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A Semilinear Pseudo-Parabolic Equation in Exterior Domains

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Dedicated to the Memory of Professor L. C. HSU on the Occasion of His 100th Birthday

Abstract This paper is concerned with large time behavior of solutions for the semilinear pseudo-parabolic equation in exterior domains. It is revealed that the inhomogeneous boundary condition may develop large variation of solutions with the evolution of time.

Keywords pseudo-parabolic equation; exterior domain; large time behavior

MR(2010) Subject Classification 35K70; 35B05; 35B40

1. Introduction

In this paper, we consider the following semilinear pseudo-parabolic equation in the exterior domain with initial and boundary condition

$$\frac{\partial u}{\partial t}-k\frac{\partial \Delta u}{\partial t}=\Delta u+u^p, \qquad x\in\Omega,\ t>0, \tag{1.1}$$

$$\frac{\partial u}{\partial \vec{n}}=f(x), \qquad x\in\partial\Omega,\ t>0, \tag{1.2}$$

$$u(x,0)=u_0(x), \qquad x\in\Omega, \tag{1.3}$$

where $k>0$, $p>1$, $\Omega\equiv\mathbb{R}^n\setminus\overline{B_1(0)}$, $B_l(0)$ is the ball centered at the origin with radius l in $\mathbb{R}^n$, $\vec{n}$ is the unit normal vector of the unit ball in $\mathbb{R}^n$, namely the unit external normal vector of Ω. Furthermore, the non-negative and non-trivial functions $u_0(x)$ and $f(x)$ are smooth, and $\partial u_0(x)/\partial\vec{n}=f(x)$ on $\partial\Omega$, namely $u_0(x)$ and $f(x)$ satisfy the compatible condition.

Equations that include a third order mixed derivatives term are called pseudo-parabolic equations [1], and appear in a variety of important physical processes [2–6]. Regardless of the physical context, many authors have used u_{xxt} as a regularizing term for ill-posed diffusion equations [7, 8]. In resent years, considerable attentions have been paid to viscous pseudo-parabolic equations, see [9–14] and references therein, where general properties of solutions, optimal control problems, traveling waves etc. were considered.

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The purpose of the present paper is devoted to the critical exponent of the semilinear pseudo-parabolic equation in the exterior domain with inhomogeneous boundary value. Critical exponent is an important topic for nonlinear partial differential equations. The related studies began in 1966 by Fujita [15], where it was shown that the Cauchy problem of the semilinear heat equation does not have any nontrivial, nonnegative global solution if $1 < p < 1 + \frac{2}{n}$, whereas if $p > 1 + \frac{2}{n}$, there exist both global (with small initial data) and blowing-up (with large initial data) solutions. In the critical case $p = 1 + \frac{2}{n}$, it was shown by Hayakawa [16] and Kobayashi et al. [17] that the problem possesses no nontrivial global solutions. For the Cauchy problem of the semilinear pseudo-parabolic equation, [18] and [19] showed that there exist both global and non-global solutions if $p > 1 + \frac{2}{n}$, depending on the size of initial data, while the solutions blow up in finite time for any nontrivial initial data whenever p in the ignition interval $(1, 1 + \frac{2}{n}]$. In this paper, we find that for the nonlinear problem (1.1)–(1.3), small inhomogeneous boundary condition has a stronger effect. In fact $f(x)$ would develop large variation of solutions with the evolution of time, resulted in enlarging the ignition region from $(1, 1 + \frac{2}{n}]$ to $(1, \frac{n}{n-2}]$ with $n \geq 3$ or $(1, \infty)$ with $n = 1, 2$. Thus the Fujita exponent of the problem (1.1)–(1.3) is $p_c = \frac{n}{n-2}$. Consequently, this conclusion is consistent with the observation for the inner inhomogeneous semilinear heat equations and pseudo-parabolic equations [20–23]. This shows that the boundary inhomogeneous term has the same effect as that of the inner inhomogeneous term, and the diffusion effect of the viscous term $k\Delta u_t$ is neither strong enough to shake the effect of sources nor to dominate the effect of the inhomogeneous term.

We would mention that, owing to the lack of self-similar feature caused by Δu_t, the most useful method for parabolic equations, e.g. constructing global self-similar supersolutions and blowing-up self-similar subsolutions, is almost impossible to apply here. Moreover since we consider the exterior domain problem, we cannot use the integral representation in the whole space and the contraction-mapping principle. In this paper, we use the monotone iteration method for the global existence results, where the supersolutions are inspired by [24, 25]. For the blowing-up results, the technic used in [19] can only deduce partial results here. Therefore, we consult the discussion in [23] for the critical case of inhomogeneous quasilinear parabolic equations to show the energy blowing-up. That is to say, we will determine the interactions among diffusions, sources and the inhomogeneous boundary condition, by a series of precise integral estimates instead of pointwise comparison. Due to the appearance of two kinds of diffusions, these blow-up conclusions are more complicated and more difficult to prove.

