Journal of Mathematical Research with Applications
Nov., 2019, Vol. 39, No. 6, pp. 657–676
DOI:10.3770/j.issn:2095-2651.2019.06.011
Http://jmre.dlut.edu.cn

The Modified Weak Galerkin Finite Element Method for Solving Brinkman Equations

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Dedicated to the Memory of Professor L. C. HSU on the Occasion of His 100th Birthday

Abstract A modified weak Galerkin (MWG) finite element method is introduced for the Brinkman equations in this paper. We approximate the model by the variational formulation based on two discrete weak gradient operators. In the MWG finite element method, discontinuous piecewise polynomials of degree k and $k-1$ are used to approximate the velocity $\boldsymbol{u}$ and the pressure p, respectively. The main idea of the MWG finite element method is to replace the boundary functions by the average of the interior functions. Therefore, the MWG finite element method has fewer degrees of freedom than the WG finite element method without loss of accuracy. The MWG finite element method satisfies the stability conditions for any polynomial with degree no more than $k-1$. The MWG finite element method is highly flexible by allowing the use of discontinuous functions on arbitrary polygons or polyhedra with certain shape regularity. Optimal order error estimates are established for the velocity and pressure approximations in H^1 and L^2 norms. Some numerical examples are presented to demonstrate the accuracy, convergence and stability of the method.

Keywords the Brinkman equations; the modified weak Galerkin finite element method; discrete weak gradient

MR(2010) Subject Classification 65N15; 65N30; 76D07

1. Introduction

The Brinkman equations provide a description of fluid flow in multi-physics environment. Modeling fluid flow in complex media with multiphysics has significant impact on many fields, such as low porosity filtration equipment, vuggy carbonate reservoirs, biomedical hydrodynamic and underground water hydrology [1, 2]. The Brinkman equations are applicable to both the Darcy flow and the Stokes flow without employing complex interface assumptions [3, 4]. Therefore, it is significant to design efficient and stable numerical schemes for the Brinkman equations

Received September 17, 2019; Accepted October 25, 2019
Supported by the Natural National Science Foundation of China (Grant Nos. 91630201; U1530116; 11726102; 11771179; 93K172018Z01; 11701210; JJKH20180113KJ; 20190103029JH), the Program for Cheung Kong Scholars of Ministry of Education of China, Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University.
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to accommodate both the Stokes and Darcy simultaneously. In this paper, we consider the Brinkman model which seeks unknown functions $\boldsymbol{u}$ and p satisfying

$$-\mu\Delta\boldsymbol{u}+\nabla p+\mu\kappa^{-1}\boldsymbol{u}=\boldsymbol{f},\quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u}=0,\quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}=\boldsymbol{g},\quad \text{on } \partial\Omega, \tag{1.3}$$

where μ is the fluid viscosity and κ denotes the permeability tensor of the porous media which occupies a polygonal or polyhedral domain Ω in $\mathbb{R}^d$ $(d=2,3)$. $\boldsymbol{u}$ and p represent the velocity and pressure of the fluid, respectively. $\boldsymbol{f}$ is the momentum source term. $\boldsymbol{g}$ is the Dirichlet boundary condition satisfying $\int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{n}=0$. For simplicity, we consider $\boldsymbol{g}=\boldsymbol{0}$ and $\mu=1$ (note that one can always scale the solution with μ). We further assume that there exist two positive numbers λ_1, λ_2 such that

$$\lambda_1\xi^t\xi\le\xi^t\kappa^{-1}\xi\le\lambda_2\xi^t\xi,\quad \forall\xi\in\mathbb{R}^d,\quad d=2,3,$$

where ξ is a column vector with the transpose ξ^t. We consider the case where λ_1 is of unit size and λ_2 is possibly of large size.

Mathematically, the Brinkman equations (1.1)–(1.3) are the combination of the Darcy and the Stokes equations. The challenge is how to construct the compatible schemes for both the Darcy problems and the Stokes problems. In [5], a traditional $H(\text{div})$ conforming mixed finite element method was introduced. Unfortunately, the numerical experiments indicate that the convergent rate deteriorates as the Brinkman equations become Darcy-dominated when certain stable Stokes elements are used, such as the P_2-P_0 element, the Mini element and the Taylor-Hood element. Similarly, the convergent rate deteriorates as Brinkman equations become Stokes-dominated when certain stable Darcy elements, such as the Raviart-Thomas element are used. Then [4] developed a modified $H(\text{div})$ conforming mixed finite element method, which is uniformly stable with respect to the coeffients.

Furthermore, [6] introduced a new family of robust non-conforming elements for the Brinkman equations (1.1)–(1.3), such that the corresponding finite element methods are robust and strongly mass-conservative. Wang and Ye [7] proposed a stable and consistent method-weak Galerkin (WG) finite element method for the Brinkman equations (1.1)–(1.3) based on the following variational formulation: find $(\boldsymbol{u};p)\in[H_0^1(\Omega)]^d\times L_0^2(\Omega)$ satisfying

$$(\nabla\boldsymbol{u},\nabla\boldsymbol{v})+(\kappa^{-1}\boldsymbol{u},\boldsymbol{v})-(\nabla\cdot\boldsymbol{v},p)=(\boldsymbol{f},\boldsymbol{v}),\quad \forall\boldsymbol{v}\in[H_0^1(\Omega)]^d, \tag{1.4}$$

$$(\nabla\cdot\boldsymbol{u},q)=0,\quad \forall q\in L_0^2(\Omega). \tag{1.5}$$

More recently, [8] applied the WG finite element method to solve the Brinkman problems (1.1)–(1.3) based on the variational formulation as follows: find $(\boldsymbol{u};p)\in[H_0^1(\Omega)]^d\times H^1(\Omega)\cap L_0^2(\Omega)$ such that

$$(\nabla\boldsymbol{u},\nabla\boldsymbol{v})+(\kappa^{-1}\boldsymbol{u},\boldsymbol{v})+(\nabla p,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad \forall\boldsymbol{v}\in[H_0^1(\Omega)]^d, \tag{1.6}$$

$$(\nabla q,\boldsymbol{u})=0,\quad \forall q\in H^1(\Omega)\cap L_0^2(\Omega), \tag{1.7}$$

which is more accurate approximating the Darcy equations and more suitable for the complex porous media with interface problems compared with the variational formulation (1.4) and (1.5). In addition, a proof of the existence and uniqueness of the solutions for (1.6) and (1.7) can be found in [9].

The WG finite element method developed in [7,8] uses the new-defined discrete weak differential operators to replace the traditional differential operators, and has many nice features. Firstly, the WG method allows the use of the discontinuous piecewise polynomials on arbitrary shape of polygons in $2D$ or polyhedra in $3D$ with certain shape regularity. Secondly, the well-posedness of the corresponding discrete system is independent of the parameters κ and μ. Thirdly, the numerical solutions maintain mass conservation in the system. The optimal orders of convergence can also be derived for the WG finite element method in different norms. The WG finite element method was firstly proposed in [10] for the second order elliptic problem [10–13], and further applied for other partial differential equations, such as the Stokes equations [14, 15], the Sobolev equations [16, 17], the Brinkman equations [18, 19], etc.

The central idea of the WG method is to use the discrete weak gradient and divergence operators instead of classical differential operators. The WG finite element formulation for the Brinkman equations in [8] is derived from the variational formulation (1.6) and (1.7) as follows: find weak functions $\boldsymbol{u}_h = \{\boldsymbol{u}_0, \boldsymbol{u}_b\} \in V_h^0$ and $p_h = \{p_0, p_b\} \in W_h^0$ satisfying

$$(\nabla_w \boldsymbol{u}_h, \nabla_w \boldsymbol{v}) + (\kappa^{-1}\boldsymbol{u}_h, \boldsymbol{v}) + (\boldsymbol{v}, \nabla_w p) + s_1(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V_h^0, \tag{1.8}$$

$$(\boldsymbol{u}_h, \nabla_w q) - s_2(p_h, q) = 0, \quad \forall q \in W_h^0, \tag{1.9}$$

where V_h^0 and W_h^0 are the corresponding weak Galerkin finite element spaces. The WG finite element method has many advantages, such as stability, high order accuracy and high flexibility. The main problem of the WG finite element method is the calculation cost. The WG method introduces additional freedoms on each edge/face in the partition, so the degree of freedom is much more than the classical finite element method. To this end, [20, 21] introduced a modified weak Galerkin (MWG) finite element method for the Stokes equations, and proved the stability and optimal order error estimates for the MWG method.

The main idea of the MWG finite element method is to use the average $\{\boldsymbol{u}_0\}$ inside of the element to replace $\boldsymbol{u}_b$ on the boundary of the element. Therefore, the MWG method not only maintains the flexibility on the selection of approximation functions, but also eliminates the unknowns associated with element boundaries. In this paper, we use the MWG finite element method based on the variational formulation (1.6) and (1.7) for solving the Brinkman problems (1.1)–(1.3). The weak Galerkin finite element spaces consist of discontinuous piecewise polynomials of degree k and $k-1$ for the vector-valued function $\boldsymbol{u}$ and the scalar function p, respectively. The convergence of the MWG finite element method is obtained in both the theoretical analysis and the numerical experiments. In addition, the numerical experiments illustrate that the MWG method is efficient and robust both for the constant and variable permeability tensor κ.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we propose the modified weak Galerkin finite element scheme and give a proof of the existence and uniqueness of the solutions. In Section 4, we derive the error equation, and further establish the optimal order error estimates in H^1 norm and L^2 norm in Section 5. Finally, we present some numerical examples in Section 6 to demonstrate the accuracy, convergence and flexibility of the MWG finite element method.

2. Preliminaries

In this section, some preparations for the modified weak Galerkin finite element method are given.

