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Mixed-Type Reverse Order Laws Associated to $\{1,3,4\}$-Inverse

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Abstract In this paper, we study the mixed-type reverse order laws to $\{1,3,4\}$-inverses for closed range operators A, B and AB. It is shown that $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$ if and only if $R(A^*AB) \subseteq R(B)$. For every $A^{(134)} \in A\{1,3,4\}$, it has $(A^{(134)}AB)\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\}$ if and only if $R(AA^*AB) \subseteq R(AB)$. As an application of our results, some new characterizations of the mixed-type reverse order laws associated to the Moore-Penrose inverse and the $\{1,3,4\}$-inverse are established.

Keywords $\{1,3,4\}$-inverse; reverse order law; generalized inverse; block-operator matrix

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1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, and abbreviate $\mathcal{B}(\mathcal{H},\mathcal{K})$ to $\mathcal{B}(\mathcal{H})$ if $\mathcal{H}=\mathcal{K}$. For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $N(A)$ and $R(A)$ are the null space and the range of A, respectively. A generalized inverse of A is an operator $G \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying some of the following four equations, which is said to be the Moore-Penrose conditions:

(1) $AGA = A$, (2) $GAG = G$, (3) $(AG)^* = AG$, (4) $(GA)^* = GA$.

Let $A\{i,j,\ldots,l\}$ denote the set of all operators $G \in \mathcal{B}(\mathcal{K},\mathcal{H})$ which satisfy equation $(i), (j), \ldots, (l)$ from the above equations. An operator $G \in A\{i,j,\ldots,l\}$ is called an $\{i,j,\ldots,l\}$-inverse of A, denoted by $A^{(ij\cdots l)}$. The unique $\{1,2,3,4\}$-inverse of A is denoted by $A^\dagger$, which is called the Moore-Penrose inverse of A. As is well known, A is the Moore-Penrose invertible if and only if $R(A)$ is closed.

The reverse order law for many types of generalized inverses has been the subject of intensive research since 1960s, and many interesting results have been obtained. For the Moore-Penrose inverse, Greville gave a classical result

$$(AB)^+ = B^\dagger A^\dagger \Longleftrightarrow R(A^*AB) \subset R(B), \quad R(BB^*A^*) \subset R(A^*)$$

for any complex matrices A and B in [1]. This result was extended to bounded linear operators on Hilbert spaces by Izumino [2] and Bouldin [3]. Following these, reverse order laws for different

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types of generalized inverses have been studied [4–12]. The mixed-type reverse-order laws for AB like

$$(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger, \quad B^\dagger(ABB^\dagger)^\dagger = (AB)^\dagger$$

have been considered in [2] and [13]. Many scholars also discussed Mixed-type reverse order laws for $(AB)^{(13)}$, $(AB)^{(123)}$, $(AB)^{(124)}$ and $(AB)^{(134)}$ (see [14–18]). In [17], the author considered the equivalent condition for $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$ in matrix algebra. By the block operator matrix technique, the author studied the equivalent condition for $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3,4\}$ in [12]. Using the similar space decompositions method in [12], Liu, et al. established the necessary and sufficient conditions for the mixed-type reverse-order laws for $B\{1,3,4\}(ABB^{(134)})\{1,3,4\} \subseteq (AB)\{1,3,4\}$ (see [14]). However, this method is not suitable for studying $(A^{(134)}AB)\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\}$.

In this paper, we shall improve the space decompositions method in [11, 12] and study the mixed-type reverse order laws associated to $\{1,3,4\}$-inverse. In Section 2, some preliminaries are given and the $\{1,3,4\}$-inverse for a class of triangular matrix is obtained. In Section 3, we derive the necessary and sufficient condition for $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$ when $R(A)$, $R(B)$ and $R(AB)$ are closed. A new equivalent condition for $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3,4\}$ is obtained. Moreover, the necessary and sufficient condition for $(A^{(134)}AB)\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\}$ is given. As an application of our results, the mixed-type reverse order laws associated to the Moore-Penrose inverse are considered.

In this section, we mainly discuss representations for generalized inverses of triangular operator matrices. Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ have closed range. It is well known that A, as an operator from $\mathcal{H} = R(A^*) \oplus N(A)$ into $\mathcal{K} = R(A) \oplus N(A^*)$, has the diagonal matrix form $A = A_1 \oplus 0$, where $A_1 \in \mathcal{B}(R(A^*), R(A))$ is invertible. In this case, the Moore-Penrose inverse $A^\dagger$ of A can be represented by $A^\dagger = A_1^{-1} \oplus 0$. In general, we have the following results.