The contents of the present paper are as follows. In Section 2, as a preliminary, we establish the necessary existence, uniqueness and comparison principle for our problem. The large time behavior of solutions are investigated in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we briefly collect some important preliminaries on local existence, uniqueness and comparison principle for the classical solutions of the problem (1.1)–(1.3). Actually, from the classical theory of the elliptic and parabolic equations, there exists a unique local classical solution for the problem (1.1)–(1.3), when $u_0(x)$ and $f(x)$ are smooth enough. Using the maximum principle for the pseudo-parabolic equations [26, 27], we can derive the comparison principle [28], here we omit the details.

Lemma 2.1 (Comparison Principle) *Assume that $p \geq 1$. Let u_1, u_2 be two classical solutions of the problem (1.1)–(1.3) with non-negative and non-trivial initial data and boundary condition. If*

$$\begin{aligned} &0 \leq u_1(x,0) \leq u_2(x,0), \quad x \in \Omega, \\ &\frac{\partial u_1}{\partial \vec{n}} \leq \frac{\partial u_2}{\partial \vec{n}}, \quad x \in \partial\Omega. \end{aligned}$$

Then

$$0 \leq u_1(x,t) \leq u_2(x,t), \quad (x,t) \in \Omega \times [0,T].$$

Our proof of blow-up results will be based on the following monotonicity property of the solutions, which can be proved by a similar argument as [24, Lemma 2.3].

Lemma 2.2 (Monotonicity Property) *Let $\underline{u}(x)$ be a non-negative and non-trivial subsolution to the stationary problem of (1.1)–(1.3). Then the non-negative and non-trivial solution $u(x,t)$ of (1.1)–(1.3) with initial data $\underline{u}(x)$ is monotone increasing to t.*

3. $1 < p \leq p_c$

In this section, we establish the blow-up results for the exterior domain problem (1.1)–(1.3) when $1 < p \leq p_c = \frac{n}{n-2}$ with $n \geq 3$ and $1 < p < \infty$ with $n = 1, 2$. Here we use the method in [19, 21, 23] to show the energy (some integral) blowing-up.

Theorem 3.1 *Suppose $n \geq 3$. If $1 < p \leq p_c = \frac{n}{n-2}$, then for any non-trivial non-negative $u_0(x)$ and $f(x)$, the solution of the problem (1.1)–(1.3) blows up in finite time.*

Proof Due to the comparison principle, $u_0(x) \geq 0$ and $f(x) \geq 0$, we can consider directly the following problem

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u + u^p, \qquad x \in \Omega,\ t > 0, \tag{3.1}$$

$$\frac{\partial u}{\partial \vec{n}} = f(x)f(t), \qquad x \in \partial\Omega,\ t > 0, \tag{3.2}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{3.3}$$

where $f(t)$ is a smooth function on $[0,\infty)$, $0 \leq f(t) \leq 1$, $f(0) = 0$ and $f(t) = 1$ when $1 \leq t < \infty$. It is obvious that 0 is a subsolution of the problem (3.1)–(3.3) and 0 does not satisfy (3.1)–(3.3). Then by Lemma 2.2, we can derive that the solution of the problem (3.1)–(3.3) is increasing with respect to t.

We shall prove the blow-up phenomenon of the problem (3.1)–(3.3) by contradiction.

First of all, we construct some useful cut-off functions. Let $\eta \in C^\infty[0,\infty)$ satisfy the following conditions

(i) $0 \le \eta(t) \le 1$; $\eta(t) = 1$, $t \in [0,1]$; $\eta(t) = 0$, $t \in [2,\infty)$;

(ii) $-C \le \eta'(t) \le 0$, where C is a positive constant.

For $T > 0$, let

$$\eta_T(t) = \eta(\frac{t}{2T}).$$

Then we can derive that

$$-\frac{C}{T} \le \eta_T'(t) \le 0, \tag{3.4}$$

where C is a positive constant independent of T. For the space variable cut-off functions, we denote

$$\psi(x) = \begin{cases} 1, & 0 \le |x| \le 1, \\ \varphi(|x|-1), & 1 < |x| < 2, \\ 0, & |x| \ge 2, \end{cases}$$

where φ is the principle eigenfunction of $-\Delta$ in the unit ball of $\mathbb{R}^n$ with homogeneous Dirichlet boundary value condition, normalized by $\|\varphi\|_{L^\infty(B_1)} = 1$. For $l > 0$, we define

$$\psi_l(x) = \psi(\frac{x}{l}), \quad x \in \mathbb{R}^n,$$

which satisfies the following properties

$$|\nabla \psi_l| \le \frac{C}{l}, \quad |\Delta \psi_l| \le \frac{C}{l^2}, \quad \frac{|\Delta \psi_l|}{\psi_l} \le \frac{C}{l^2}, \quad x \in B_{2l} \setminus B_l, \tag{3.5}$$

where C is a positive constant independent of l, B_l is the ball in $\mathbb{R}^n$ with radius l and centered at the origin.

