{d}x = \int_{\mathbb{S}} u_0(x)\mathrm{d}x.$$

Or, more precisely, we have

$$\int_{\mathbb{S}} u_x^2 \mathrm{d}x = -\int_{\mathbb{S}} u\mathrm{d}x = K,$$

here $K \geq 0$ is a constant only depending on u_0, and as a consequence for any $(x,t) \in \mathbb{S} \times [0,T)$,

$$-K - \sqrt{K} \leq u(x,t) \leq -K + \sqrt{K}.$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \mapsto u(\cdot, u_0) : H^s \to C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$$

is continuous.

Proof Existence, uniqueness, and continuous dependence in $H^s, s > 3/2$ can be proved by performing the same argument as in [27] (up to a slight modification), so the proof is omitted here. To prove the conservation laws, multiplying the first equation of (1.5) by u_x, and integrating by parts in the unit circle $\mathbb{S}$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_{L^2(\mathbb{S})}^2 &= \int_{\mathbb{S}} u_x u_{xt}\mathrm{d}x = \int_{\mathbb{S}} u_x(u - 2uu_{xx} - u_x^2)\mathrm{d}x \\ &= \int_{\mathbb{S}} uu_x\mathrm{d}x - \int_{\mathbb{S}} 2uu_xu_{xx}\mathrm{d}x - \int_{\mathbb{S}} u_x^3\mathrm{d}x = 0,\end{aligned}$$

which implies the desired conserved quantity for $\|u_x\|_{L^2}$. On the other hand, integrating the first equation of (1.5) directly over $\mathbb{S}$ yields

$$\int_{\mathbb{S}} u_{xt}\mathrm{d}x = \int_{\mathbb{S}}(u - 2uu_{xx} - u_x^2)\mathrm{d}x = \int_{\mathbb{S}} u\mathrm{d}x + \int_{\mathbb{S}} u_x^2\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} u_x\mathrm{d}x = 0.$$

By the conserved quantity $\|u_x\|_{L^2}$, we obtain

$$-\int_{\mathbb{S}} u(x,t)\mathrm{d}x = \int_{\mathbb{S}} u_x^2\mathrm{d}x = \int_{\mathbb{S}} u_{0,x}^2\mathrm{d}x = K. \tag{2.1}$$

Obviously, we get $K \geq 0$ only depending on u_0. Then we have

$$\begin{aligned}\Big|u(x,t) - \int_{\mathbb{S}} u(y,t)\mathrm{d}y\Big| =& \Big|\int_{\mathbb{S}}(u(x,t) - u(y,t))\mathrm{d}y\Big| = \Big|\int_{\mathbb{S}}\Big[\int_y^x u_z(z,t)\mathrm{d}z\Big]\mathrm{d}y\Big| \\ \leq& \sup_{y\in\mathbb{S}}\Big|\int_y^x u_z(z,t)\mathrm{d}z\Big| \leq \int_{\mathbb{S}}|u_z(z,t)|\mathrm{d}z \leq \Big(\int_{\mathbb{S}} u_z^2(z,t)\mathrm{d}z\Big)^{1/2}.\end{aligned}$$

According to the equality (2.1), we get $|u(x,t) + K| \leq \sqrt{K}$, for any $(x,t) \in \mathbb{S} \times [0,T)$. This completes the proof. □

Using the above conservation law of $\|u_x\|_{L^2}$ and the boundedness of $\|u\|_{L^\infty}$, the H^1-norm of u can be controlled in the periodic case. We now show a precise blow-up criterion for (1.5) as follows.

Theorem 2.3 *Let $u_0(x) \in H^s, s \geq 2$, and let T be the maximal existence time of the solution $u(x,t)$ to (1.5) with the initial data $u_0(x)$. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{S}} u_x(t,x)\} = -\infty.$$

Proof Applying the local well-posedness and a simple density argument, it suffices to consider the case when $u \in C_0^\infty$. To begin with, for the H^1-norm of u, we have

$$\|u(\cdot,t)\|_{H^1(\mathbb{S})}^2 = \int_{\mathbb{S}} u^2 + u_x^2\mathrm{d}x \leq \|u\|_{L^\infty}^2 + \int_{\mathbb{S}} u_x^2\mathrm{d}x \leq (K + \sqrt{K})^2 + K \leq C.$$