Lemma 2.1 *For a given operator $C \in \mathcal{B}(\mathcal{H})$, let $M_0 = \left(\begin{smallmatrix} C & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. Then the following statements hold.*

(1) ([14]) If C is invertible, then the $\{1,3\}$-inverse $M_0^{(13)}$ and the $\{1,3,4\}$-inverse $M_0^{(134)}$ of M_0 can be represented by

$$M_0{}^{(13)} = \begin{pmatrix} C^{-1} & 0 \\ G_{21} & G_{22} \end{pmatrix} \quad \text{and} \quad M_0{}^{(134)} = \begin{pmatrix} C^{-1} & 0 \\ 0 & G_{22} \end{pmatrix},$$

respectively, where $G_{21} \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $G_{22} \in \mathcal{B}(\mathcal{K})$ are arbitrary.

(2) If C^* is surjective, then the $\{1,3,4\}$-inverse $M_0^{(134)}$ of M_0 can be represented by

$$M_0^{(134)} = \begin{pmatrix} C^\dagger & 0 \\ G_{21} & G_{22} \end{pmatrix}, \tag{2.1}$$

where $G_{22} \in \mathcal{B}(\mathcal{K})$ is arbitrary. $G_{21} = X(I_\mathcal{H} - CC^\dagger)$ for arbitrary $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$.

Proof We only need to prove the statement (2). Since C^* is surjective, M_0 can be represented by

$$M_0 = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} R(C) \\ N(C^*) \\ \mathcal{K} \end{pmatrix},$$

where $C = \binom{C_1}{0}$ such that C_1 is invertible. By the statement (1), the $\{1,3,4\}$-inverse $M_0^{(134)}$ of M_0 has the matrix form as

$$M_0{}^{(134)} = \begin{pmatrix} C_1{}^{-1} & 0 & 0 \\ 0 & G_{21}^{''} & G_{22} \end{pmatrix} : \begin{pmatrix} R(C) \\ N(C^*) \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

in which $G_{21}^{''} \in \mathcal{B}(N(C^*), \mathcal{K}), G_{22} \in \mathcal{B}(\mathcal{K})$ are arbitrary. Here $(C_1^{-1}\ 0) = C^\dagger$ and

$$G_{21} = \begin{pmatrix} 0 & G_{21}^{''} \end{pmatrix} = X(I_{\mathcal{H}} - CC^\dagger), \quad \forall X \in \mathcal{B}(\mathcal{H}, \mathcal{K}).$$

Therefore, (2.1) holds. The proof is completed. □

Moreover, for given operators $C \in \mathcal{B}(\mathcal{H})$ and $D \in \mathcal{B}(\mathcal{K})$, denote by $M_F = \begin{pmatrix} C & 0 \\ F & D \end{pmatrix}$, where $F \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then we have the following results.

Lemma 2.2 *Let* $C \in \mathcal{B}(\mathcal{H})$ *and* $D \in \mathcal{B}(\mathcal{K})$ *be given such that* C *is invertible and* D^* *is surjective. Then the* $\{1,3,4\}$*-inverse* $M_F^{(134)}$ *is unique and can be formulated by*

$$M_F^{(134)} = M_F^{+} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

where

$$\begin{cases} G_{11} = C^{-1}\left(I_{\mathcal{H}} + C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger)FC^{-1}\right)^{-1}, \\ G_{12} = C^{-1}\left(I_{\mathcal{H}} + C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger)FC^{-1}\right)^{-1}C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger), \\ G_{21} = -D^\dagger FC^{-1}\left(I_{\mathcal{H}} + C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger)FC^{-1}\right)^{-1}, \\ G_{22} = D^\dagger - D^\dagger FG_{12}. \end{cases} \tag{2.2}$$

Moreover, $G_{12} = 0$ *if and only if* $R(F) \subseteq R(D)$.