Secondly, we choose the suitable time and spacial region. For $l > 1$ and $T > 1$, we let

$$Q_{l,T} = (B_{2l}(0) \cap \Omega) \times [0, 4T].$$

Notice that these sets Q_l increase with respect to l and T, and $\cup_{l>1/2,T>0} Q_l = \Omega \times [0,\infty)$.

Now we suppose $u(x,t)$ is the non-negative and non-trivial global solution of the problem (3.1)–(3.3). Then for any $t \ge 0$, u satisfies

$$\begin{aligned} &\int_0^t \int_\Omega \frac{\partial u}{\partial s} \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s - k \int_0^t \int_\Omega \frac{\partial \Delta u}{\partial s} \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s \\ &\quad = \int_0^t \int_\Omega \Delta u \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega u^p \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s, \end{aligned} \tag{3.6}$$

where $r > 1$ is a constant to be determined. Set

$$I_l \equiv \int_{Q_{l,T}} u^p(x,t) \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s.$$

According to the definition of the cut-off functions ψ_l and η_T, if we choose $t > 4T$ in (3.6), then

$$\begin{aligned} I_l &= \int_{Q_{l,T}} \frac{\partial u}{\partial s} \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s - k \int_{Q_{l,T}} \frac{\partial \Delta u}{\partial s} \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s - \int_{Q_{l,T}} \Delta u \psi_l^r \eta_T^r \mathrm{d}x\mathrm{d}s \\ &= J_1 + J_2 + J_3. \end{aligned}$$

In what follows, we estimate J_1, J_2 and J_3. Integrating by parts, we have

$$
\begin{aligned}
J_1 =& \int_{B_{2l}(0)\setminus\overline{B_1(0)}} u(x,s)\psi_l^r\eta_T^r \mathrm{d}x\Big|_0^{4T} - \int_{Q_{l,T}} u(x,s)\psi_l^r(\eta_T^r)'\mathrm{d}x\mathrm{d}s,\\
J_2 =& -k\int_{B_{2l}(0)\setminus\overline{B_1(0)}} \Delta u(x,s)\psi_l^r\eta_l^r\mathrm{d}x\Big|_0^{4T} + k\int_{Q_{l,T}} \Delta u(x,s)\psi_l^r(\eta_T^r)'\mathrm{d}x\mathrm{d}s\\
=& -k\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} \frac{\partial u}{\partial\vec{n}}\psi_l^r\eta_T^r\mathrm{d}\sigma\Big|_0^{4T} + k\int_{B_{2l}(0)\setminus\overline{B_1(0)}} \nabla u(x,s)\cdot\nabla\psi_l^r\eta_T^r\mathrm{d}x\Big|_0^{4T} +\\
& k\int_0^{4T}\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} \frac{\partial u}{\partial\vec{n}}\psi_l^r(\eta_T^r)'\mathrm{d}\sigma\mathrm{d}s - k\int_{Q_{l,T}} \nabla u(x,s)\cdot\nabla\psi_l^r(\eta_T^r)'\mathrm{d}x\mathrm{d}s\\
=& -k\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} \frac{\partial u}{\partial\vec{n}}\psi_l^r\eta_T^r\mathrm{d}\sigma\Big|_0^{4T} - k\int_{B_{2l}(0)\setminus\overline{B_1(0)}} u(x,s)\Delta\psi_l^r\eta_T^r\mathrm{d}x\Big|_0^{4T} +\\
& k\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} u\frac{\partial\psi_l^r}{\partial\nu}\eta_T^r\mathrm{d}\sigma\Big|_0^{4T} + k\int_0^{4T}\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} \frac{\partial u}{\partial\vec{n}}\psi_l^r(\eta_T^r)'\mathrm{d}\sigma\mathrm{d}s+\\
& k\int_{Q_{l,T}} u(x,s)\Delta\psi_l^r(\eta_T^r)'\mathrm{d}x\mathrm{d}s - k\int_0^{4T}\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} u(x,s)\frac{\partial\psi_l^r}{\partial\vec{n}}\eta_T^r\mathrm{d}\sigma\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
J_3 =& -\int_0^{4T}\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} \frac{\partial u}{\partial\vec{n}}\psi_l^r\eta_T^r\mathrm{d}\sigma\mathrm{d}s + \int_{Q_{l,T}} \nabla u\cdot\nabla\psi_l^r\eta_T^r\mathrm{d}x\mathrm{d}s\\
=& -\int_0^{4T}\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} \frac{\partial u}{\partial\vec{n}}\psi_l^r\eta_T^r\mathrm{d}\sigma\mathrm{d}s - \int_{Q_{l,T}} u\Delta\psi_l^r\eta_T^r\mathrm{d}x\mathrm{d}s+\\
& \int_0^{4T}\int_{\partial(B_{2l}(0)\setminus\overline{B_1(0)})} u\frac{\partial\psi_l^r}{\partial\vec{n}}\eta_T^r\mathrm{d}\sigma\mathrm{d}s.
\end{aligned}
$$