Differentiating both sides of (1.5) with respect to x, taking L^2 inner product with u_{xx} and then integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} u_{xx}^2\mathrm{d}x =& \int_{\mathbb{S}} u_{xx}u_{xxt}\mathrm{d}x = \int_{\mathbb{S}} u_{xx}(u_x - 4u_xu_{xx} - 2uu_{xxx})\mathrm{d}x \\ =& \int_{\mathbb{S}}(u_xu_{xx} - 4u_xu_{xx}^2 - 2uu_{xx}u_{xxx})\mathrm{d}x = -3\int_{\mathbb{S}} u_xu_{xx}^2\mathrm{d}x.\end{aligned}$$

Suppose that u_x is bounded from below on $[0,T)\times\mathbb{S}$, i.e., there exists $M > 0$ such that $u_x \geq -M$ on $[0,T)\times\mathbb{S}$, then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} u_{xx}^2\mathrm{d}x \leq 6M\int_{\mathbb{S}} u_{xx}^2\mathrm{d}x.$$

An application of Gronwall's inequality yields

$$\int_{\mathbb{S}} u_{xx}^2\mathrm{d}x \leq e^{6Mt}\int_{\mathbb{S}} u_{0,xx}^2\mathrm{d}x.$$

Therefore, we obtain

$$\|u(\cdot,t)\|^2_{H^2(\mathbb{S})} \le C + e^{6Mt}\int_{\mathbb{S}} u^2_{0,xx}\mathrm{d}x \le C + e^{6Mt}\|u_0\|_{H^2(\mathbb{S})}.$$

This contradicts the assumption that $T<\infty$ is the maximal existence time, which completes the proof of the theorem. □

A sufficient condition for the wave breaking in the Cauchy problem (1.5) is established as follows.

Theorem 2.4 *Assume that $u_0(x)\in H^s(\mathbb{S}), s\ge 2$. If u_0 satisfies*

$$\int_{\mathbb{S}}(u_0'(x))^3\mathrm{d}x < -(3(K+\sqrt{K}))^{3/2},$$

then the solution $u(t,x)$ of the Cauchy problem (1.5) blows up in finite time $T\in(0,\infty)$.

Proof Let $T>0$ be the maximal time of existence of solution $u(x,t)$ in the well-posedness result. Then we obtain the priori differential estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} u_x^3\mathrm{d}x &= \int_{\mathbb{S}} 3u_x^2u_{xt}\mathrm{d}x = \int_{\mathbb{S}} 3u_x^2(u-2uu_{xx}-u_x^2)\mathrm{d}x\\ &= \int_{\mathbb{S}} 3uu_x^2\mathrm{d}x - \int_{\mathbb{S}} u_x^4\mathrm{d}x \le 3\|u\|_{L^\infty}\int_{\mathbb{S}} u_x^2\mathrm{d}x - \int_{\mathbb{S}} u_x^4\mathrm{d}x\\ &\le 3(K+\sqrt{K})\|u_x\|^2_{L^4} - \|u_x\|^4_{L^4} = -(\|u_x\|^2_{L^4} - \frac{3(K+\sqrt{K})}{2})^2 + \frac{9(K+\sqrt{K})^2}{4},\end{aligned}$$

where we have used the Cauchy-Schwartz inequality. An application of Hölder inequality yields

$$\|u_x\|^3_{L^3} \le \|u_x\|^3_{L^4}.$$

Let $V(t)=\int_{\mathbb{S}} u_x^3(t,x)\mathrm{d}x$ for all $t\in[0,T)$, and consider the assumption, we have that

$$V(0) < -(3(K+\sqrt{K}))^{3/2} < 0.$$

Then we have

$$\|u_x\|^2_{L^4} - \frac{3(K+\sqrt{K})}{2} \ge \|u_x\|^2_{L^3} - \frac{3(K+\sqrt{K})}{2} \ge |V|^{2/3} - \frac{3(K+\sqrt{K})}{2},$$

so that

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &\le -(|V|^{2/3} - \frac{3(K+\sqrt{K})}{2})^2 + \frac{9(K+\sqrt{K})^2}{4}\\ &= -|V|^{2/3}(|V|^{2/3} - 3(K+\sqrt{K})).\end{aligned}$$