Proof Since the range of M_F^* is surjective, by Lemma 2.1(2), we have the $\{1,3,4\}$-inverse $M_F^{(134)}$ of M_F must be equal to the Moore-Penrose inverse M_F^{+}. Set $\mathcal{K} = R(D) \oplus N(D^*)$. Then M_F has the matrix form

$$M_F = \begin{pmatrix} C & 0 \\ F_1 & D_1 \\ F_2 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ R(D) \\ N(D^*) \end{pmatrix},$$

where C, D_1 are invertible. Let M_F^{+} have the matrix form

$$M_F^{+} = \begin{pmatrix} G_{11} & G_{12}^{'} & G_{12}^{''} \\ G_{21} & G_{22}^{'} & G_{22}^{''} \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ R(D) \\ N(D^*) \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

where $(G'_{12}\ G''_{12}) = G_{12}$ and $(G'_{22}\ G''_{22}) = G_{22}$. Set $A = \begin{pmatrix} C & 0 \\ F_1 & D_1 \end{pmatrix}$ and $B = (F_2\ 0)$, then $A^{-1} = \begin{pmatrix} C^{-1} & 0 \\ -D_1^{-1}F_1C^{-1} & {D_1}^{-1} \end{pmatrix}$ and $BA^{-1} = (F_2C^{-1}\ 0)$. We have

$$\begin{aligned}
{M_F}^{\dagger} &= \left[\begin{pmatrix} A^* & B^* \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}\right]^{\dagger}\begin{pmatrix} A^* & B^* \end{pmatrix} \\
&= [A^*A + B^*B]^{\dagger}\begin{pmatrix} A^* & B^* \end{pmatrix} \\
&= A^{-1}\left[I_{\mathcal{H}\oplus R(D)} + (BA^{-1})^*(BA^{-1})\right]^{\dagger}\begin{pmatrix} I_{\mathcal{H}\oplus R(D)} & (BA^{-1})^* \end{pmatrix} \\
&= \begin{pmatrix} C^{-1} & 0 \\ -D_1^{-1}F_1C^{-1} & {D_1}^{-1} \end{pmatrix}\begin{pmatrix} \left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1} & 0 \\ 0 & I_{R(D)} \end{pmatrix}\begin{pmatrix} I_{\mathcal{H}} & 0 & {C^{-1}}^*F_2^* \\ 0 & I_{R(D)} & 0 \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1} & 0 & C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1}{C^*}^{-1}{F_2}^* \\ -{D_1}^{-1}F_1C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1} & {D_1}^{-1} & -{D_1}^{-1}F_1C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1}{C^*}^{-1}{F_2}^* \end{pmatrix}.
\end{aligned}$$

Therefore,

$$G_{11} = C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F^*(I_{\mathcal{K}} - DD^{\dagger})FC^{-1}\right)^{-1},$$

$$\begin{aligned}
G_{12} &= \begin{pmatrix} 0 & C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1}{C^*}^{-1}{F_2}^* \end{pmatrix} \\
&= C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F^*(I_{\mathcal{K}} - DD^{\dagger})FC^{-1}\right)^{-1}{C^*}^{-1}F^*(I_{\mathcal{K}} - DD^{\dagger}),
\end{aligned}$$

$$G_{21} = -{D_1}^{-1}F_1C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1} = -D^{\dagger}FC^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F^*(I_{\mathcal{K}} - DD^{\dagger})FC^{-1}\right)^{-1}$$

and

$$\begin{aligned}
G_{22} &= \begin{pmatrix} G'_{22} & G''_{22} \end{pmatrix} = \begin{pmatrix} D_1^{-1} & -{D_1}^{-1}F_1C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1}{C^*}^{-1}{F_2}^* \end{pmatrix} \\
&= \begin{pmatrix} D_1^{-1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & {D_1}^{-1}F_1C^{-1}\left(I_{\mathcal{H}} + {C^*}^{-1}F_2^*F_2C^{-1}\right)^{-1}{C^*}^{-1}{F_2}^* \end{pmatrix} \\
&= D^{\dagger} - D^{\dagger}FG_{12}.
\end{aligned}$$

This shows that equalities in (2.2) hold.

Moreover, $G_{12} = 0$ if and only if $F^*(I_{\mathcal{K}} - DD^{\dagger}) = 0$, which is equivalent to $R(F) \subseteq R(D)$. The proof is completed. □

3. Mixed-reverse order laws associated to $\{1,3,4\}$-inverse

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{J}$ be complex Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H},\mathcal{J})$, $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and AB have closed ranges. In this section, we will give necessary and sufficient conditions for

$$B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$$

and

$$(A^{(134)}AB)\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\}.$$

Firstly, we give some space decompositions. Denote

$$\begin{cases} \mathcal{H}_1 &= R(A^*)\ominus(R(A^*)\cap N(B^*)),\\ \mathcal{H}_2 &= N(A)\ominus(N(A)\cap N(B^*)),\\ \mathcal{H}_3 &= R(A^*)\cap N(B^*),\\ \mathcal{H}_4 &= N(A)\cap N(B^*),\end{cases}\quad \begin{cases} \mathcal{K}_1 &= R(B^*A^*),\\ \mathcal{K}_2 &= R(B^*)\ominus R(B^*A^*),\\ \mathcal{K}_3 &= N(B),\end{cases}\quad \begin{cases} \mathcal{J}_1 &= (A^*)^{\dagger}\mathcal{H}_3,\\ \mathcal{J}_2 &= R(A)\ominus(A^*)^{\dagger}\mathcal{H}_3,\\ \mathcal{J}_3 &= N(A^*).\end{cases}$$