Actually, from the definition of η_T and ψ_l, there holds $\eta_T(0) = 1$, $\eta_T(4T) = 0$, $\psi_l(2l) = 0$, $\frac{\partial\psi_l^r}{\partial\vec{n}} = r\psi_l^{r-1}\nabla\psi\cdot\nu = 0$ on $\partial(B_{2l}(0)\setminus\overline{B_1(0)})$. Using the initial and inhomogeneous boundary conditions (3.2) and (3.3), we can get

$$
\begin{aligned}
J_1+J_2+J_3 =& -\int_{Q_{l,T}} u(x,s)\psi_l^r(\eta_T^r)'\mathrm{d}x\mathrm{d}s + k\int_0^{4T}\int_{\partial B_1(0)} f(x)f(s)\psi_l^r(\eta_T^r)'\mathrm{d}\sigma\mathrm{d}s+\\
& k\int_{Q_{l,T}} u(x,s)\Delta\psi_l^r(\eta_T^r)'\mathrm{d}x\mathrm{d}s - \int_{Q_{l,T}} u\Delta\psi_l^r\eta_T^r\mathrm{d}x\mathrm{d}s-\\
& \int_0^{4T}\int_{\partial B_1(0)} f(x)f(s)\psi_l^r\eta_T^r\mathrm{d}\sigma\mathrm{d}s.
\end{aligned}
$$

Substituting $\Delta\psi_l^r = r\psi_l^{r-1}\Delta\psi_l + r(r-1)\psi_l^{r-2}|\nabla\psi_l|^2$ and $(\eta_T^r)' = r\eta_T^{r-1}\eta_T'$ into the above equation leads to

$$
\begin{aligned}
J_1+J_2+J_3 =& -\int_{Q_l} u(x,s)\psi_l^r r\eta_T^{r-1}\eta_T'\mathrm{d}x\mathrm{d}s + k\int_0^{4T}\int_{\partial B_1(0)} f(x)f(s)\psi_l^r r\eta_T^{r-1}\eta_T'\mathrm{d}\sigma\mathrm{d}s+\\
& k\int_{Q_l} u(x,s)(r\psi_l^{r-1}\Delta\psi_l + r(r-1)\psi_l^{r-2}|\nabla\psi_l|^2)r\eta_T^{r-1}\eta_T'\mathrm{d}x\mathrm{d}s-\\
& \int_{Q_l} u(r\psi_l^{r-1}\Delta\psi_l + r(r-1)\psi_l^{r-2}|\nabla\psi_l|^2)\eta_T^r\mathrm{d}x\mathrm{d}s-
\end{aligned}
$$

$$\int_0^{4T}\int_{\partial B_1(0)} f(x)f(s)\psi_l^r\eta_T^r \mathrm{d}\sigma\mathrm{d}s.$$

Since $u(x,t)\geq 0$, $f(x)f(t)\geq 0$, $\Delta\psi_l\leq 0$, $\eta_T'\leq 0$, we can derive

$$\begin{aligned}&J_1+J_2+J_3\\ &\leq -\int_{Q_{l,T}} u(x,s)\psi_l^r r\eta_T^{r-1}\eta_T' \mathrm{d}x\mathrm{d}s + k\int_{Q_{l,T}} u(x,s) r\psi_l^{r-1}\Delta\psi_l r\eta_T^{r-1}\eta_T'\mathrm{d}x\mathrm{d}s-\\ &\quad\int_{Q_{l,T}} ur\psi_l^{r-1}\Delta\psi_l\eta_T^r\mathrm{d}x\mathrm{d}s - \int_0^{4T}\int_{\partial B_1(0)} f(x)f(s)\psi_l^r\eta_T^r\mathrm{d}\sigma\mathrm{d}s.\end{aligned}$$

Since $f(x)\geq 0$ and $f(x)\not\equiv 0$ on $\partial\Omega$, there exists a $\delta>0$, such that $\int_{\partial\Omega} f(x)\mathrm{d}\sigma>\delta$. Using (3.4), (3.5) and the facts ψ_l, $\eta_l\leq 1$, we have

$$\begin{aligned}&J_1+J_2+J_3\\ &\leq \frac{C}{T}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} u(x,s)\psi_l^r\eta_T^{r-1}\mathrm{d}x\mathrm{d}s + \frac{C}{Tl^2}k\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} u(x,s)\psi_l^{r-1}\eta_T^{r-1}\mathrm{d}x\mathrm{d}s+\\ &\quad\frac{C}{l^2}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} u(x,s)\psi_l^{r-1}\eta_T^r\mathrm{d}x\mathrm{d}s - T\delta\\ &\leq \frac{C}{T}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} u(x,s)\psi_l^{r-1}\eta_T^{r-1}\mathrm{d}x\mathrm{d}s + \frac{C}{Tl^2}k\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} u(x,s)\psi_l^{r-1}\eta_T^{r-1}\mathrm{d}x\mathrm{d}s+\\ &\quad\frac{C}{l^2}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} u(x,s)\psi_l^{r-1}\eta_T^{r-1}\mathrm{d}x\mathrm{d}s - T\delta.\end{aligned}$$