2.1. Space and partition

We first introduce the standard definitions for the Sobolev space $H^s(\Omega)(s \geq 0)$, where Ω is a closed convex polygon in $\mathbb{R}^2$ or polyhedron or $\mathbb{R}^3$ with the Lipschitz continuous boundary. $(\cdot,\cdot)_{s,\Omega}$ and $\|\cdot\|_{s,\Omega}$ denote the inner products and norms of the Sobolev space $H^s(\Omega)$, respectively. The space $H^0(\Omega)$ coincides with the space $L^2(\Omega)$. In this case we drop the subscript s of above notations without confusion. In particular, we have the following function spaces

$$[H_0^1(\Omega)]^d = \{\boldsymbol{v} \in [H^1(\Omega)]^d,\ \boldsymbol{v}|_{\partial\Omega} = 0\},$$

and

$$L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q\mathrm{d}x = 0\}.$$

The space $H(\mathrm{div};\Omega)$ is defined as the set of vector-valued functions $\boldsymbol{v}$, which together with their divergence are square integrable, i.e.,

$$H(\mathrm{div},\Omega) = \{\boldsymbol{v} : \boldsymbol{v} \in [L^2(\Omega)]^d, \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\}.$$

The norm in $H(\mathrm{div},\Omega)$ is given by

$$\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)} = (\|\boldsymbol{v}\|_\Omega^2 + \|\nabla\cdot\boldsymbol{v}\|_\Omega^2)^{\frac{1}{2}}.$$

Let $\mathcal{T}_h$ be a partition of the domain Ω satisfying the shape regular conditions [22], and T be each element with ∂T as its boundary. Denote by $\mathbb{E}_h$ the set of all edges/flats in $\mathcal{T}_h$, and $\mathbb{E}_h^0 = \mathbb{E}_h \setminus \partial\Omega$ the set of all interior edges/faces in $\mathcal{T}_h$. For each $T \in \mathcal{T}_h$, denote by h_T the diameter of T, and $h = \max_{T\in\mathcal{T}_h} h_T$ is the mesh size of $\mathcal{T}_h$. Similarly, the diameter of e is given by h_e.

2.2. Average and jump

We define the average $\{\cdot\}$ and jump $[\![\cdot]\!]$ on edges for the scalar function q, the vector-valued function $\boldsymbol{v}$ and the matrix-valued function τ, respectively.

For the interior edge e, define

$$\{q\} = \frac{1}{2}(q|_{T_1} + q|_{T_2}),\quad [\![q]\!] = q|_{T_1}\boldsymbol{n}_1 + q|_{T_2}\boldsymbol{n}_2,$$
$$\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}|_{T_1} + \boldsymbol{v}|_{T_2}),\quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_{T_1}\cdot\boldsymbol{n}_1 + \boldsymbol{v}|_{T_2}\cdot\boldsymbol{n}_2,$$
$$\{\tau\} = \frac{1}{2}(\tau|_{T_1} + \tau|_{T_2}),\quad [\![\tau]\!] = \boldsymbol{n}_1\cdot\tau|_{T_1} + \boldsymbol{n}_2\cdot\tau|_{T_2},$$

where T_1 and T_2 are two partition elements sharing a common edge e. $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the unit outward normal vectors on e, associated with T_1 and T_2, respectively.

We also define a matrix-valued jump $[\cdot]$ for the vector-valued function $\boldsymbol{v}$ by

$$[\boldsymbol{v}] = \boldsymbol{v}|_{T_1} \otimes \boldsymbol{n}_1 + \boldsymbol{v}|_{T_2} \otimes \boldsymbol{n}_2,$$

where $\otimes$ denotes the tensors product of the two vectors.

If e is a boundary edge, the above definitions need to be adjusted accordingly so that both the average and the jump are equal to the one-sided values on e. That is,

$$\begin{aligned} &\{q\} = q|_e, \quad [\![q]\!] = q|_e \boldsymbol{n}, \quad \{\boldsymbol{v}\} = \boldsymbol{v}|_e, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_e \cdot \boldsymbol{n}, \\ &\{\tau\} = \tau|_e, \quad [\![\tau]\!] = \boldsymbol{n} \cdot \tau|_e, \quad [\boldsymbol{v}] = \boldsymbol{v}|_e \otimes \boldsymbol{n}, \end{aligned}$$

where $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$.

It is obvious that we can get the following properties.

Lemma 2.1 ([20]) *For any smooth scalar function q, vector-valued function $\boldsymbol{v}$, and matrix-valued function τ, there holds*

$$\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v}\cdot\boldsymbol{n}, q\rangle_{\partial T} = \sum_{e\in\mathbb{E}_h^0} \langle \{\boldsymbol{v}\}, [\![q]\!]\rangle_e + \sum_{e\in\mathbb{E}_h} \langle [\![\boldsymbol{v}]\!], \{q\}\rangle_e,$$

$$\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v}, \tau\cdot\boldsymbol{n}\rangle_{\partial T} = \sum_{e\in\mathbb{E}_h^0} \langle \{\boldsymbol{v}\}, [\![\tau]\!]\rangle_e + \sum_{e\in\mathbb{E}_h} \langle [\boldsymbol{v}], \{\tau\}\rangle_e,$$

and then

$$\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v}\cdot\boldsymbol{n}, q-\{q\}\rangle_{\partial T} = \sum_{e\in\mathbb{E}_h^0} \langle \{\boldsymbol{v}\}, [\![q]\!]\rangle_e, \tag{2.1}$$

$$\sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v}-\{\boldsymbol{v}\}, \tau\cdot\boldsymbol{n}\rangle_{\partial T} = \sum_{e\in\mathbb{E}_h^0} \langle [\boldsymbol{v}], \{\tau\}\rangle_e, \tag{2.2}$$

where $\boldsymbol{n}$ is unit and normal to the edge e.

We then introduce the corresponding weak Galerkin finite element spaces for the scalar function and the vector-valued function on the partition $\mathcal{T}_h$. In the MWG method, the weak function of scalar function q has the form $q = \{q, \{q\}\}$, and can be denoted by q without generating any confusions. The corresponding weak function space is

$$W_h = \{q \in L^2(\Omega) : q|_T \in P_{k-1}(T),\ \forall\, T \in \mathcal{T}_h\}.$$

Denote by W_h^0 the subspace of W_h satisfying

$$W_h^0 = \Big\{q \in W_h : \int_\Omega q \mathrm{d}x = 0\Big\}.$$

Following the Definition 2.1 in [8], we define the discrete weak gradient for the scalar function $q \in W_h$ as follows.

Definition 2.2 *For any $q \in W_h$, denote the discrete weak gradient of q by $\tilde{\nabla}_{w,k,T} q$, which is determined by*

$$(\tilde{\nabla}_{w,k} q, \varphi)_T := -(q, \nabla\cdot\varphi)_T + \langle \{q\}, \varphi\cdot\boldsymbol{n}\rangle_{\partial T}, \quad \forall\, \varphi \in [P_k(T)]^d,\ \forall T \in \mathcal{T}_h,$$

where $\boldsymbol{n}$ denotes the unit outward normal on ∂T, $[P_k(T)]^d$ is the set of vector-valued polynomials with degree no more than k on T.

Similarly, the weak function of the velocity function $\boldsymbol{v}$ is $\boldsymbol{v} = \{\boldsymbol{v}, \{\boldsymbol{v}\}\}$ denoted also by $\boldsymbol{v}$ without any confusions. We define the weak function space by

$$V_h = \{\boldsymbol{v} \in [L^2(\Omega)]^d : \ \boldsymbol{v}|_T \in [P_k(T)]^d, \ \ \forall\, T \in \mathcal{T}_h\},$$

and the subspace of V_h by V_h^0, which is

$$V_h^0 = \{\boldsymbol{v} \in V_h : \ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\}.$$

Based on the Definition 2.1 in the paper [8], the discrete weak gradient for the vector-valued function $\boldsymbol{v} \in V_h$ is defined as follows.

Definition 2.3 *For any $\boldsymbol{v} \in V_h$, denote the discrete weak gradient of $\boldsymbol{v}$ by $\nabla_{w,k-1}\boldsymbol{v}$, which is determined by*

$$(\nabla_{w,k-1,T}\boldsymbol{v}, \tau)_T := -(\boldsymbol{v}, \nabla\cdot\tau)_T + \langle\{\boldsymbol{v}\}, \tau\cdot\boldsymbol{n}\rangle_{\partial T}, \ \ \forall \tau \in [P_{k-1}(T)]^{d\times d}, \ \forall T \in \mathcal{T}_h.$$

For simplicity, we will drop the subscripts $k-1$, k and T in the following, and use $\tilde{\nabla}_w$ and ∇_w to denote $\tilde{\nabla}_{w,k,T}$ and $\nabla_{w,k-1,T}$, respectively.

For any $\boldsymbol{v}, \boldsymbol{w} \in V_h$ and $p, q \in W_h$, we define

$$(\nabla_w\boldsymbol{v}, \nabla_w\boldsymbol{w}) = \sum_{T\in\mathcal{T}_h} (\nabla_w\boldsymbol{v}, \nabla_w\boldsymbol{w})_T,$$

and

$$(\boldsymbol{v}, \tilde{\nabla}_w q) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{v}, \tilde{\nabla}_w q)_T.$$

Then, four bilinear forms are defined as follows

$$\begin{aligned}
s_1(\boldsymbol{v}, \boldsymbol{w}) &= \sum_{e\in\mathbb{E}_h} h^{-1}\langle[\boldsymbol{v}], [\boldsymbol{w}]\rangle_e,\\
a(\boldsymbol{v}, \boldsymbol{w}) &= (\nabla_w\boldsymbol{v}, \nabla_w\boldsymbol{w}) + (\kappa^{-1}\boldsymbol{v}, \boldsymbol{w}) + s_1(\boldsymbol{v}, \boldsymbol{w}),\\
b(\boldsymbol{v}, q) &= (\boldsymbol{v}, \tilde{\nabla}_w q),\\
s_2(q, r) &= \sum_{e\in\mathbb{E}_h} h\langle[\![p]\!], [\![r]\!]\rangle_e.
\end{aligned}$$

For convenience, we finally define the norms for any $\boldsymbol{v} \in V_h^0$ and $q \in W_h^0$, and the semi-norm for $q \in W_h^0$ as follows

$$\begin{aligned}
|\!|\!|\boldsymbol{v}|\!|\!|^2 &= a(\boldsymbol{v}, \boldsymbol{v}) = \|\kappa^{-\frac{1}{2}}\boldsymbol{v}\|^2 + \|\nabla_w\boldsymbol{v}\|^2 + \sum_{e\in\mathbb{E}_h} h^{-1}\|[\boldsymbol{v}]\|_e^2,\\
|\!|\!|q|\!|\!|_1^2 &= \|\kappa^{\frac{1}{2}}\tilde{\nabla}_w q\|^2 + \sum_{e\in\mathbb{E}_h} h^{-1}\|[\![q]\!]\|_e^2,\\
|\!|\!|q|\!|\!|_h^2 &= s_2(q, q) = \sum_{e\in\mathbb{E}_h} h\|[\![q]\!]\|_e^2.
\end{aligned}$$

It is easy to verify that the definitions of the norms and the semi-norm above are reasonable.