When $t=0$, the right hand side of the above inequality is negative. By the continuation argument, $V(t)$ is decreasing on $[0,T)$ so that $V(t)\le V(0)<0$. To prove that T is finite and $\lim\limits_{t\uparrow T} V(t) = -\infty$. Let $y=|V|^{1/3}$ and obtain

$$-3y^2\frac{\mathrm{d}y}{\mathrm{d}t} \le -y^2(y^2 - 3(K+\sqrt{K})),$$

i.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}t} \ge \frac{1}{3}(y^2 - 3(K+\sqrt{K})),$$

where the right hand side is positive at $t = 0$. By the comparison principle for differential equations, $y(t) \geq y^+(t)$ for all $t \in [0, T)$, where $y^+(t)$ solves the differential equation

$$\begin{cases} \frac{\mathrm{d}y^+}{\mathrm{d}t} = \frac{1}{3}((y^+)^2 - 3(K + \sqrt{K}), \\ y^+(0) = y(0). \end{cases}$$

Since $y(0) > (\frac{3(K+\sqrt{K})}{2})^{1/2}$, there is a finite time $T^+ \in (0, \infty)$ such that $\lim\limits_{t \uparrow T^+} y^+(t) = +\infty$, and thus there is a time $T \in (0, T^+)$ such that $\lim\limits_{t \uparrow T} y(t) = +\infty$, which implies $\lim\limits_{t \uparrow T} V(t) = -\infty$. Consider that in the above case,

$$\inf_{x \in \mathbb{S}} u_x^3(t, x) \leq \int_{\mathbb{S}} u_x^3 \mathrm{d}x \equiv V(t),$$

it implies immediately that

$$\liminf_{t \uparrow T} \inf_{x \in \mathbb{S}} u_x(t, x) = -\infty.$$

This completes the proof of the theorem. □

3. Blow-up rate of wave breaking

We shall study here the blow-up rate of the wave-breaking for solutions of the Cauchy problem (1.5), which can be transformed into a transport-like equation

$$\begin{cases} u_t + 2uu_x = \partial_x^{-1}(u + u_x^2), \quad t > 0, \; x \in \mathbb{S}, \\ u(0, x) = u_0(x), \quad x \in \mathbb{S}. \end{cases} \tag{3.1}$$

Here ∂_x^{-1} is the mean-zero antiderivative in the sense of

$$\partial_x^{-1} f = \int_0^x f(t, x')\mathrm{d}x' - \int_{\mathbb{S}} \int_0^x f(t, x')\mathrm{d}x'\mathrm{d}x.$$

Let $T > 0$ be the maximal time of existence of the solution $u(x, t)$ of the Cauchy problem (3.1) with the initial data $u_0 \in H^s(\mathbb{S})$ for $s \geq 2$. For all $t \in [0, T)$ and $\xi \in \mathbb{S}$, define

$$x = X(t, \xi), \; u(t, x) = U(t, \xi), \; \partial_x^{-1}(u + u_x^2)(t, x) = G(t, \xi),$$

so that

$$\dot{X}(t) = 2U, \; X(0) = \xi, \; \dot{U}(t) = G, \; U(0) = u_0(\xi), \tag{3.2}$$

where dots denote derivatives with respect to time t on a particular characteristic $x = X(t, \xi)$ for a fixed $\xi \in \mathbb{S}$. Applying classical results in the theory of ODEs, we obtain the following two useful results on the solutions of the initial-value problem (3.2).