From the Young inequality it follows

$$\begin{aligned}&J_1+J_2+J_3\\ &\leq \frac{3}{4}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} u^p(x,s)\psi_l^r\eta_T^r\mathrm{d}x\mathrm{d}s+\\ &\quad C\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} \psi_l^{r-p/(p-1)}\eta_T^{r-p/(p-1)}T^{-p/(p-1)}\mathrm{d}x\mathrm{d}s+\\ &\quad Ck^{p/(p-1)}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} \psi_l^{r-p/(p-1)}\eta_T^{r-p/(p-1)}(Tl^2)^{-p/(p-1)}\mathrm{d}x\mathrm{d}s+\\ &\quad C\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} \psi_l^{r-p/(p-1)}\eta_T^{r-p/(p-1)}l^{-2p/(p-1)}\mathrm{d}x\mathrm{d}s - T\delta.\end{aligned}$$

If r is selected large enough such that $r-1-r/p>0$ and $r>2$, then the above inequality becomes

$$\begin{aligned}&J_1+J_2+J_3\\ &\leq \frac{3}{4}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} u^p(x,s)\psi_l^r\eta_T^r\mathrm{d}x\mathrm{d}s + C\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} T^{-p/(p-1)}\mathrm{d}x\mathrm{d}s+\\ &\quad Ck^{p/(p-1)}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} (Tl^2)^{-p/(p-1)}\mathrm{d}x\mathrm{d}s+\end{aligned}$$

$$
\begin{aligned}
&C\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} l^{-2p/(p-1)}\mathrm{d}x\mathrm{d}s - T\delta\\
\le\ &\frac{3}{4}\int_{2T}^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} u^p(x,s)\psi_l^r\eta_T^r\mathrm{d}x\mathrm{d}s + Cl^nT^{1-p/(p-1)}+\\
&Ck^{p/(p-1)}l^{n-2p/(p-1)}T^{1-p/(p-1)} + CTl^{n-2p/(p-1)} - T\delta.
\end{aligned}
$$

Thus we get

$$
\frac{1}{4}I_l \le T(Cl^nT^{-p/(p-1)} + Ck^{p/(p-1)}l^{n-2p/(p-1)}T^{-p/(p-1)} + Cl^{n-2p/(p-1)}) - T\delta. \tag{3.7}
$$

It follows from $n \ge 3$ and $1 < p \le n/(n-2)$ that

$$
n - \frac{2p}{p-1} = \frac{(n-2)p-n}{p-1} \le 0.
$$

Set $T \ge l^{n(p-1)/p}$ such that $l^nT^{-p/(p-1)} \le 1$, then (3.7) becomes

$$
I_l \le CT - T\delta. \tag{3.8}
$$

If δ is chosen small enough, then (3.8) is

$$
\int_0^{4T}\int_{B_{2l}(0)\backslash\overline{B_1(0)}} u^p\psi_l^r\eta_T^r\mathrm{d}x\mathrm{d}t \le CT.
$$

Hence we have

$$
\int_T^{2T}\int_{B_l(0)\backslash\overline{B_1(0)}} u^p\mathrm{d}x\mathrm{d}t \le CT.
$$

By the integral mean value theorem, there exists $t_1 \in [T, 2T]$ such that

$$
\int_{B_l(0)\backslash\overline{B_1(0)}} u^p(x,t_1)\mathrm{d}x \le C,
$$

where C is a positive constant independent of l and T. We deduce that, for any fixed $l > 1$ and any $t \ge 0$, there holds $\int_{B_l(0)\backslash\overline{B_1(0)}} u^p(x,t)\mathrm{d}x \le C$. If there exists t_2 such that $\int_{B_l(0)\backslash\overline{B_1(0)}} u^p(x,t_2)\mathrm{d}x > C$, then from the monotone increasing property of $u(x,t)$ with respect to t, $\int_{B_l(0)\backslash\overline{B_1(0)}} u^p(x,t)\mathrm{d}x > C$, for any $t > t_2$. However, if we choose $T > \max(t_2, l^{n(p-1)/p})$, then from the above process, there exists $t_3 \in [T, 2T]$ such that $\int_{B_l(0)\backslash\overline{B_1(0)}} u^p(x,t_3)\mathrm{d}x \le C$, which is a contradiction. Because $u(x,t)$ is increasing with respect to t, then $\int_{B_l(0)\backslash\overline{B_1(0)}} u^p(x,t)\mathrm{d}x$ is increasing with respect to t, which yields the existence of $I_l^\infty = \lim_{t\to\infty}\int_{B_l(0)\backslash\overline{B_1(0)}} u^p(x,t)\mathrm{d}x$ and $I_l^\infty \le C$. Due to the non-negativity of $u(x,t)$, I_l^∞ is increasing with respect to l. Hence the limitation $\lim_{l\to\infty} I_l^\infty$ exists and $\lim_{l\to\infty} I_l^\infty \le C$. Thus for any small $\varepsilon > 0$, there exists $l_\varepsilon > 1$, such that for $l > l_\varepsilon$,

$$
\lim_{t\to\infty}\int_{B_{2l}(0)\backslash\overline{B_l(0)}} u^p(x,t)\mathrm{d}x = I_{2l}^\infty - I_l^\infty < \varepsilon.
$$