For simplicity of analysis, we assume that the permeability parameter κ is a piecewise constant on each element $T \in \mathcal{T}_h$ in the following sections, that is, κ is a piecewise constant on Ω. The results can be easily extended to the case of piecewise smooth parameter κ.

3. Modified weak Galerkin finite element method

In this section, we introduce modified weak Galerkin finite element algorithms to solve the problems (1.1)–(1.3) and demonstrate the existence and uniqueness of the solution.

Algorithm 1. Find $(\boldsymbol{u}_h; q_h) \in V_h^0 \times W_h^0$, such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}), \tag{3.1}$$

$$b(\boldsymbol{u}_h, q) - s_2(p_h, q) = 0, \tag{3.2}$$

for any $(\boldsymbol{v}\ ; q) \in V_h^0 \times W_h^0$.

Theorem 3.1 *The schemes (3.1) and (3.2) have a unique solution.*

Proof Because (3.1) and (3.2) is a linear system, we just need to verify the uniqueness of the homogeneous problems, that is, prove that $\boldsymbol{u}_h = \boldsymbol{0}$ and $p_h = 0$ when $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$.

Taking $\boldsymbol{v} = \boldsymbol{u}_h$ and $q = p_h$ in (3.1), (3.2) and subtracting (3.2) from (3.1), with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$, we obtain

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) + s_2(p_h, p_h) = 0.$$

Note that both $a(\boldsymbol{u}_h, \boldsymbol{u}_h)$ and $s_2(p_h, p_h)$ are non-negative, then from the definitions of $|\!|\!| \cdot |\!|\!|$ and $\| \cdot \|_h$, we have

$$|\!|\!| \boldsymbol{u}_h |\!|\!|^2 + \|p_h\|_h^2 = 0,$$

which implies that $|\!|\!| \boldsymbol{u}_h |\!|\!| = 0$ and $\|p_h\|_h = 0$. Combining with the boundary conditions, we obtain $\boldsymbol{u}_h = \boldsymbol{0}$ and $[\![p_h]\!] = 0$. To show $p_h = 0$, we use the fact that $\boldsymbol{u}_h = \boldsymbol{0}$ and $\boldsymbol{f} = \boldsymbol{0}$, and obtain $b(\boldsymbol{v}, p_h) = 0$ for any $\boldsymbol{v} \in V_h^0$. From the definitions of $b(\cdot, \cdot)$, $\tilde{\nabla}_w$ and $[\![p_h]\!]$, and Lemma 2.1, we derive that

$$\begin{aligned} 0 = b(\boldsymbol{v}, p_h) = (\boldsymbol{v}, \tilde{\nabla}_w p_h) &= \sum_{T \in \mathcal{T}_h} -(p_h, \nabla \cdot \boldsymbol{v})_T + \sum_{T \in \mathcal{T}_h} \langle \{p_h\}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (\nabla p_h, \boldsymbol{v})_T - \sum_{T \in \mathcal{T}_h} \langle p_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} \langle \{p_h\}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (\nabla p_h, \boldsymbol{v})_T - \sum_{e \in \mathbb{E}_h^0} \langle \{\boldsymbol{v}\}, [\![p_h]\!] \rangle_e = \sum_{T \in \mathcal{T}_h} (\nabla p_h, \boldsymbol{v})_T. \end{aligned}$$

It follows that $\nabla p_h = 0$ for all $T \in \mathcal{T}_h$. p_h is continuous and $p_h \in W_h \subset L_0^2(\Omega)$, thus $p_h = 0$, which completes the proof. $\Box$

4. Error equation

In this section, we establish and prove the error equations for the MWG finite element schemes.

4.1. Projection operator

First, we define several projection operators. For the vector-valued function $\boldsymbol{u}$, the two L^2 projection operators are denoted by

$$Q_h\boldsymbol{u} = \{Q_0\boldsymbol{u}, Q_b\boldsymbol{u}\}, \quad \hat{Q}_h\boldsymbol{u} = \{\hat{Q}_0\boldsymbol{u}, \hat{Q}_b\boldsymbol{u}\},$$

where Q_0 denotes the L^2 projection from $[L_2(T)]^d$ onto $[P_k(T)]^d$ on each element $T \in \mathcal{T}_h$ and Q_b denotes the L^2 projection from $[L_2(e)]^d$ onto $[P_k(e)]^d$ on each edge/face $e \in \mathbb{E}_h$. We have the fact that $\hat{Q}_0\boldsymbol{u} = Q_0\boldsymbol{u}$ on each element T, $\hat{Q}_b\boldsymbol{u} = \{Q_0\boldsymbol{u}\}$ on each interior edge $e \in \mathbb{E}_h^0$, $\hat{Q}_b\boldsymbol{u} = Q_b\boldsymbol{g}$ on the outer boundaries $e \in \partial\Omega$.

Similarly, for the scalar function p, define

$$\mathbb{Q}_hp = \{\mathbb{Q}_0p, \mathbb{Q}_bp\}, \quad \hat{\mathbb{Q}}_hp = \{\hat{\mathbb{Q}}_0p, \hat{\mathbb{Q}}_bp\},$$

where $\mathbb{Q}_0$ denotes the L^2 projection from $L_2(T)$ onto $P_{k-1}(T)$ on each element $T \in \mathcal{T}_h$ and $\mathbb{Q}_b$ denotes L^2 projection from $L_2(e)$ onto $P_{k-1}(e)$ on each edge/face $e \in \mathbb{E}_h$. We have the fact that $\hat{\mathbb{Q}}_0p = \mathbb{Q}_0p$ on each element T and $\hat{\mathbb{Q}}_bp = \{\mathbb{Q}_0p\}$ on each edge $e \in \mathbb{E}_h$.

In addition, we define $\mathbf{Q}_h$ the matrix local L^2 projection operator from $[L^2(T)]^{d\times d}$ onto polynomials space $[P_{k-1}(T)]^{d\times d}$.

It is easy to verify that above projection operators have the following features.

Lemma 4.1 ([8]) *For any $\boldsymbol{v} \in H(\mathrm{div}, \Omega)$, $q \in H^1(\Omega)$, and $\boldsymbol{w} \in [P_k(T)]^d$, we have*

$$\nabla_w(Q_h\boldsymbol{v}) = \mathbf{Q}_h(\nabla\boldsymbol{v}), \tag{4.1}$$

$$(\tilde{\nabla}_w(\mathbb{Q}_hq), \boldsymbol{w}) = (Q_h(\nabla q), \boldsymbol{w}) - \sum_{T\in\mathcal{T}_h} \langle q - \mathbb{Q}_bq, \boldsymbol{w}\cdot\boldsymbol{n}\rangle_{\partial T}. \tag{4.2}$$

4.2. Error equation

Let $\boldsymbol{e}_h = \hat{Q}_h\boldsymbol{u} - \boldsymbol{u}_h$ and $\epsilon_h = \hat{\mathbb{Q}}_hp - p_h$ denote the errors between the projection of the exact solution and the MWG numerical solution for the velocity and the pressure function, respectively. we establish the following error equations.

Theorem 4.2 *Let $(\boldsymbol{u}_h; p_h) \in V_h^0 \times W_h^0$ be the numerical solution of the variational problems (3.1), (3.2) and $(\boldsymbol{u}; p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ be the exact solution of the Brinkman problems (1.1)–(1.3). Then, we have*

$$a(\boldsymbol{e}_h, \boldsymbol{v}) + b(\boldsymbol{v}, \epsilon_h) = \varphi_{\boldsymbol{u},p}(\boldsymbol{v}), \tag{4.3}$$

$$b(\boldsymbol{e}_h, q) - s_2(\epsilon_h, q) = \phi_{\boldsymbol{u},p}(q), \tag{4.4}$$

for any $(\mathbf{v}, q) \in V_h^0 \times W_h^0$, where

$$\begin{aligned}
\varphi_{\boldsymbol{u},p}(\boldsymbol{v}) &= l_1(\boldsymbol{v}, \boldsymbol{u}) - l_2(\boldsymbol{v}, p) + l_3(\boldsymbol{v}, \boldsymbol{u}) + l_4(\boldsymbol{v}, p) + s_1(\hat{Q}_h\boldsymbol{u}, \boldsymbol{v}),\\
\phi_{\boldsymbol{u},p}(q) &= l_5(\boldsymbol{u}, q) - s_2(\hat{\mathbb{Q}}_hp, q),\\
l_1(\boldsymbol{v}, \boldsymbol{u}) &= \sum_{T\in\mathcal{T}_h} \langle \boldsymbol{v} - \{\boldsymbol{v}\}, (\nabla\boldsymbol{u} - \mathbf{Q}_h(\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\rangle_{\partial T},
\end{aligned}$$

$$l_2(\boldsymbol{v},p)=\sum_{T\in\mathcal{T}_h}\langle(\boldsymbol{v}-\{\boldsymbol{v}\})\cdot\boldsymbol{n},p-\mathbb{Q}_bp\rangle_{\partial T},$$
$$l_3(\boldsymbol{v},\boldsymbol{u})=\sum_{T\in\mathcal{T}_h}(\nabla_w(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})_T,$$
$$l_4(\boldsymbol{v},p)=\sum_{T\in\mathcal{T}_h}(\boldsymbol{v},\tilde{\nabla}_w(\hat{\mathbb{Q}}_hp-\mathbb{Q}_hp))_T,$$
$$l_5(\boldsymbol{u},q)=\sum_{T\in\mathcal{T}_h}\langle q-\{q\},(\boldsymbol{u}-Q_h\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}.$$