Then

$$\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4,\quad \mathcal{K}=\mathcal{K}_1\oplus\mathcal{K}_2\oplus\mathcal{K}_3\quad\text{and}\quad \mathcal{J}=\mathcal{J}_1\oplus\mathcal{J}_2\oplus\mathcal{J}_3.$$

Lemma 3.1 *Let $A\in\mathcal{B}(\mathcal{H},\mathcal{J})$ and $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Suppose that $AB\neq\{0\}$. The following statements hold.*

(1) If $\mathcal{H}_3=R(A^*)\cap N(B^*)\neq\{0\}$, then A and B have the following matrix forms

$$A=\begin{pmatrix} 0 & 0 & A_{13} & 0\\ A_{21} & 0 & A_{23} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix}\to\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\\ \mathcal{J}_3\end{pmatrix}\tag{3.1}$$

and

$$B=\begin{pmatrix} B_{11} & 0 & 0\\ B_{21} & B_{22} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{K}_1\\ \mathcal{K}_2\\ \mathcal{K}_3\end{pmatrix}\to\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix}\tag{3.2}$$

such that A_{13}, A_{21},B_{11} are invertible and B_{22}^* is surjective.

(2) If $\mathcal{H}_3=R(A^*)\cap N(B^*)=\{0\}$, then A and B have the following matrix forms

$$A=\begin{pmatrix} A_{21} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_4\end{pmatrix}\to\begin{pmatrix}\mathcal{J}_2\\ \mathcal{J}_3\end{pmatrix},\tag{3.3}$$

$$B=\begin{pmatrix} B_{11} & 0 & 0\\ B_{21} & B_{22} & 0\\ 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{K}_1\\ \mathcal{K}_2\\ \mathcal{K}_3\end{pmatrix}\to\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_4\end{pmatrix}\tag{3.4}$$

such that A_{21},B_{11} are invertible and B_{22}^* is surjective.

Proof (1) According to space decompositions of $\mathcal{H}$ and $\mathcal{J}$, it is clear that A^* has matrix form as follows,

$$A^*=\begin{pmatrix} A_{11}^* & A_{21}^* & 0\\ 0 & 0 & 0\\ A_{13}^* & A_{23}^* & 0\\ 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\\ \mathcal{J}_3\end{pmatrix}\to\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix}.$$

Since $A^*\mathcal{J}_1=A^*A^{*\dagger}\mathcal{H}_3=\mathcal{H}_3$, it is obvious that $A_{11}^*=0$ and A_{13}^* is surjective. Moreover, $N(A^*)=\mathcal{J}_3$, this implies that A_{13}^* is injective. So A_{13}^* is invertible. This infers A_{21}^* is surjective, sine $R(A^*)=\mathcal{H}_1\oplus\mathcal{H}_3$. In fact, A_{21}^* is also injective. Otherwise there is a non-zero element $x\in\mathcal{J}_2$ such that $A_{21}^*x=0$. Then there exists an element $y\in\mathcal{J}_1$ such that $A_{13}^*y=-A_{23}^*x$ since A_{13}^* is invertible. This follows that $A^*(y\oplus x)=0$ which is a contradiction with $N(A^*)=\mathcal{J}_3$. Therefore, A_{21}^* is invertible. And then A has the matrix form (3.1) where A_{13} and A_{21} are invertible.

By space decompositions $\mathcal{H}$ and $\mathcal{K}$, it is elementary that

$$B^*=\begin{pmatrix} B_{11}^* & B_{21}^* & 0 & 0\\ 0 & B_{22}^* & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix}\to\begin{pmatrix}\mathcal{K}_1\\ \mathcal{K}_2\\ \mathcal{K}_3\end{pmatrix},$$

where B_{11}^* is invertible and B_{22}^* is surjective. So B has the matrix form (3.2). Similarly, the statement (2) holds. The proof is completed. $\square$

Theorem 3.2 *Let $A\in\mathcal{B}(\mathcal{H},\mathcal{J})$ and $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then*

$$B\{1,3,4\}A\{1,3,4\}\subseteq(AB)\{1,3\}\Longleftrightarrow R(A^*AB)\subset R(B).$$

Proof The result naturally holds when $AB=0$, since $(AB)\{1,3\}=\mathcal{B}(\mathcal{J},\mathcal{K})$ and $R(A^*AB)=\{0\}$ in this case. Assume that $AB\neq 0$. We divide the proof into two cases.