Then for sufficiently large $l > \max(1, l_\varepsilon)$, we have

$$
\int_{B_{2l}(0)\backslash\overline{B_l(0)}} u^p(x,t)\mathrm{d}x \le \varepsilon, \quad t \ge 0.
$$

Multiplying both sides of (3.1) by $\psi_l(x) = \phi(\frac{x}{l})$ and integrating in Ω, we get

$$
\int_\Omega \frac{\partial u}{\partial t}\psi_l\mathrm{d}x - k\int_\Omega \frac{\partial\Delta u}{\partial t}\psi_l\mathrm{d}x = \int_\Omega \Delta u\phi_l\mathrm{d}x + \int_\Omega u^p\psi_l\mathrm{d}x. \tag{3.9}
$$

Integrating by parts leads to

$$\int_\Omega \frac{\partial u}{\partial t}\psi_l \mathrm{d}x + k\int_\Omega \frac{\partial \nabla u}{\partial t}\cdot\nabla\psi_l \mathrm{d}x - k\int_{\partial(B_{2l}\setminus\overline{B_1(0)})}\frac{\partial u_t}{\partial\vec{n}}\psi_l\mathrm{d}x$$
$$= -\int_\Omega \nabla u\cdot\nabla\phi_l\mathrm{d}x + \int_{\partial(B_{2l}\setminus\overline{B_1(0)})}\frac{\partial u}{\partial\vec{n}}\phi_l\mathrm{d}x + \int_\Omega u^p\phi_l\mathrm{d}x.$$

Furthermore

$$\int_\Omega \frac{\partial u}{\partial t}\psi_l\mathrm{d}x - k\int_\Omega\frac{\partial u}{\partial t}\Delta\psi_l\mathrm{d}x + k\int_{\partial(B_{2l}\setminus\overline{B_1(0)})}\frac{\partial u}{\partial t}\frac{\partial\psi_l}{\partial\vec{n}}\mathrm{d}x - k\int_{\partial(B_{2l}\setminus\overline{B_1(0)})}\frac{\partial u_t}{\partial\vec{n}}\psi_l\mathrm{d}x$$
$$= \int_\Omega u\Delta\psi_l\mathrm{d}x - \int_{\partial(B_{2l}\setminus\overline{B_1(0)})} u\frac{\partial\psi_l}{\partial\vec{n}}\mathrm{d}x + \int_{\partial(B_{2l}\setminus\overline{B_1(0)})}\frac{\partial u}{\partial\vec{n}}\psi_l\mathrm{d}x + \int_\Omega u^p\psi_l\mathrm{d}x.$$

Integrating the above equality in $(0,t)$ and using the boundary condition (3.3), we derive

$$\int_\Omega u(x,t)\psi_l\mathrm{d}x - k\int_\Omega u(x,t)\Delta\psi_l\mathrm{d}x + k\int_{\partial(B_{2l}\setminus\overline{B_1(0)})} u(x,t)\frac{\partial\psi_l}{\partial\vec{n}}\mathrm{d}x - k\int_{\partial(B_{2l}\setminus\overline{B_1(0)})} f(x)f(t)\psi_l\mathrm{d}x$$
$$= \int_0^t\int_\Omega u\Delta\psi_l\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\partial(B_{2l}\setminus\overline{B_1(0)})} u\frac{\partial\psi_l}{\partial\vec{n}}\mathrm{d}x\mathrm{d}s +$$
$$\int_0^t\int_{\partial(B_{2l}\setminus\overline{B_1(0)})} f(x)f(t)\psi_l\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega u^p\psi_l\mathrm{d}x\mathrm{d}s.$$