Proof Using (4.1), the definition of ∇_w, the integration by parts, the definition of projection operator, and $\boldsymbol{v}|_{\partial\Omega}=0$, we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})_T&=\sum_{T\in\mathcal{T}_h}(\mathbf{Q_h}(\nabla\boldsymbol{u}),\nabla_w\boldsymbol{v})_T\\&=\sum_{T\in\mathcal{T}_h}-(\boldsymbol{v},\nabla\cdot\mathbf{Q_h}(\nabla\boldsymbol{u}))_T+\sum_{T\in\mathcal{T}_h}\langle\{\boldsymbol{v}\},\mathbf{Q_h}(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\&=\sum_{T\in\mathcal{T}_h}(\nabla\boldsymbol{v},\mathbf{Q_h}(\nabla\boldsymbol{u}))_T-\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}-\{\boldsymbol{v}\},\mathbf{Q_h}(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\&=\sum_{T\in\mathcal{T}_h}-(\boldsymbol{v},\nabla\cdot(\nabla\boldsymbol{u}))_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v},\nabla\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T}-\\&\quad\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}-\{\boldsymbol{v}\},\mathbf{Q_h}(\nabla\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\\&=\sum_{T\in\mathcal{T}_h}-(\Delta\boldsymbol{u},\boldsymbol{v})_T+\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}-\{\boldsymbol{v}\},(\nabla\boldsymbol{u}-\mathbf{Q_h}(\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\rangle_{\partial T},\end{aligned}$$

which implies

$$(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})=-(\Delta\boldsymbol{u},\boldsymbol{v})+l_1(\boldsymbol{v},\boldsymbol{u}).\tag{4.5}$$

Additionally, using the property (4.2), the definition of Q_h, and the fact $\boldsymbol{v}|_{\partial\Omega}=0$, we arrive at

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}(\boldsymbol{v},\tilde{\nabla}_w(\mathbb{Q}_hp))_T&=\sum_{T\in\mathcal{T}_h}(Q_h(\nabla p),\boldsymbol{v})_T-\sum_{T\in\mathcal{T}_h}\langle p-\mathbb{Q}_bp,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T}\\&=\sum_{T\in\mathcal{T}_h}(\nabla p,\boldsymbol{v})_T-\sum_{T\in\mathcal{T}_h}\langle p-\mathbb{Q}_bp,(\boldsymbol{v}-\{\boldsymbol{v}\})\cdot\boldsymbol{n}\rangle_{\partial T}.\end{aligned}$$

That is

$$(\boldsymbol{v},\tilde{\nabla}_w(\mathbb{Q}_hp))=(\nabla p,\boldsymbol{v})-l_2(\boldsymbol{v},p).\tag{4.6}$$

Furthermore, testing the equation (1.1) by $\boldsymbol{v}\in V_h^0$, we have

$$-(\Delta\boldsymbol{u},\boldsymbol{v})+(\nabla p,\boldsymbol{v})+(\kappa^{-1}\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}).$$

Substituting (4.5) and (4.6) to both sides of above equation, we obtain

$$(\nabla_w(Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})-l_1(\boldsymbol{v},\boldsymbol{u})+(\boldsymbol{v},\tilde{\nabla}_w(\mathbb{Q}_hp))+l_2(\boldsymbol{v},p)+(\kappa^{-1}\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}).\tag{4.7}$$

Adding $(\nabla_w(\hat{Q}_h\boldsymbol{u}),\nabla_w\boldsymbol{v})$, $s_1(\hat{Q}_h\boldsymbol{u},\boldsymbol{v})$, and $(\boldsymbol{v},\tilde{\nabla}_w(\hat{\mathbb{Q}}_hp))$ to both sides of the equation (4.7) yields

$$(\nabla_w(\hat{Q}_h\boldsymbol{u}),\nabla_w\boldsymbol{v})+(\kappa^{-1}\boldsymbol{u},\boldsymbol{v})+s_1(\hat{Q}_h\boldsymbol{u},\boldsymbol{v})+(\boldsymbol{v},\tilde{\nabla}_w(\hat{\mathbb{Q}}_hp))$$

$$= l_1(\boldsymbol{v},\boldsymbol{u}) - l_2(\boldsymbol{v},p) + \sum_{T\in\mathcal{T}_h} (\nabla_w(\hat{Q}_h\boldsymbol{u} - Q_h\boldsymbol{u}), \nabla_w\boldsymbol{v})_T +$$
$$\sum_{T\in\mathcal{T}_h} (\boldsymbol{v}, \tilde{\nabla}_w(\hat{\mathbb{Q}}_h p - \mathbb{Q}_h p))_T + (\boldsymbol{f},\boldsymbol{v}) + s_1(\hat{Q}_h\boldsymbol{u},\boldsymbol{v}).$$

It follows from $(\kappa^{-1}\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\kappa^{-1}\boldsymbol{v}) = (\hat{Q}_h\boldsymbol{u},\kappa^{-1}\boldsymbol{v}) = (\kappa^{-1}\hat{Q}_h\boldsymbol{u},\boldsymbol{v})$ that

$$a(\nabla_w(\hat{Q}_h\boldsymbol{u}),\boldsymbol{v}) + b(\boldsymbol{v},\hat{\mathbb{Q}}_h p) = \varphi_{\boldsymbol{u},p}(\boldsymbol{v}) + (\boldsymbol{f},\boldsymbol{v}). \tag{4.8}$$

Combining with (3.1), we obtain the equation (4.3).

In addition, from the definition of $\tilde{\nabla}_w$, integration by parts, the definition of projection operator, and the fact $\boldsymbol{u}|_{\partial\Omega} = 0$, we arrive at

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} (Q_h\boldsymbol{u},\tilde{\nabla}_w q)_T &= \sum_{T\in\mathcal{T}_h} (-(q,\nabla\cdot(Q_h\boldsymbol{u}))_T + \langle\{q\}, Q_h\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T})\\ &= \sum_{T\in\mathcal{T}_h} (\nabla q, Q_h\boldsymbol{u})_T - \sum_{T\in\mathcal{T}_h} \langle q-\{q\}, Q_h\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= \sum_{T\in\mathcal{T}_h} -(q,\nabla\cdot\boldsymbol{u})_T + \langle q,\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle q-\{q\}, Q_h\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial T}\\ &= \sum_{T\in\mathcal{T}_h} -(q,\nabla\cdot\boldsymbol{u})_T + \sum_{T\in\mathcal{T}_h} \langle q-\{q\}, (\boldsymbol{u}-Q_h\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T},\end{aligned}$$

then

$$(Q_h\boldsymbol{u},\tilde{\nabla}_w q) - \sum_{T\in\mathcal{T}_h} \langle q-\{q\}, (\boldsymbol{u}-Q_h\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T} = -(\nabla\cdot\boldsymbol{u}, q).$$

Similarity, testing the equation (1.2) by $q\in W_h$, we obtain $(\nabla\cdot\boldsymbol{u},q) = 0$, and

$$(Q_h\boldsymbol{u},\tilde{\nabla}_w q) = \sum_{T\in\mathcal{T}_h} \langle q-\{q\}, (\boldsymbol{u}-Q_h\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}.$$

Adding $(\hat{Q}_h\boldsymbol{u},\tilde{\nabla}_w q) - s_2(\hat{\mathbb{Q}}_h p, q)$ to both sides of the equation above, we have

$$(\hat{Q}_h\boldsymbol{u},\tilde{\nabla}_w q) - s_2(\hat{\mathbb{Q}}_h p, q) = \sum_{T\in\mathcal{T}_h} (\hat{Q}_h\boldsymbol{u} - Q_h\boldsymbol{u},\tilde{\nabla}_w q)_T + \sum_{T\in\mathcal{T}_h} \langle q-\{q\}, (\boldsymbol{u}-Q_h\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T} -$$
$$s_2(\hat{\mathbb{Q}}_h p, q).$$

We observe that $\hat{Q}_h = Q_h$ in the interior of T, it follows that

$$b(\hat{Q}_h\boldsymbol{u}, q) - s_2(\hat{\mathbb{Q}}_h p, q) = l_5(\boldsymbol{u}, q) - s_2(\hat{\mathbb{Q}}_h p, q).$$

Combining with (3.2) makes the proof of this theorem completed. □

5. Error estimates

The error equations for the vector-valued function $\boldsymbol{u}$ and scalar function p are established in the previous section. In this section, we show the detail analysis of the error estimates in H^1 and L^2 norms for the function $\boldsymbol{u}$ and H^1 norm for the function p, respectively. The optimal convergence orders of the MWG finite element method are derived in several theorems.

5.1. Preparation for estimating

Before the error estimation, we first verify the following inf-sup condition.