Lemma 3.1 *Let $u_0(x) \in H^s(\mathbb{S})$ with $s \geq 2$, and $T > 0$ be the maximal time of existence of solution $u(t, x)$. Then there exists a unique solution $X(t, \xi) \in C^1([0, T) \times \mathbb{S})$ to the initial-value problem (3.2). Moreover, the map $X(t, \cdot) : \mathbb{S} \to \mathbb{R}$ is an increasing diffeomorphism with*

$$\partial_\xi X(t, \xi) = \exp\Big(\int_0^t 2u_x(s, X(s, \xi))\mathrm{d}s\Big) > 0, \quad \forall t \in [0, T), \; \forall x \in \mathbb{S}.$$

Proof Consider the integral equation

$$X(t,\xi)=\xi+\int_0^t 2u(s,X(s,\xi))\mathrm{d}s,\quad t\in[0,T),$$

where $u(t,x)\in C([0,T);H^s(\mathbb{S}))\cap C^1([0,T);H^{s-1}(\mathbb{S}))$ for $s\geq 2$. By the ODE theory, there exists a unique solution $X(t,\xi)\in C^1([0,T);H^{s-1}(\mathbb{S}))$ of the integral equation above. Using the chain rule, we obtain

$$\partial_\xi \dot{X}=\frac{\partial}{\partial\xi}(2u(t,X(t,\xi)))=2u_x(t,X(t,\xi))\partial_\xi X,$$

then

$$\partial_\xi X(t,\xi)=\exp\Big(\int_0^t 2u_x(s,X(s,\xi))\mathrm{d}s\Big).$$

So that $\partial_\xi X(t,\xi)>0$ for all $(t,x)\in[0,T)\times\mathbb{S}$. □

Lemma 3.2 *Let $u_0(x)\in H^s(\mathbb{S})$ with $s\geq 2$, and $T>0$ be the maximal time of existence of solution $u(t,x)$ in the Theorem 2.1. Then the solution $u(t,x)$ satisfies*

$$\sup_{s\in[0,t]}|u(s,\cdot)|\leq \|u_0\|_{L^\infty}+t(\|u_{0,x}\|_{L^2}^2+2\|u_{0,x}\|_{L^2}),\quad \forall t\in[0,T).$$

Proof By the lemma above, the function $x=X(t,\xi)$ is invertible in $\xi\in\mathbb{S}$ for any $t\in[0,T)$. Then, we have

$$\sup_{s\in[0,t]}\sup_{x\in\mathbb{S}}|u(s,x)|=\sup_{s\in[0,t]}\sup_{x\in\mathbb{S}}|U(s,\xi)|,\quad t\in[0,T).$$

Since $\partial_x^{-1}(u+u_x^2)(t,x)\in C([0,T);H^s(\mathbb{S}))$ is the mean-zero periodic function of x for each $t\in[0,T)$, there exists a $\xi_t\in\mathbb{S}$ such that $\partial_x^{-1}(u+u_x^2)(t,\xi_t)=0$. Then for any $x\in\mathbb{S}$ and $t\in[0,T)$, we have that

$$\begin{aligned}|\partial_x^{-1}(u+u_x^2)(t,x)|&\leq\Big|\int_{\xi_t}^x(u+u_x^2)(t,x)\mathrm{d}x\Big|\leq\int_{\mathbb{S}}|u+u_x^2|\mathrm{d}x\\&\leq\int_{\mathbb{S}}|u|\mathrm{d}x+\int_{\mathbb{S}}u_x^2\mathrm{d}x\leq\|u\|_{L^\infty}+\|u_{0,x}^2\|_{L^2},\end{aligned}$$

where we have used the Cauchy-Schwartz inequality and the conserved quantity $\|u_x\|_{L^2}$. Using the integral equation,

$$U(t,\xi)=u_0(\xi)+\int_0^t G(s,\xi)\mathrm{d}s,\quad t\in[0,T),$$

we obtain

$$\begin{aligned}\sup_{s\in[0,t]}\sup_{x\in\mathbb{S}}|u(s,x)|&\leq\|u_0\|_{L^\infty}+t\sup_{s\in[0,t]}\sup_{\xi\in\mathbb{S}}|G(s,\xi)|\\&\leq\|u_0\|_{L^\infty}+t\sup_{s\in[0,t]}\sup_{x\in\mathbb{S}}|\partial_x^{-1}(u+u_x^2)(s,x)|\\&\leq\|u_0\|_{L^\infty}+t(\|u_{0,x}\|_{L^2}^2+2\|u_{0,x}\|_{L^2}),\quad t\in[0,T),\end{aligned}$$

and the lemma is proved. □

Using the method of characteristics, we obtain a sufficient condition for the wave breaking in the Cauchy problem (1.5) that is different from the sufficient condition of Theorem 2.4.