From the properties of ψ_l and the non-negativity of $u(x,t)$, we have

$$\int_\Omega u(x,t)\psi_l\mathrm{d}x - k\int_\Omega u(x,t)\Delta\psi_l\mathrm{d}x$$
$$= -k\int_{\partial B_{2l}(0)} u(x,t)\frac{\partial\psi_l}{\partial\vec{n}}\mathrm{d}x + k\int_{\partial B_1(0)} f(x)f(t)\psi_l\mathrm{d}x +$$
$$\int_0^t\int_\Omega u\Delta\psi_l\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\partial B_{2l}(0)} u\frac{\partial\psi_l}{\partial\vec{n}}\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\partial B_1(0)} f(x)f(t)\psi_l\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega u^p\psi_l\mathrm{d}x\mathrm{d}s$$
$$\geq \int_0^t\int_\Omega u\Delta\psi_l\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\partial B_1(0)} f(x)f(t)\psi_l\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega u^p\psi_l\mathrm{d}x\mathrm{d}s,$$

namely

$$\int_\Omega u(x,t)\psi_l\mathrm{d}x \geq -Ckl^{-2}\int_\Omega u(x,t)\psi_l\mathrm{d}x - \int_0^t\int_\Omega u|\Delta\psi_l|\mathrm{d}x\mathrm{d}t +$$
$$\int_0^t\int_{\partial B_1(0)} f(x)f(t)\psi_l\mathrm{d}\sigma\mathrm{d}t + \int_0^t\int_\Omega u^p\psi_l\mathrm{d}x\mathrm{d}t.$$

Using the Hölder inequality and noticing $n-2-\frac{n}{p}\leq 0$, we get that

$$\int_{B_{2l}(0)\setminus\overline{B_l(0)}} u|\Delta\psi_l|\mathrm{d}x \leq C\Big(\int_{B_{2l}(0)\setminus\overline{B_l(0)}} u^p\Big)^{1/p} l^{n-2-n/p} < C\varepsilon^{1/p}.$$

Combining the above two inequalities leads to

$$(1+Ckl^{-2})\int_\Omega u(x,t)\psi_l\mathrm{d}x \geq t\delta - tC\varepsilon^{1/p} + \int_0^t\int_\Omega u^p\psi_l\mathrm{d}x\mathrm{d}t.$$

Choose ε small enough such that $C\varepsilon^{\frac{1}{p}} < \delta$, then

$$(1+Ckl^{-2})\int_\Omega u(x,t)\phi_l \mathrm{d}x \geq \int_0^t \int_\Omega u^p \phi_l \mathrm{d}x\mathrm{d}t.$$

Set $F_l(t) = \int_\Omega u\psi_l \mathrm{d}x$, $G(t) = \int_\Omega u^p \psi_l \mathrm{d}x$. Then we have

$$(1+Ckl^{-2})F_l(t) \geq \int_0^t G(t)\mathrm{d}t.$$

From the Hölder inequality, there holds

$$\int_{t_0}^t (F_l(s))^p \mathrm{d}s \leq Cl^{(1-1/p)n} \int_{t_0}^t G_l(s)\mathrm{d}s \leq Cl^{(1-1/p)n}(1+Ckl^{-2})F_l(t),$$

namely,

$$F_l(t) \geq Cl^{-(1-1/p)n}(1+Ckl^{-2})^{-1} \int_{t_0}^t (F_l(s))^p \mathrm{d}s. \tag{3.10}$$

Let $g(t) = \int_{t_0}^t (F_l(s))^p \mathrm{d}s$. Then we have

$$\begin{aligned} g'(t) = (F_l(t))^p &\geq Cl^{-(1-1/p)np}(1+Ckl^{-2})^{-p}\Big(\int_{t_0}^t (F_l(s))^p \mathrm{d}s\Big)^p \\ &= Cl^{-(1-1/p)np}(1+Ckl^{-2})^{-p} g^p(t). \end{aligned}$$

Set $t_1 > 0$ such that $g(t_1) > 0$. Since $p > 1$, by solving the above equation, we have

$$\lim_{t \to T_1(k)} g(t) = +\infty,$$

where

$$T_1(k) = \frac{g^{1-p}(t_4)}{C(1+Ckl^{-2})^{-p}(1-p)l^{-(1-1/p)np}} + t_1. \tag{3.11}$$

Combining this with (3.10), we know that $F_l(t)$ blows up in finite time, then u blows up in finite time which contradicts our assumptions. Hence every solution of the problem (1.1)–(1.3) blows up in finite time. □

Remark 3.2 In the proof of Theorem 3.1, we can find that when $n = 1, 2$ and $p > 1$, there holds

$$n - \frac{2p}{p-1} = \frac{(n-2)p - n}{p-1} < 0,$$

which would guarantee the validity of (3.8). Thus we can deduce that when $n = 1, 2$ and $p > 1$, every solution of the problem (1.1)–(1.3) blows up in finite time.

4. The case $p > p_c$

In this section, we treat the case $p > p_c = \frac{n}{n-2}$. We will give two theorems to show that when the initial data $u_0(x)$ and the inhomogeneous boundary condition $f(x)$ are small enough, then the solution of the problem (1.1)–(1.3) exists globally. Otherwise, the solution blows up in finite time provided that one of $u_0(x)$ and $f(x)$ is large enough.

Theorem 4.1 *Suppose $n \geq 3$. If $p > p_c = \frac{n}{n-2}$, then for sufficiently small $u_0(x) \geq 0$ and*

$f(x) \geq 0$, the problem (1.1)–(1.3) admits a global solution.