Lemma 5.1 *For any $q \in W_h \subset L_0^2(\Omega)$, there exist two constants C_1 and C_2 inde-pendent of the mesh size h such that*

$$\sup_{v \in V_h} \frac{|b(\boldsymbol{v}, q)|}{|||\boldsymbol{v}|||} \geq C_1 h |||q|||_1 - C_2 \|q\|_h. \tag{5.1}$$

Proof For any $\boldsymbol{v} \in V_h^0$, it follows from the definition of $\nabla_w \boldsymbol{v}$, the integration by parts, the Cauchy-Schwarz inequality, the triangle inequality, the trace inequality, the inverse inequality, and the Young inequality that

$$\begin{aligned}
\|\nabla_w \boldsymbol{v}\|^2 &= \sum_{T \in \mathcal{T}_h} (\nabla_w \boldsymbol{v}, \nabla_w \boldsymbol{v})_T = \sum_{T \in \mathcal{T}_h} ((\nabla \boldsymbol{v}, \nabla_w \boldsymbol{v})_T - \langle \boldsymbol{v} - \{\boldsymbol{v}\}, \nabla_w \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial T}) \\
&= \sum_{T \in \mathcal{T}_h} ((\nabla \boldsymbol{v}, \nabla \boldsymbol{v})_T - \langle \boldsymbol{v} - \{\boldsymbol{v}\}, (\nabla_w \boldsymbol{v} + \nabla \boldsymbol{v}) \cdot \boldsymbol{n} \rangle_{\partial T}) \\
&\leq \sum_{T \in \mathcal{T}_h} (\|\nabla \boldsymbol{v}\|_T^2 + |\langle \boldsymbol{v} - \{\boldsymbol{v}\}, (\nabla_w \boldsymbol{v} + \nabla \boldsymbol{v}) \cdot \boldsymbol{n} \rangle_{\partial T}|) \\
&\leq \sum_{T \in \mathcal{T}_h} (\|\nabla \boldsymbol{v}\|_T^2 + \|\boldsymbol{v} - \{\boldsymbol{v}\}\|_{\partial T} \|\nabla_w \boldsymbol{v} + \nabla \boldsymbol{v}\|_{\partial T}) \\
&\leq \sum_{T \in \mathcal{T}_h} (\|\nabla \boldsymbol{v}\|_T^2 + \|\boldsymbol{v}\|_{\partial T} \|\nabla_w \boldsymbol{v} + \nabla \boldsymbol{v}\|_{\partial T}) \\
&\leq \sum_{T \in \mathcal{T}_h} (\|\nabla \boldsymbol{v}\|_T^2 + \|\boldsymbol{v}\|_{\partial T} (\|\nabla_w \boldsymbol{v}\|_{\partial T} + \|\nabla \boldsymbol{v}\|_{\partial T})) \\
&\leq \sum_{T \in \mathcal{T}_h} (\|\nabla \boldsymbol{v}\|_T^2 + h^{-1} \|\boldsymbol{v}\|_T (\|\nabla_w \boldsymbol{v}\|_T + \|\nabla \boldsymbol{v}\|_T)) \\
&\leq C h^{-2} \|\boldsymbol{v}\|^2 + \frac{1}{2} \|\nabla_w \boldsymbol{v}\|^2.
\end{aligned}$$

Replacing above equation by taking $\boldsymbol{v} = \kappa \tilde{\nabla}_w q \in V_h^0$, we have

$$\|\nabla_w \boldsymbol{v}\|^2 \leq C h^{-2} \|\boldsymbol{v}\|^2 = C h^{-2} \|\kappa \tilde{\nabla}_w q\|^2 \leq C h^{-2} \|\kappa^{\frac{1}{2}} \tilde{\nabla}_w q\|^2 \leq C h^{-2} |||q|||_1^2. \tag{5.2}$$

For the same $\boldsymbol{v}$, we obtain

$$\|\kappa^{-\frac{1}{2}} \boldsymbol{v}\|^2 = \|\kappa^{\frac{1}{2}} \tilde{\nabla}_w q\|^2 \leq C |||q|||_1^2. \tag{5.3}$$

From the definition of $[\boldsymbol{v}]$, the triangle inequality, the trace inequality, and the inverse inequality, we have

$$\begin{aligned}
\sum_{e \in \mathbb{E}_h} h^{-1} \|[\boldsymbol{v}]\|_e^2 &= \sum_{e \in \mathbb{E}_h} h^{-1} \|\boldsymbol{v}|_{T_1} \otimes \boldsymbol{n}_1 + \boldsymbol{v}|_{T_2} \otimes \boldsymbol{n}_2\|_e^2 \\
&\leq C \sum_{e \in \mathbb{E}_h} h^{-1} \|\boldsymbol{v}\|_e^2 \leq C h^{-2} \|\boldsymbol{v}\|^2 \leq C h^{-2} |||q|||_1^2.
\end{aligned} \tag{5.4}$$

Combining (5.2)–(5.4) yields

$$|||\boldsymbol{v}||| \leq C h^{-1} |||q|||_1. \tag{5.5}$$

In addition, using the definition of $b(\boldsymbol{v}, q)$, $|||q|||_1$, and $\|q\|_h$ with above function $\boldsymbol{v}$, we get

$$b(\boldsymbol{v}, q) = (\boldsymbol{v}, \tilde{\nabla}_w q) = \|\kappa^{\frac{1}{2}} \tilde{\nabla}_w q\|^2 = |||q|||_1^2 - \sum_{e \in \mathbb{E}_h} h^{-1} \|[\![q]\!]\|_e^2 \geq |||q|||_1^2 - C h^{-1} \|q\|_h |||q|||_1.$$

Therefore, we have

$$\sup_{v\in V_h}\frac{|b(\boldsymbol{v},q)|}{\|\!|\boldsymbol{v}|\!\|}\geq\frac{\|\!|q|\!\|_1^2-Ch^{-1}\|q\|_h\|\!|q|\!\|_1}{Ch^{-1}\|\!|q|\!\|_1}\geq C_1h\|\!|q|\!\|_1-C_2\|q\|_h.$$

Thus, we complete the proof of the inf-sup condition. □

Theorem 5.2 *Let $(\boldsymbol{u};p)\in[H^2(\Omega)\cap H^{k+1}(\Omega)]^d\times L_0^2(\Omega)\cap H^k(\Omega)$ with $k\geq 1$ be the solution of the Brinkman problems (1.1)–(1.3). Then for any $(\boldsymbol{v};q)\in V_h^0\times W_h^0$, there is a constant C independent of h such that*

$$|\varphi_{\boldsymbol{u},p}(\boldsymbol{v})|\leq Ch^k(\|\boldsymbol{u}\|_{k+1}+\|p\|_k)\|\!|\boldsymbol{v}|\!\|, \tag{5.6}$$

$$|\phi_{\boldsymbol{u},p}(q)|\leq Ch^k(\|\boldsymbol{u}\|_{k+1}+\|p\|_k)\|q\|_h. \tag{5.7}$$

Proof It follows from the triangle inequality that

$$\begin{aligned}|\varphi_{\boldsymbol{u},p}(\boldsymbol{v})|&\leq|l_1(\boldsymbol{v},\boldsymbol{u})|+|l_2(\boldsymbol{v},p)|+|l_3(\boldsymbol{v},\boldsymbol{u})|+|l_4(\boldsymbol{v},p)|+|s_1(\hat{Q}_h\boldsymbol{u},\boldsymbol{v})|,\\ |\phi_{\boldsymbol{u},p}(q)|&\leq|l_5(\boldsymbol{u},q)|+|s_2(\hat{\mathbb{Q}}_hp,q)|.\end{aligned}$$

The terms on the right-hand side of the above inequalities are estimated one by one. Using the equation (2.2), the Cauchy-Schwarz inequality, the definition of $\|\!|\cdot|\!\|$, the trace inequality, and the projection inequality, we have

$$\begin{aligned}|l_1(\boldsymbol{v},\boldsymbol{u})|&=\Big|\sum_{T\in\mathcal{T}_h}\langle\boldsymbol{v}-\{\boldsymbol{v}\},(\nabla\boldsymbol{u}-\mathbf{Q}_h(\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\rangle_{\partial T}\Big|\\ &=\Big|\sum_{e\in\mathbb{E}_h}\langle[\boldsymbol{v}],\{\nabla\boldsymbol{u}-\mathbf{Q}_h(\nabla\boldsymbol{u})\}\rangle_e\Big|\\ &\leq C\Big(\sum_{e\in\mathbb{E}_h}h^{-1}\|[\boldsymbol{v}]\|_e^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}h\|\nabla\boldsymbol{u}-\mathbf{Q}_h(\nabla\boldsymbol{u})\|_{\partial T}^2\Big)^{\frac12}\\ &\leq C\|\!|\boldsymbol{v}|\!\|\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{u}-\mathbf{Q}_h(\nabla\boldsymbol{u})\|_T^2+h^2\|\nabla(\nabla\boldsymbol{u}-\mathbf{Q}_h(\nabla\boldsymbol{u}))\|_T^2\Big)^{\frac12}\\ &\leq Ch^k\|\boldsymbol{u}\|_{k+1}\|\!|\boldsymbol{v}|\!\|.\end{aligned}$$

Similarly, we have

$$|l_2(\boldsymbol{v},p)|\leq Ch^k\|p\|_k\|\!|\boldsymbol{v}|\!\|.$$

Applying the definition of ∇_w and $\hat{Q}_h$, adding and subtracting $\boldsymbol{u}$ to the right-hand side of the equation above yields

$$\begin{aligned}|l_3(\boldsymbol{v},\boldsymbol{u})|&=\Big|\sum_{T\in\mathcal{T}_h}(\nabla_w(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}),\nabla_w\boldsymbol{v})_T\Big|=\Big|\sum_{T\in\mathcal{T}_h}\langle\{Q_0\boldsymbol{u}\}-Q_b\boldsymbol{u},\nabla_w\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T}\Big|\\ &=\Big|\sum_{T\in\mathcal{T}_h}\langle\{Q_0\boldsymbol{u}-\boldsymbol{u}\}+\boldsymbol{u}-Q_b\boldsymbol{u},\nabla_w\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial T}\Big|\\ &\leq C\Big(\sum_{T\in\mathcal{T}_h}h^{-1}(\|\{Q_0\boldsymbol{u}-\boldsymbol{u}\}\|_{\partial T}^2+\|\boldsymbol{u}-Q_b\boldsymbol{u}\|_{\partial T}^2)\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}h\|\nabla_w\boldsymbol{v}\|_{\partial T}^2)\Big)^{\frac12}\\ &\leq Ch^k\|\boldsymbol{u}\|_{k+1}\|\!|\boldsymbol{v}|\!\|.\end{aligned}$$