Theorem 3.3 *Let $\epsilon > 0$ and let $u_0(x) \in H^s(\mathbb{S}), s \geq 2$. Let T_1 be the smallest positive root of*

$$2T_1[\|u_0\|_{L^\infty} + T_1(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2})]^{1/2} = \ln(1 + \frac{2}{\epsilon}),$$

and assume that there is a $x_0 \in \mathbb{S}$ such that

$$u_0'(x_0) \leq -(1+\epsilon)[\|u_0\|_{L^\infty} + T_1(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2})]^{1/2}. \tag{3.3}$$

Then the solution $u(t,x)$ blows up in a finite time $T \in (0, T_1)$ in the sense of Theorem 2.3.

Proof Define $V(x,t) = u_x(t, X(t,\xi))$. By the well-posedness Theorem 2.1 and Lemma 2.1, $V(t,\xi)$ is absolutely continuous on $[0,T) \times \mathbb{S}$ and a.e., differentiable on $(0,T) \times \mathbb{S}$, so that

$$\dot{V} = (u_{xt} + 2uu_{xx})|_{x=X(t,\xi)} = (u - u_x^2)|_{x=X(t,\xi)} = -V^2 + U \text{ a.e., } \xi \in \mathbb{S}, \quad t \in (0,T).$$

By Lemma 3.2, we obtain the a priori differential estimate

$$\dot{V} = -V^2 + U \leq -V^2 + [\|u_0\|_{L^\infty} + t(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2})] \text{ a.e., } \xi \in \mathbb{S}, \quad t \in [0,T). \tag{3.4}$$

Since $u_0'(x)$ is a continuous, mean-zero, periodic function of x on $\mathbb{S}$, and assumption (3.3) is satisfied for fixed $\epsilon > 0$, there exists $\tilde{x}_0$ such that

$$V(0, \tilde{x}_0) = -(1+\epsilon)h(T_1),$$

where

$$h(T_1) = [\|u_0\|_{L^\infty} + T_1(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2})]^{1/2}.$$

Thanks to the a priori estimate (3.4), $V(t) := V(t, \tilde{x}_0)$ satisfies

$$\begin{cases} \dot{V}(t) \leq -V^2 + h^2(T_1), \text{ a.e., } t \in [0, T_1) \cap (0,T), \\ V(0) = -(1+\epsilon)h(T_1). \end{cases}$$

By the comparison principle for ODES, we have

$$V(t) \leq V_+(t) < 0, \quad t \in [0, T_1) \cap (0,T),$$

where $V_+(t)$ solves the equation

$$\begin{cases} \dot{V}_+(t) \leq -V_+^2(t) + h^2(T_1), \quad t \in [0, T_1), \\ V_+(0) = V(0). \end{cases}$$

The above equation admits an implicit solution:

$$\frac{V_+(t) + h(T_1)}{V_+(t) - h(T_1)} = \frac{V(0) + h(T_1)}{V(0) - h(T_1)} e^{2h(T_1)t}, \quad t \in [0, T_1).$$

If T_1 is the smallest positive root of

$$2T_1[\|u_0\|_{L^\infty} + T_1(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2}]^{1/2} = \ln(1 + \frac{2}{\epsilon}),$$

then

$$\frac{V_+(t) + h(T_1)}{V_+(t) - h(T_1)} = \frac{\epsilon}{2+\epsilon} e^{2h(T_1)t} \uparrow 1, \text{ as } t \uparrow T_1,$$

so that $\lim\limits_{t \uparrow T_1} V_+(t) = -\infty$. Thus there is $T \in (0, T_1)$ such that $\lim\limits_{t \uparrow T} V(t) = -\infty$. $\square$

Remark 3.4 Note that if $\epsilon \to \infty$ and the assumption of Theorem 3.1 still holds, then $T \to 0$. This means that the steeper the slope of the initial data $u_0(x)$, the quicker the solution $u(t,x)$ blows up.

Finally, we specifies the wave breaks rate in the Cauchy problem (3.1). By the well-posedness result, the initial data $u_0(x)$ can be considered in $H^3(\mathbb{S})$.