Proof Since the problem (1.1)–(1.3) is in the exterior domain, we cannot use the kernel functions $\mathcal{G}(t)$ and $\mathcal{H}(t)$ of the pseudo-parabolic equation in [19] to represent the solution of the problem (1.1)–(1.3). Thus making use of the integral representation which is effective for the parabolic and pseudo-parabolic equations in the whole space $\mathbb{R}^n$ (see [19, 21, 23]) is not active here to derive precise and thorough L^q estimates for the global existence. Here we can use the comparison principle and the monotone iteration method to prove the global existence results. The supersolutions and the subsolutions are inspired by [24, 25]. Set $\hat{u} = \lambda(1+|x|^2)^{-1/(p-1)}$, where $\lambda > 0$ is an undetermined constant. After a simple computation, we can have

$$-\Delta\hat{u} = (\frac{2\lambda}{p-1})(n - \frac{2p}{p-1} + \frac{2p}{(p-1)(1+|x|^2)})(1+|x|^2)^{-p/(p-1)}, \quad x \in \Omega,$$

$$\frac{\partial\hat{u}}{\partial\vec{n}} = -\frac{\partial\hat{u}}{\partial|x|} = (\frac{2\lambda}{p-1})(1+|x|^2)^{-p/(p-1)}|x| > 0, \quad |x| = 1,$$

Since $p > p_c = \frac{n}{n-2}$, we can get

$$n - \frac{2p}{p-1} > 0.$$

Thus if we choose λ small enough such that

$$(\frac{2\lambda}{p-1})(n - \frac{2p}{p-1} + \frac{2p}{(p-1)(1+|x|^2)}) \geq \lambda^p,$$

then there holds

$$-\Delta\hat{u} \geq \hat{u}^p, \quad x \in \Omega.$$

Hence, if we choose $u_0(x) \leq \hat{u}(x)$ and $f(x) \leq (\frac{2\lambda}{p-1})(1+|x|^2)^{-p/(p-1)}|x|$, then $\hat{u}(x)$ is a global supersolution of the problem (1.1)–(1.3). It is obvious that 0 is a subsolution of the problem (1.1)–(1.3). Therefore, by the iterative process and the comparison principle Lemma 2.1, the problem (1.1)–(1.3) admits a global solution. □

Theorem 4.2 *Suppose $n \geq 3$. If $p > p_c = \frac{n}{n-2}$, then for sufficiently large $u_0 \geq 0$ or $f(x) \geq 0$, the solution of the problem (1.1)–(1.2) blows up in finite time.*

Proof We divide the proof into two parts. In the first place, we consider the case that the initial data $u_0(x)$ is large enough. In fact, similar to the proof in [19], when $p > 1 + \frac{2}{n}$ and the initial data $u_0(x)$ is large enough, the following homogeneous boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u + u^p, && x \in \Omega,\ t > 0, \\ &\frac{\partial u}{\partial \vec{n}} = 0, && x \in \partial\Omega,\ t > 0, \\ &u(x,0) = u_0(x), && x \in \Omega, \end{aligned}$$

possesses no global solutions. The solution of the above problem is just the subsolution of the problem (1.1)–(1.3). Thus from the comparison, when $p > 1 + \frac{2}{n}$ and the initial data $u_0(x)$ is large enough, the solution of the problem (1.1)–(1.3) blows up in finite time. Notice that when

$n \geq 3$, $\frac{n}{n-2} > 1 + \frac{2}{n}$. Then for $p > p_c = \frac{n}{n-2}$, if the initial data $u_0(x)$ is large enough, the problem (1.1)–(1.3) possesses no global solutions.

Next, we take into account the case that $f(x)$ is large enough. Reviewing the proof of Theorem 3.1, we find that (3.7) is

$$\begin{aligned}\frac{1}{4}I_l \leq & T(Cl^n T^{-p/(p-1)} + Ck^{p/(p-1)} l^{n-2p/(p-1)} T^{-p/(p-1)} + Cl^{n-2p/(p-1)}) - \\ & T\int_{\partial B_1(0)} f(x)\mathrm{d}x. \end{aligned} \tag{4.1}$$

When $n \geq 3$ and $p > p_c = \frac{n}{n-2}$, we have

$$n - \frac{2p}{p-1} = \frac{(n-2)p - n}{p-1} > 0.$$

Let $T > l^{n(p-1)/p}$ such that $l^n T^{-p/(p-1)} < 1$. Then (3.8) becomes

$$I_l \leq TC + CTl^{n-2p/(p-1)} - T\int_{\partial B_1(0)} f(x)\mathrm{d}x. \tag{4.2}$$

It is obvious that if $f(x)$ is large enough such that $Cl^{n-2p/(p-1)} \leq \int_{\partial B_1(0)} f(x)\mathrm{d}x$, then from (4.2), we can get

$$\int_T^{2T}\int_\Omega u^p \mathrm{d}x\mathrm{d}t \leq CT.$$

Similar to the proof of Theorem 3.1, we can deduce that the solution blows up in finite time. □

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