Case 1. $\mathcal{H}_3\neq\{0\}$. By Lemma 3.1, A,B can be represented by (3.1) and (3.2), respectively, where operators A_{13}, A_{21}, B_{11} are invertible and B_{22}^* is surjective. This implies that

$$AB=\begin{pmatrix} 0 & 0 & 0\\ A_{21}B_{11} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{K}_1\\ \mathcal{K}_2\\ \mathcal{K}_3\end{pmatrix}\to\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\\ \mathcal{J}_3\end{pmatrix}, \tag{3.5}$$

and

$$A^*AB=\begin{pmatrix} A_{21}^*A_{21}B_{11} & 0 & 0\\ 0 & 0 & 0\\ A_{23}^*A_{21}B_{11} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{K}_1\\ \mathcal{K}_2\\ \mathcal{K}_3\end{pmatrix}\to\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix}. \tag{3.6}$$

For any $A^{(134)}\in A\{1,3,4\}$, by Lemma 2.1(1), $A^{(134)}$ has the matrix form

$$A^{(134)}=\begin{pmatrix} -A_{21}^{-1}A_{23}A_{13}^{-1} & A_{21}^{-1} & 0\\ 0 & 0 & G_1\\ A_{13}^{-1} & 0 & 0\\ 0 & 0 & G_2\end{pmatrix}:\begin{pmatrix}\mathcal{J}_1\\ \mathcal{J}_2\\ \mathcal{J}_3\end{pmatrix}\to\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix}, \tag{3.7}$$

where $G_1\in\mathcal{B}(\mathcal{J}_3,\mathcal{H}_2)$ and $G_2\in\mathcal{B}(\mathcal{J}_3,\mathcal{H}_4)$. Combining Lemma 2.1(2) with Lemma 2.2, we conclude that the $\{1,3,4\}$-inverse $B^{(134)}$ of B has the matrix form as follows:

$$B^{(134)} = \begin{pmatrix} G_{11} & G_{12} & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 \\ G_{31} & G_{32} & G_{33} & G_{34} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix}, \tag{3.8}$$

where $G_{33} \in \mathcal{B}(\mathcal{H}_3, \mathcal{K}_3)$, $G_{43} \in \mathcal{B}(\mathcal{H}_4, \mathcal{K}_3)$ and

$$\begin{cases} G_{11} = B_{11}{}^{-1}(I_{\mathcal{K}_1} + B_{11}^{*}{}^{-1}B_{21}{}^{*}(I_{\mathcal{H}_2} - B_{22}B_{22}^{\dagger})B_{21}B_{11}{}^{-1})^{-1}, \\ G_{12} = B_{11}{}^{-1}(I_{\mathcal{K}_1} + B_{11}^{*}{}^{-1}B_{21}{}^{*}(I_{\mathcal{H}_2} - B_{22}B_{22}^{\dagger})B_{21}B_{11}{}^{-1})^{-1}B_{11}^{*}{}^{-1}B_{21}{}^{*}(I_{\mathcal{H}_2} - B_{22}B_{22}^{\dagger}), \\ G_{21} = -B_{22}^{+}B_{21}G_{11}, \\ G_{22} = B_{22}^{\dagger} - B_{22}^{\dagger}B_{21}G_{12}. \end{cases} \tag{3.9}$$

Then it follows from matrix forms of (3.7) and (3.8) that

$$B^{(134)}A^{(134)} = \begin{pmatrix} -G_{11}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{11}A_{21}^{-1} & G_{12}G_1 \\ -G_{21}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{21}A_{21}^{-1} & G_{22}G_1 \\ -G_{31}A_{21}^{-1}A_{23}A_{13}^{-1} + G_{32}A_{13}^{-1} & G_{31}A_{21}^{-1} & G_{32}G_1 + G_{34}G_2 \end{pmatrix}. \tag{3.10}$$

Using Lemma 2.1(1), we get

$$(AB)^{(13)} = \begin{pmatrix} 0 & B_{11}^{-1}A_{21}^{-1} & 0 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \tag{3.11}$$

where $M_{ij} \in \mathcal{B}(\mathcal{K}_j, \mathcal{J}_i)$, $i \in \{2,3\}$, $j \in \{1,2,3\}$.