According to the definition of $\tilde{\nabla}_w$, $\hat{\mathbb{Q}}_h$, and $[\![\cdot]\!]$, we have

$$\begin{aligned}|l_4(\boldsymbol{v},p)| &= \Big|\sum_{T\in\mathcal{T}_h}(\boldsymbol{v},\tilde{\nabla}_w(\hat{\mathbb{Q}}_hp-\mathbb{Q}_hp))_T\Big| = \Big|\sum_{T\in\mathcal{T}_h}\langle \boldsymbol{v}\cdot\boldsymbol{n},\{\mathbb{Q}_0p\}-\mathbb{Q}_bp\rangle_{\partial T}\Big|\\ &= \Big|\sum_{e\in\mathbb{E}_h}\langle [\![\boldsymbol{v}]\!],\{\mathbb{Q}_0p-p\}+p-\mathbb{Q}_bp\rangle_e\Big| \le \Big|\sum_{e\in\mathbb{E}_h}\langle [\boldsymbol{v}],\{\mathbb{Q}_0p-p\}+p-\mathbb{Q}_bp\rangle_e\Big|\\ &\le C\Big(\sum_{e\in\mathbb{E}_h}h^{-1}\|[\boldsymbol{v}]\|_e^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}h(\|\{\mathbb{Q}_0p-p\}\|_{\partial T}^2+\|p-\mathbb{Q}_bp\|_{\partial T}^2)\Big)^{\frac{1}{2}}\\ &\le Ch^k\|p\|_k|\!|\!|\boldsymbol{v}|\!|\!|.\end{aligned}$$

From the definition of $s_1(\cdot,\cdot)$ and $\boldsymbol{u}\in[H_0^1(\Omega)]^d$, we obtain

$$\begin{aligned}|s_1(\hat{Q}_h\boldsymbol{u},\boldsymbol{v})| &= \Big|\sum_{e\in\mathbb{E}_h^0}h^{-1}\langle[Q_0\boldsymbol{u}],[\boldsymbol{v}]\rangle_e\Big| = \Big|\sum_{e\in\mathbb{E}_h^0}h^{-1}\langle[Q_0\boldsymbol{u}-\boldsymbol{u}],[\boldsymbol{v}]\rangle_e\Big|\\ &\le C\Big(\sum_{e\in\mathbb{E}_h^0}h^{-1}\|[\boldsymbol{v}]\|_e^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}h^{-1}\|[Q_0\boldsymbol{u}-\boldsymbol{u}]\|_{\partial T}^2\Big)^{\frac{1}{2}}\\ &\le Ch^k\|\boldsymbol{u}\|_{k+1}|\!|\!|\boldsymbol{v}|\!|\!|.\end{aligned}$$

Summing all the above inequalities, we get (5.6).

In addition, it follows from the property (2.1) and the definition of $\hat{\mathbb{Q}}_h$ that

$$\begin{aligned}|l_5(\boldsymbol{u},q)| &= \Big|\sum_{T\in\mathcal{T}_h}\langle q-\{q\},(\boldsymbol{u}-Q_h\boldsymbol{u})\cdot\boldsymbol{n}\rangle_{\partial T}\Big|\\ &= \Big|\sum_{e\in\mathbb{E}_h^0}\langle[\![q]\!],\{\boldsymbol{u}-Q_h\boldsymbol{u}\}\rangle_e\Big|\\ &\le C\Big(\sum_{e\in\mathbb{E}_h^0}h\|[\![q]\!]\|_e^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}h^{-1}\|\{\boldsymbol{u}-Q_h\boldsymbol{u}\}\|_{\partial T}^2\Big)^{\frac{1}{2}}\\ &\le Ch^k\|\boldsymbol{u}\|_{k+1}\|q\|_h.\end{aligned}$$

Using the definition of $s_2(\cdot,\cdot)$, $p\in L_0^2(\Omega)$, and $\|\cdot\|_h$, we get

$$\begin{aligned}|s_2(\hat{\mathbb{Q}}_hp,q)| &= \Big|\sum_{e\in\mathbb{E}_h^0}h\langle[\![\hat{\mathbb{Q}}_hp]\!],[\![q]\!]\rangle_e\Big| = \Big|\sum_{e\in\mathbb{E}_h^0}h\langle[\![\hat{\mathbb{Q}}_hp-p]\!],[\![q]\!]\rangle_e\Big|\\ &\le C\Big(\sum_{e\in\mathbb{E}_h^0}h\|[\![q]\!]\|_e^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}h\|[\![\mathbb{Q}_hp-p]\!]\|_{\partial T}^2\Big)^{\frac{1}{2}}\\ &\le Ch^k\|p\|_k\|q\|_h.\end{aligned}$$

Thus, we obtain (5.7) and complete the proof. □

5.2. Error estimate in H^1 and L^2 norms

We give the error estimates between the numerical solution and the projection of the exact solution in H^1 and L^2 norms, respectively.

Theorem 5.3 *Assume* $(\boldsymbol{u}_h;p_h)\in V_h^0\times W_h$ *and* $(\boldsymbol{u};p)\in[H_0^1(\Omega)]^d\cap[H^{k+1}(\Omega)]^d\times L_0^2(\Omega)\cap H^k(\Omega)$

with $k \geq 1$ are the solutions of the MWG schemes (3.1), (3.2) and the Brinkman problems (1.1)–(1.3), respectively. Then, there exists a constant C independent of h, such that

$$|\!|\!| \boldsymbol{e}_h |\!|\!| + \|\epsilon_h\|_h \leq Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k), \tag{5.8}$$

$$|\!|\!| \epsilon_h |\!|\!|_1 \leq Ch^{k-1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k). \tag{5.9}$$

Proof Let $\boldsymbol{v} = \boldsymbol{e}_h$ and $q = \epsilon_h$ in (4.3) and (4.4), respectively. From the definition of $|\!|\!| \cdot |\!|\!|$ and $\|\cdot\|_h$, we have

$$|\!|\!| \boldsymbol{e}_h |\!|\!|^2 + \|\epsilon_h\|_h^2 = \varphi_{\boldsymbol{u},p}(\boldsymbol{e}_h) - \phi_{\boldsymbol{u},p}(\epsilon_h).$$

It follows from the estimations (5.6) and (5.7) that

$$|\varphi_{\boldsymbol{u},p}(\boldsymbol{e}_h)| \leq Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)|\!|\!| \boldsymbol{e}_h |\!|\!|,$$

$$|\phi_{\boldsymbol{u},p}(\epsilon_h)| \leq Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)\|\epsilon_h\|_h.$$

Thus

$$|\!|\!| \boldsymbol{e}_h |\!|\!|^2 + \|\epsilon_h\|_h^2 \leq Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)(|\!|\!| \boldsymbol{e}_h |\!|\!| + \|\epsilon_h\|_h).$$

Furthermore, we obtain (5.8).

In addition, using (4.3), we have

$$b(\boldsymbol{v}, \epsilon_h) = -a(\boldsymbol{e}_h, \boldsymbol{v}) + \varphi_{\boldsymbol{u},p}(\boldsymbol{v}).$$

From the boundedness of $a(\cdot,\cdot)$ and the estimation (5.6), we obtain

$$|b(\boldsymbol{v}, \epsilon_h)| \leq |\!|\!| \boldsymbol{e}_h |\!|\!| |\!|\!| \boldsymbol{v} |\!|\!| + Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)|\!|\!| \boldsymbol{v} |\!|\!| \tag{5.10}$$

$$\leq (|\!|\!| \boldsymbol{e}_h |\!|\!| + Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k))|\!|\!| \boldsymbol{v} |\!|\!|. \tag{5.11}$$

Using the inf-sup condition (5.1), we have

$$|b(\boldsymbol{v}, \epsilon_h)| \geq |\!|\!| \boldsymbol{v} |\!|\!|(C_1 h |\!|\!| \epsilon_h |\!|\!|_1 - C_2\|\epsilon_h\|_h).$$

Thus

$$h|\!|\!| \epsilon_h |\!|\!|_1 \leq C(|\!|\!| \boldsymbol{e}_h |\!|\!| + \|\epsilon_h\|_h) + Ch^k(\|\boldsymbol{u}\|_{k+1} + \|p\|_k).$$

Finally, we obtain (5.9) from (5.8). The proof is completed. □

In order to get the L^2 estimation for velocity function $\boldsymbol{u}$, we consider the following dual problems. Find $(\Psi; \xi) \in [H^2(\Omega)]^d \times H^1(\Omega)$ satisfying

$$-\Delta\Psi + \kappa^{-1}\Psi + \nabla\xi = \boldsymbol{e}_0, \quad \text{in } \Omega, \tag{5.12}$$

$$\nabla \cdot \Psi = 0, \quad \text{in } \Omega, \tag{5.13}$$

$$\Psi = \boldsymbol{0}, \quad \text{on } \partial\Omega. \tag{5.14}$$

Furthermore, we assume the solution of the equations above satisfy the regularity condition $\|\Psi\|_2 + \|\xi\|_1 \leq C\|\boldsymbol{e}_0\|$.

Theorem 5.4 *Let $(\boldsymbol{u}; p) \in [H_0^2(\Omega)]^d \cap [H^{k+1}(\Omega)]^d \times L_0^2(\Omega) \cap H^k(\Omega)$ with $k \geq 1$ be the exact solutions of the Brinkman problems (1.1)–(1.3), and $(\boldsymbol{u}_h; p_h) \in V_h^0 \times W_h^0$ be the numerical* *solutions of the MWG schemes (3.1) and (3.2). Then there exists a constant C independent of h, such that*

$$\|\boldsymbol{e}_0\| \le Ch^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k). \tag{5.15}$$

Proof Taking $\boldsymbol{v} = \hat{Q}_h\Psi$ and $q = \hat{\mathbb{Q}}_h\xi$ in (4.3) and (4.4), respectively, yields

$$a(\boldsymbol{e}_h, \hat{Q}_h\Psi) + b(\hat{Q}_h\Psi, \epsilon_h) = \varphi_{\boldsymbol{u},p}(\hat{Q}_h\Psi), \tag{5.16}$$

$$b(\boldsymbol{e}_h, \hat{\mathbb{Q}}_h\xi) - s_2(\epsilon_h, \hat{\mathbb{Q}}_h\xi) = \phi_{\boldsymbol{u},p}(\hat{\mathbb{Q}}_h\xi). \tag{5.17}$$