Theorem 3.5 *Let $u_0(x) \in H^3(\mathbb{S})$, and let $T \in (0,\infty)$ be the finite blow-up time of the solution $u(t,x)$ in the local well-posedness Theorem 2.1. Then we have*

$$\lim_{t\uparrow T}((T-t)\inf_{x\in\mathbb{S}} u_x(t,x)) = -1 \tag{3.5}$$

and

$$\lim_{t\uparrow T}((T-t)\sup_{x\in\mathbb{S}} u_x(t,x)) = 0. \tag{3.6}$$

Proof Let $m(t) = \inf\limits_{x\in\mathbb{S}} u_x(t,x)$. By Lemma 2.1, for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{S}$ such that $m(t) := u_x(t,\xi(t))$ and $u_{xx}(t,\xi(t)) = 0$. Moreover, $m(t)$ and $\xi(t)$ is absolutely continuous on $[0,T)$, is a.e., differentiable on $(0,T)$, and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}m(t) = u_{xt}(t,\xi(t)) = -m^2(t) + u(t,\xi(t)), \text{ a.e., } t \in (0,T).$$

Set $N(t) = \|u_0\|_{L^\infty} + T_1\big(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2}\big)(>0)$. By Lemma 3.2, we get

$$-m^2(t) - N(T) \le \frac{\mathrm{d}}{\mathrm{d}t}m(t) \le -m^2(t) + N(T), \text{ a.e., } t \in (0,T).$$

Let us choose $\epsilon \in (0,1)$. From Theorem 2.2, there exists $t_0 \in (0,T)$ such that

$$m(t_0) < -\sqrt{N(T) + \frac{N(T)}{\epsilon}}.$$

Notice that $m(t)$ is absolutely continuous on $[0,T)$, it then follows from the above inequality that $m(t)$ is decreasing on $[t_0,T)$ and satisfies that

$$m(t) \le m(t_0) < -\sqrt{N(T) + \frac{N(T)}{\epsilon}} < -\sqrt{\frac{N(T)}{\epsilon}}, \quad t \in [t_0,T).$$

It then follows that $\lim\limits_{t\uparrow T} m(t) = -\infty$, and

$$1 - \epsilon \le \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{m(t)} = -\frac{1}{m^2(t)}\frac{\mathrm{d}m(t)}{\mathrm{d}t} \le 1 + \epsilon.$$

Integrating the above relation on (t,T) with $t \in [t_0,T)$, and noticing that $\lim\limits_{t\uparrow T} m(t) = -\infty$, we can get

$$(1-\epsilon)(T-t) \le -\frac{1}{m(t)} \le (1+\epsilon)(T-t).$$

Since $\epsilon \in (0,1)$ is arbitrary, in view of the definition of $m(t)$, the above inequality in the limit $\epsilon \to 0$ implies the desired result (3.5).

Now let $M(t) := \sup\limits_{x\in\mathbb{S}} u_x(t,x)$. By the same Lemma 2.1, for every $t \in [0,T)$, there exists at least one point $\eta(t) \in \mathbb{S}$ such that $M(t) = u_x(t,\eta(t))$ and $u_{xx}(t,\eta(t)) = 0$. Repeating the same arguments, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) = -M^2(t) + u(t,\eta(t)) \leq \|u_0\|_{L^\infty} + t(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2})t, \quad t \in (0,T),$$

so that

$$M(t) \leq \sup_{x\in\mathbb{S}} u_{0,x}(x) + T\|u_0\|_{L^\infty} + \frac{T^2}{2}(\|u_{0,x}\|_{L^2}^2 + 2\|u_{0,x}\|_{L^2}) < +\infty. \tag{3.7}$$

Consider $u(t,x)$ is periodic on $\mathbb{S}$ for all $t \in [0,T)$ and belongs to $C([0,T);H^3(\mathbb{S}))$, there exists $\xi_0(t) \in \mathbb{S}$ for every $t \in [0,T)$ such that $u_x(t,\xi_0(t)) = 0$. Therefore, $M(t) \geq u_x(t,x) = 0$ for all $t \in [0,T)$, so that bound (3.7) yields the desired result (3.6). This completes the proof of the theorem. □

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