Assume that $B\{1,3,4\}A\{1,3,4\} \subseteq AB\{1,3\}$, it follows from (3.10) and (3.11) that

$$G_{11} = B_{11}^{-1}, \quad A_{23} = 0, \quad G_{12}G_1 = 0,$$

since M_{ij}, $i \in \{2,3\}$, $j \in \{1,2,3\}$ are arbitrary. Combining $G_{11} = B_{11}^{-1}$ with the equality $G_{11} = B_{11}^{-1} - G_{12}B_{21}B_{11}^{-1}$ in (3.9), we can obtain that $G_{12}B_{21} = 0$, and consequently $R(G_{12}^*) \subseteq N(B_{21}^*)$. From the relation $G_{12}B_{22} = 0$ in (3.9), we can infer that $R(G_{12}^*) \subseteq N(B_{22}^*)$. This shows $R(G_{12}^*) \subseteq N(B_{22}^*) \cap N(B_{21}^*)$ and so $G_{12} = 0$ by the definition of $\mathcal{H}_2$. From Lemma 2.2 again, it is obvious that $R(B_{21}) \subseteq R(B_{22})$. This infers that $N(B_{22}^*) \subseteq N(B_{21}^*)$ and so B_{22}^* is injective since $N(B_{22}^*) \cap N(B_{21}^*) = \{0\}$. Therefore, B_{22} is invertible. Combining (3.6) with (3.2), we have $R(A^*AB) = \mathcal{H}_1 \subseteq R(B)$ since $A_{23} = 0$ and B_{22} is invertible.

On the contrary, suppose that $R(A^*AB) \subseteq R(B)$. It is evident from the formula (3.6) that $A_{23}^*A_{21}B_{11} = 0$, and so $A_{23} = 0$. This infers $R(A^*AB) = \mathcal{H}_1 = R(B_{11}) \subseteq R\begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$. It follows that $R(B_{21}) \subseteq R(B_{22})$ and then $N(B_{22}^*) \subseteq N(B_{21}^*)$. Thus B_{22}^* is injective by the definition of $\mathcal{H}_2$ and so B_{22} is invertible. From Lemma 2.1 again, $G_{11} = B_{11}^{-1}, G_{12} = 0$ in (3.8). Combining formulae (3.10) with (3.11), we infer that

$$B\{1,3,4\}A\{1,3,4\} \subseteq AB\{1,3\}$$

by the arbitrariness of M_{2i}, M_{3i}, $i \in \{1,2,3\}$.

Case 2. $\mathcal{H}_3 = \{0\}$. In this case, A, B can be represented by (3.3) and (3.4), respectively, by Lemma 3.1. Similar to Case 1, it is clear that $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3\}$ if and only if

$R(A^*AB) \subseteq R(B)$. The proof is completed. □

In view of the relationship of $\{1,3\}$-inverse and $\{1,4\}$-inverse, we have the following result.

Corollary 3.3 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then*

$$B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,4\} \Longleftrightarrow R(BB^*A^*) \subseteq R(A^*).$$

Proof For any $E \in A\{1,3,4\}$ and $F \in B\{1,3,4\}$, then $E^* \in A^*\{1,3,4\}$ and $F^* \in B^*\{1,3,4\}$. It is clear that $FE \in (AB)\{1,4\}$ if and only if $E^*F^* \in (B^*A^*)\{1,3\}$ by Moore-Penrose equations (1), (3) and (4). Therefore, $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,4\}$ if and only if $A^*\{1,3,4\}B^*\{1,3,4\} \subseteq (B^*A^*)\{1,3\}$. Moreover, from Theorem 3.2, we have that

$$A^*\{1,3,4\}B^*\{1,3,4\} \subseteq (B^*A^*)\{1,3\} \Longleftrightarrow R(BB^*A^*) \subseteq R(A^*).$$

Hence,

$$B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,4\} \Longleftrightarrow R(BB^*A^*) \subseteq R(A^*).$$

The proof is completed. □

Corollary 3.4 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent,*

(1) $B\{1,3,4\}A\{1,3,4\} \subseteq (AB)\{1,3,4\}$;

(2) $R(A^*AB) \subseteq R(B)$ *and* $R(BB^*A^*) \subseteq R(A^*)$;

(3) $BB^\dagger A^*AB = A^*AB$ *and* $ABB^*A^\dagger A = ABB^*$.