Let Ψ_h, ξ_h be the numerical solution of the dual problems (5.12)–(5.14) in the MWG finite element scheme. Replacing $(\boldsymbol{u}_h, p_h)$ by (Ψ_h, ξ_h) in (3.1), (3.2) and taking $\boldsymbol{v} = \boldsymbol{e}_h$, $\boldsymbol{f} = \boldsymbol{e}_0$, and $q = \epsilon_h$, we obtain

$$a(\Psi_h, \boldsymbol{e}_h) + b(\boldsymbol{e}_h, \xi_h) = \|\boldsymbol{e}_0\|^2, \tag{5.18}$$

$$b(\Psi_h, \epsilon_h) - s_2(\xi_h, \epsilon_h) = 0. \tag{5.19}$$

Noticing that $\hat{Q}_h\Psi - \Psi$ and $\hat{\mathbb{Q}}_h\xi - \xi$ satisfy (4.3), (4.4), (5.18) and (5.19), respectively, we get

$$\begin{aligned} a(\hat{Q}_h\Psi, \boldsymbol{e}_h) + b(\boldsymbol{e}_h, \hat{\mathbb{Q}}_h\xi) &= a(\hat{Q}_h\Psi - \Psi_h, \boldsymbol{e}_h) + b(\boldsymbol{e}_h, \hat{\mathbb{Q}}_h\xi - \xi_h) + \|\boldsymbol{e}_0\|^2 \\ &= \varphi_{\Psi,\xi}(\boldsymbol{e}_h) + \|\boldsymbol{e}_0\|^2, \end{aligned} \tag{5.20}$$

$$\begin{aligned} b(\hat{Q}_h\Psi, \epsilon_h) - s_2(\hat{\mathbb{Q}}_h\xi, \epsilon_h) &= b(\hat{Q}_h\Psi - \Psi_h, \epsilon_h) - s_2(\hat{\mathbb{Q}}_h\xi - \xi_h, \epsilon_h) \\ &= \phi_{\Psi,\xi}(\epsilon_h). \end{aligned} \tag{5.21}$$

Combining equations (5.16), (5.17), (5.20) and (5.21) yields

$$\|\boldsymbol{e}_0\|^2 = \varphi_{\boldsymbol{u},p}(\hat{Q}_h\Psi) + \phi_{\boldsymbol{u},p}(\hat{\mathbb{Q}}_h\xi) - \varphi_{\Psi,\xi}(\boldsymbol{e}_h) - \phi_{\Psi,\xi}(\epsilon_h). \tag{5.22}$$

From the estimation of H^1 and $k = 1$, we obtain

$$|\varphi_{\Psi,\xi}(\boldsymbol{e}_h)| \le Ch^k(\|\Psi\|_{k+1} + \|\xi\|_k)|\!|\!|\boldsymbol{e}_h|\!|\!| \le Ch(\|\Psi\|_2 + \|\xi\|_1)|\!|\!|\boldsymbol{e}_h|\!|\!|, \tag{5.23}$$

$$|\phi_{\Psi,\xi}(\epsilon_h)| \le Ch(\|\Psi\|_2 + \|\xi\|_1)\|\epsilon_h\|_h. \tag{5.24}$$

In addition, using the definition of $\varphi_{\boldsymbol{u},p}$ and the triangle inequality, we arrive at

$$\begin{aligned} |\varphi_{\boldsymbol{u},p}(\hat{Q}_h\Psi)| \le &|l_1(\hat{Q}_h\Psi, \boldsymbol{u})| + |l_2(\hat{Q}_h\Psi, p)| + |l_3(\hat{Q}_h\Psi, \boldsymbol{u})| + \\ &|l_4(\hat{Q}_h\Psi, p)| + |s_1(\hat{Q}_h\boldsymbol{u}, \hat{Q}_h\Psi)|. \end{aligned}$$

From the definition of l_1, the Cauchy-Schwarz inequality, the triangle inequality, the trace inequality, and the projection inequality with $k = 1$, we obtain

$$\begin{aligned} |l_1(\hat{Q}_h\Psi, \boldsymbol{u})| =&\Big|\sum_{T\in\mathcal{T}_h} \langle \hat{Q}_h\Psi - \{\hat{Q}_h\Psi\}, (\nabla\boldsymbol{u} - \mathbf{Q}_h(\nabla\boldsymbol{u}))\cdot\boldsymbol{n}\rangle_{\partial T}\Big| \\ =&\Big|\sum_{T\in\mathcal{T}_h} \langle \hat{Q}_h\Psi - \Psi + \{\Psi - \hat{Q}_h\Psi\}, \{\nabla\boldsymbol{u} - \mathbf{Q}_h(\nabla\boldsymbol{u})\}\rangle_{\partial T}\Big| \\ \le& C\Big(\sum_{T\in\mathcal{T}_h} h^{-1}(\|\hat{Q}_h\Psi - \Psi\|_{\partial T}^2 + \|\{\Psi - \hat{Q}_h\Psi\}\|_{\partial T}^2)\Big)^{\frac{1}{2}}. \end{aligned}$$

$$\Big(\sum_{T\in\mathcal{T}_h} h\|\nabla \boldsymbol{u}-\mathbf{Q}_h(\nabla \boldsymbol{u})\|_{\partial T}^2\Big)^{\frac{1}{2}}$$
$$\leq Ch^k\|\boldsymbol{u}\|_{k+1}h^k\|\Psi\|_{k+1}\leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\Psi\|_2.$$

Similarly, we have

$$\begin{aligned}|l_2(\hat{Q}_h\Psi,p)|&=\Big|\sum_{T\in\mathcal{T}_h}\langle(\hat{Q}_h\Psi-\{\hat{Q}_h\Psi\})\cdot\boldsymbol{n},p-\mathbb{Q}_b p\rangle_{\partial T}\Big|\\&\leq Ch^k\|p\|_k h^k\|\Psi\|_{k+1}\leq Ch^{k+1}\|p\|_k\|\Psi\|_2,\end{aligned}$$

and

$$|l_3(\hat{Q}_h\Psi,\boldsymbol{u})|=\Big|\sum_{T\in\mathcal{T}_h}(\nabla_w(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}),\nabla_w\hat{Q}_h\Psi)_T\Big|.$$

Replacing $\boldsymbol{u}$ and p by $\hat{Q}_h\Psi$ and ξ in the equation (4.7), respectively, and taking $\boldsymbol{v}=\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}$ and $\boldsymbol{f}=\boldsymbol{e}_0$, we obtain

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}(\nabla_w(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}),\nabla_w\hat{Q}_h\Psi)_T=(\nabla_w(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}),\nabla_w\hat{Q}_h\Psi)\\&=l_1(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u},\hat{Q}_h\Psi)-(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u},\tilde{\nabla}_w(\mathbb{Q}_h\xi))-l_2(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u},\xi)-\\&\quad(\kappa^{-1}\hat{Q}_h\Psi,\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u})+(\boldsymbol{e}_0,\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}).\end{aligned}$$

Noticing that $\hat{Q}_h\boldsymbol{u}=Q_h\boldsymbol{u}$ for any element T, hence

$$\sum_{T\in\mathcal{T}_h}(\nabla_w(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u}),\nabla_w\hat{Q}_h\Psi)_T=l_1(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u},\hat{Q}_h\Psi)-l_2(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u},\xi).$$

Similar to the H^1 estimation, we obtain

$$\begin{aligned}|l_1(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u},\hat{Q}_h\Psi)|&=\Big|\sum_{T\in\mathcal{T}_h}\langle\{Q_b\boldsymbol{u}\}-Q_b\boldsymbol{u},(\nabla(\hat{Q}_h\Psi)-\mathbf{Q}_h(\nabla\hat{Q}_h\Psi))\cdot\boldsymbol{n}\rangle_{\partial T}\Big|\\&\leq Ch^k\|\boldsymbol{u}\|_{k+1}h^k\|\hat{Q}_h\Psi\|_{k+1}\leq Ch^k\|\boldsymbol{u}\|_{k+1}h^k\|\Psi\|_{k+1}\\&\leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\Psi\|_2,\\|l_2(\hat{Q}_h\boldsymbol{u}-Q_h\boldsymbol{u},\xi)|&=\Big|\sum_{T\in\mathcal{T}_h}\langle(\{Q_b\boldsymbol{u}\}-Q_b\boldsymbol{u})\cdot\boldsymbol{n},\xi-\mathbb{Q}_b\xi\rangle_{\partial T}\Big|\\&\leq Ch^k\|\xi\|_k h^k\|\boldsymbol{u}\|_{k+1}\leq Ch^{k+1}\|\xi\|_1\|\boldsymbol{u}\|_{k+1},\end{aligned}$$

and

$$|l_3(\hat{Q}_h\Psi,\boldsymbol{u})|\leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}(\|\xi\|_1+\|\Psi\|_2).$$

From the definition of l_4, $\tilde{\nabla}_w$, and $\hat{Q}_h$, $\Phi|_{\partial\Omega}=0$, the Cauchy-Schwarz inequality, the triangle inequality, the trace inequality, and the projection inequality with $k=1$, we obtain

$$\begin{aligned}|l_4(\hat{Q}_h\Psi,p)|&=\Big|\sum_{T\in\mathcal{T}_h}(\hat{Q}_h\Psi,\tilde{\nabla}_w(\hat{\mathbb{Q}}_h p-\mathbb{Q}_h p))_T\Big|\\&=\Big|\sum_{T\in\mathcal{T}_h}\langle\{Q_0\Psi\}\cdot\boldsymbol{n},\{\mathbb{Q}_0 p\}-\mathbb{Q}_b p\rangle_{\partial T}\Big|\\&=\Big|\sum_{e\in\mathbb{E}_h}\langle\{Q_0\Psi-\Psi\}\cdot\boldsymbol{n},\{\mathbb{Q}_0 p-p\}+p-\mathbb{Q}_b p\rangle_{\partial T}\Big|\end{aligned}$$

$$\leq Ch^k \|p\|_k h^k \|\Psi\|_{k+1} \leq Ch^{k+1} \|p\|_k \|\Psi\|_2.$$

Similarly, from the definition of $s_1(\cdot,\cdot)$ and the fact of $\boldsymbol{u}, \Psi \in [H_0^1(\Omega)]^d$, we arrive at

$$\begin{aligned}
|s_1(\hat{Q}_h \boldsymbol{u}, \hat{Q}_h \Psi)| &= \Big| \sum_{e\in\mathbb{E}_h} h^{-1} \langle [\hat{Q}_0 \boldsymbol{u}], [\hat{Q}_0 \Psi] \rangle_e \Big| \\
&= \Big| \sum_{e\in\mathbb{E}_h} h^{-1} \langle [\hat{Q}_0 \boldsymbol{u} - \boldsymbol{u}], [\hat{Q}_0 \Psi - \Psi] \rangle_e \Big| \\
&\leq C\Big(\sum_{e\in\mathbb{E}_h} h^{-1} \|[\hat{Q}_0 \boldsymbol{u} - \boldsymbol{u}]\|_e^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathbb{E}_h} h^{-1} \|[Q_0 \Psi - \Psi]\|_e^2 \Big)^{\frac{1}{2}} \\
&\leq Ch^{k+1} \|\boldsymbol{u}\|_{k+1} \|\Psi\|_2.
\end{aligned}$$