Proof From Theorem 3.2 and Corollary 3.3, it is easy to get that statements (1) and (2) are equivalent. It follows from matrix forms of A^*AB, BB^*A^*, A^* and B that

$$R(A^*AB) \subseteq R(B),\ R(BB^*A^*) \subseteq R(A^*) \Longleftrightarrow A_{23} = 0,\ B_{21} = 0.$$

While $A_{23} = 0, B_{21} = 0$ is equivalent to $BB^\dagger A^*AB = A^*AB$, $ABB^*A^\dagger A = ABB^*$. □

The reverse order law for $\{1,3,4\}$-inverse was considered in [12]. Under the premise condition in Corollary 3.4, it was given therein

$$B\{1,3,4\}A\{1,3,4\} \subseteq AB\{1,3,4\}$$

if and only if

$$R(A^*AB) = R(B) \ominus (R(B) \cap N(A)) \text{ and } B^*(R(B) \cap N(A)) = B^\dagger(R(B) \cap N(A)),$$

which is equivalent to the statement (2) in Corollary 3.4 from Lemma 3.1 and matrix forms of A^*AB and BB^*A^*. Moreover, the statement (3) was also obtained by Cvetković -Ilić in [5] for

$$B\{1,3,4\}A\{1,3,4\} \subseteq AB\{1,3,4\}$$

under the condition A, B, AB, $A(I - BB^\dagger)$ and $(I - A^\dagger A)B$ are generalized invertible. Here, Corollary 3.4 is a refinement of the related result in [5].

Theorem 3.5 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent:*

(1) $(A^{(134)}AB)\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\}$ *for any* $A^{(134)} \in A\{1,3,4\}$;

(2) $R(AA^*AB) \subseteq R(AB)$.

Proof If $AB = 0$, the conclusion holds. Suppose that $AB \neq 0$. In the case of $\mathcal{H}_3 \neq \{0\}$, A, B and $A^{(134)}$ have the matrix forms (3.1), (3.2) and (3.7), respectively. Direct computation yields

$$A^{(134)}AB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which gives by Lemma 2.1(1) that

$$(AB)^{(134)} = \begin{pmatrix} 0 & B_{11}^{-1}A_{21}^{-1} & 0 \\ M_1 & 0 & M_2 \\ M_3 & 0 & M_4 \end{pmatrix}, \quad (A^{(134)}AB)^{(134)} = \begin{pmatrix} B_{11}^{-1} & 0 & 0 & 0 \\ 0 & F_1 & F_2 & F_3 \\ 0 & F_4 & F_5 & F_6 \end{pmatrix}, \quad (3.12)$$

where $M_i, i = 1,2,3,4$, $F_k, k = 1,2,\ldots,6$ are arbitrary. Then

$$(A^{(134)}AB)^{(134)}A^{(134)} = \begin{pmatrix} -B_{11}^{-1}A_{21}^{-1}A_{23}A_{13}^{-1} & B_{11}^{-1}A_{21}^{-1} & 0 \\ F_2A_{13}^{-1} & 0 & F_1G_1 + F_3G_2 \\ F_5A_{13}^{-1} & 0 & F_3G_1 + F_6G_2 \end{pmatrix}. \quad (3.13)$$

Compare (3.12) and (3.13), we get

$$(A^{(134)}AB)\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\} \Longleftrightarrow A_{23} = 0$$

by the arbitrariness of G_1, G_2, $M_i, i = 1,2,3,4$ and F_k, $k = 1,2,\ldots,6$. Moreover,

$$AA^*AB = \begin{pmatrix} A_{13}A_{23}^*A_{21}B_{11} & 0 & 0 \\ A_{21}A_{21}^*A_{21}B_{11} + A_{23}A_{23}^*A_{21}B_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \to \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \end{pmatrix}.$$

This shows that

$$R(AA^*AB) \subseteq R(AB) \Longleftrightarrow A_{23} = 0,$$

since A_{13}, A_{21} and B_{11} are all invertible. Therefore,

$$(A^{(134)}AB)\{1,3,4\}A\{1,3,4\} = (AB)\{1,3,4\} \Longleftrightarrow R(AA^*AB) \subseteq R(AB).$$

In the similar way, the conclusion also holds in the case of $\mathcal{H}_3 = \{0\}$. The proof is completed. □

Next, we will study the mixed-type reverse order laws associated to the Moore-Penrose inverse and the $\{1,3,4\}$-inverse.

Theorem 3.6 *Let* $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *and* $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ *be such that all ranges* $R(A)$, $R(B)$ *and* $R(AB)$ *are closed. Then the following statements are equivalent:*

(1) $(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger$;

(2) $(A^\dagger AB)^\dagger A^\dagger \in (AB)\{1,3,4\}$;

(3) $R(AA^*AB) \subseteq R(AB)$.

Proof By Theorem 3.5, (1)⇒(2) and (3)⇒(2) hold. Next, we prove the implications (2)⇒(1)

and (2)⇒(3). It needs only to consider the case when $AB \neq 0$ and $\mathcal{H}_3 \neq \{0\}$. Here A, B, AB and $(AB)^{(134)}$ have matrix forms as in (3.1) ,(3.2), (3.5) and (3.12), respectively. It is easy to know that Moore-Penrose inverses $A^\dagger$ and $(AB)^\dagger$ have matrix forms

$$A^\dagger = \begin{pmatrix} -A_{21}^{-1}A_{23}A_{13}^{-1} & A_{21}^{-1} & 0 \\ 0 & 0 & 0 \\ A_{13}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } (AB)^\dagger = \begin{pmatrix} 0 & B_{11}^{-1}A_{21}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.14}$$

respectively. By direct computation, we have

$$A^\dagger AB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } (A^\dagger AB)^\dagger = \begin{pmatrix} B_{11}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

Therefore,

$$(A^\dagger AB)^\dagger A^\dagger = \begin{pmatrix} -B_{11}^{-1}A_{21}^{-1}A_{23}A_{13}^{-1} & B_{11}^{-1}A_{21}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

Suppose that the statement (2) of this theorem holds. Comparing (3.16) and the matrix form of $(AB)^{(134)}$ in (3.12), we have $B_{11}^{-1}A_{21}^{-1}A_{23}A_{13}^{-1} = 0$ and so $A_{23} = 0$. Thus combining (3.16) with the matrix form of $(AB)^\dagger$ in (3.14), we have $(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger$. So the statement (1) holds. Moreover, it follows from $A_{23} = 0$ that AA^*AB has the matrix form as follows:

$$AA^*AB = \begin{pmatrix} 0 & 0 & 0 \\ A_{21}A_{21}^*A_{21}B_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that $R(AA^*AB) \subseteq R(AB)$. Therefore, the statement (3) holds. The proof is completed. □

Theorem 3.7 *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent:*

(1) $B^\dagger A\{1,3,4\} \subseteq (AB)\{1,3,4\}$;

(2) $B^\dagger A^\dagger \in (AB)\{1,3,4\}$;

(3) $R(BB^*A^*) \subseteq R(A^*)$ and $R(A^*AB) \subseteq R(B)$.

Proof By Corollary 3.4, (1)⇒(2) and (3)⇒(1) hold. We need only to prove (2)⇒(3) when $AB \neq 0$ and the case $\mathcal{H}_3 \neq \{0\}$, since the case $\mathcal{H}_3 = \{0\}$ is similar. Here A, $A^\dagger$ and B have the matrix form as in (3.1), (3.14) and (3.2), respectively. The Moore-Penrose inverse $B^\dagger$ can be represented by

$$B^\dagger = \begin{pmatrix} G_{11} & G_{12} & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.17}$$

where $G_{ij}, i \in \{1,2\}$, $j \in \{1,2\}$ satisfy the system of equations (3.9) by Lemma 2.2. Combining (3.14) with (3.17), we get

$$B^\dagger A^\dagger = \begin{pmatrix} -G_{11}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{11}A_{21}^{-1} & 0 \\ -G_{21}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{21}A_{21}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.18}$$

If $B^\dagger A^\dagger \in (AB)\{1,3,4\}$ holds, then from (3.18) and (3.12), we obtain $-G_{11}A_{21}^{-1}A_{23}A_{13}^{-1} = 0$, $G_{11}A_{21}^{-1} = B_{11}^{-1}A_{21}^{-1}$ and $G_{21}A_{21}^{-1} = 0$. This shows that $A_{23} = 0$, $G_{11} = B_{11}^{-1}$ and $G_{21} = 0$. Using the same technique in the proof of theorem 3.2, we know that $R(A^*AB) \subseteq R(B)$ and B_{22} is invertible. Notice that $G_{21} = -B_{22}^{-1}B_{21}B_{11}^{-1}$, it follows that $B_{21} = 0$. This means that $R(BB^*A^*) \subseteq R(A^*)$ by (3.1) and (3.2). The proof is completed. □

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