Combining all the equations above, we have

$$|\varphi_{\boldsymbol{u},p}(\hat{Q}_h \Psi)| \leq Ch^{k+1} \|\boldsymbol{u}\|_{k+1} (\|\xi\|_1 + \|\Psi\|_2). \tag{5.25}$$

Similarly, using the definition of $\phi_{\boldsymbol{u},p}$, we obtain

$$|\phi_{\boldsymbol{u},p}(\hat{\mathbb{Q}}_h \xi)| \leq |l_5(\boldsymbol{u}, \hat{\mathbb{Q}}_h \xi)| + |s_2(\hat{\mathbb{Q}}_h p, \hat{\mathbb{Q}}_h \xi)|.$$

Applying the definition of $\hat{\mathbb{Q}}_h$, the Cauchy-Schwarz inequality, the trace inequality, and the projection inequality with $k = 1$ yields

$$\begin{aligned}
|l_5(\boldsymbol{u}, \hat{\mathbb{Q}}_h \xi)| &= \Big| \sum_{T\in\mathcal{T}_h} \langle \mathbb{Q}_0 \xi - \{\mathbb{Q}_0 \xi\}, (\boldsymbol{u} - Q_b \boldsymbol{u}) \cdot \boldsymbol{n} \rangle_{\partial T} \Big| \\
&= \Big| \sum_{T\in\mathcal{T}_h} \langle \mathbb{Q}_0 \xi - \xi + \{\xi - \mathbb{Q}_0 \xi\}, (\boldsymbol{u} - Q_b \boldsymbol{u}) \cdot \boldsymbol{n} \rangle_{\partial T} \Big| \\
&\leq Ch^k \|\boldsymbol{u}\|_{k+1} h^k \|\xi\|_k \leq Ch^{k+1} \|\boldsymbol{u}\|_{k+1} \|\xi\|_1.
\end{aligned}$$

It follows from the definition of $\hat{\mathbb{Q}}_h$, $p \in L_0^2(\Omega)$, and $\xi \in H^1(\Omega)$ that

$$\begin{aligned}
|s_2(\hat{\mathbb{Q}}_h p, \hat{\mathbb{Q}}_h \xi)| &= \Big| \sum_{e\in\mathbb{E}_h^0} h \langle [\![\mathbb{Q}_b p]\!], [\![\mathbb{Q}_0 \xi]\!] \rangle_e \Big| = \Big| \sum_{e\in\mathbb{E}_h^0} h \langle [\![\mathbb{Q}_b p - p]\!], [\![\mathbb{Q}_0 \xi - \xi]\!] \rangle_e \Big| \\
&\leq C\Big(\sum_{e\in\mathbb{E}_h^0} h \|[\![\mathbb{Q}_b p - p]\!]\|_e^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathbb{E}_h^0} h \|[\![\mathbb{Q}_0 \xi - \xi]\!]\|_e^2 \Big)^{\frac{1}{2}} \\
&\leq Ch^k \|p\|_k h^k \|\xi\|_k \leq Ch^{k+1} \|p\|_k \|\xi\|_1.
\end{aligned}$$

Thus, we obtain

$$|\phi_{\boldsymbol{u},p}(\hat{\mathbb{Q}}_h \xi)| \leq Ch^{k+1} (\|\boldsymbol{u}\|_{k+1} + \|p\|_k) \|\xi\|_1. \tag{5.26}$$

Combining (5.22)–(5.26), applying (5.8) and the assumption of regularity, we have

$$\begin{aligned}
\|\boldsymbol{e}_0\|^2 &\leq Ch(\|\Psi\|_2 + \|\xi\|_1)(\|\!|\boldsymbol{e}_h|\!\| + \|\epsilon_h\|_h) + Ch^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k)(\|\Psi\|_2 + \|\xi\|_1) \\
&\leq Ch^{k+1}(\|\boldsymbol{u}\|_{k+1} + \|p\|_k) \|\boldsymbol{e}_0\|.
\end{aligned}$$

Eliminating the same terms, we get (5.15). This completes the proof of the theorem. □

6. Numerical results

In this section, we present the numerical experiments to verify the convergence and the stability of the modified weak Galerkin finite element method established in Section 3.

Example 6.1 In this example, the MWG finite element space is $V_h = \{\boldsymbol{v} \mid_T \in [L^2(\Omega)]^2 : \boldsymbol{v} \in [P_1(T)]^2\}$ and $W_h = \{q \in L_0^2(\Omega) : q \mid_T \in P_0(T)\}$. Let $\Omega = (0,1) \times (0,1)$. h denotes the mesh size of the triangulation partition on Ω. We further assume that the parameter $\kappa^{-1} = 1$. The exact solution is as follows

$$\boldsymbol{u} = \begin{pmatrix} \sin(2\pi x)\cos(2\pi y) \\ -\cos(2\pi x)\sin(2\pi y) \end{pmatrix} \text{ and } p = x^2y^2 - \frac{1}{9}.$$

The forcing term $\boldsymbol{f}$ is easily computed to match the exact solution $\boldsymbol{u}$ and p. Tables 1 and 2 show the errors and the convergence rates with respect to h, when $\mu = 1$ and $\mu = 0.01$, respectively.

h	$\lVert\!\lvert \boldsymbol{e}_h \rvert\!\rVert$	order	$\lVert \boldsymbol{e}_0 \rVert$	order	$\lVert \epsilon_h \rVert$	order
1/16	7.2143e-01		1.9953e-02		6.2139e-02	
1/24	4.6733e-01	1.0709	8.8897e-03	1.9939	3.2807e-02	1.5753
1/32	3.4451e-01	1.0599	4.9881e-03	2.0085	2.0468e-02	1.6398
1/48	2.2530e-01	1.0474	2.2039e-03	2.0145	1.0520e-02	1.6416
1/56	1.9199e-01	1.0381	1.6152e-03	2.0160	8.2092e-03	1.6091
1/64	1.6723e-01	1.0338	1.2340e-03	2.0158	6.6483e-03	1.5793

Table 1 Numerical errors and orders for Example 6.1 when $\mu = 1$

h	$\lVert\!\lvert \boldsymbol{e}_h \rvert\!\rVert$	order	$\lVert \boldsymbol{e}_0 \rVert$	order	$\lVert \epsilon_h \rVert$	order
1/16	3.38285e-01		1.2752e-01		4.4309e-02	
1/24	1.8749e-01	1.4552	5.9225e-02	1.8915	2.9384e-02	1.0130
1/32	1.2370e-01	1.4456	3.3711e-02	1.9589	2.1769e-02	1.0428
1/48	6.9405e-02	1.4253	1.5036e-02	1.9912	1.4221e-02	1.0500
1/56	5.5898e-02	1.4040	1.1040e-02	2.0042	1.2099e-02	1.0483
1/64	4.6421e-02	1.3913	8.4437e-03	2.0075	1.0522e-02	1.0457

Table 2 Numerical errors and orders for Example 6.1 when $\mu = 0.01$

Tables 1 and 2 show that the errors and the convergence rates for the velocity function $\boldsymbol{u}$ are of order$O(h)$ and $O(h^2)$ in $\lVert\!\lvert \cdot \rvert\!\rVert$ norm and L^2 norm, respectively. The errors and the convergence rates for the pressure function p are of order $O(h)$ in L^2 norm for $\mu = 1$ and $\mu = 0.01$. Therefore, we obtain the optimal orders of convergence for the velocity function and the pressure function in variational norms, which coincide with the theoretical analysis.

Example 6.2 This Example is based on Example 6.1 by taking $\kappa^{-1} = 10^4(\sin(2\pi x) + 1.1)$, Tables 3 and 4 show the errors and the convergence rates with respect to h, when $\mu = 1$ and $\mu = 0.01$, respectively.

We observe from Tables 3 and 4 that the convergence rates for the velocity function are of order $O(h)$ and $O(h^2)$ in $\lVert\!\lvert \cdot \rvert\!\rVert$ norm and L^2 norm, and for the pressure function are of order $O(h)$ in L^2 norm, which agrees well with the theoretical conclusions.

The two examples indicate that the modified weak Galerkin finite element method is stable, consistent and convergent, whenever the parameter κ is a constant or a function.

h	$\|\|\|\boldsymbol{e}_h\|\|\|$	order	$\|\|\boldsymbol{e}_0\|$	order	$\|\|\epsilon_h\|\|$	order
1/32	3.1367e-01		2.7631e-03		1.3452e-01	
1/48	2.1498e-01	0.9318	1.2953e-03	1.8685	1.0532e-01	0.6036
1/64	1.6271e-01	0.9683	7.5004e-04	1.8992	8.3841e-02	0.7929
1/80	1.3059e-01	0.9858	4.8886e-04	1.9183	6.7899e-02	0.9451
1/96	1.0893e-01	0.9947	3.4383e-04	1.9302	5.5826e-02	1.0739
1/120	8.7137e-02	1.0002	2.2298e-03	1.9408	4.2616e-02	1.2100

Table 3 Numerical errors and orders for Example 6.2 when $\mu = 1$

h	$\|\|\|\boldsymbol{e}_h\|\|\|$	order	$\|\|\boldsymbol{e}_0\|\|$	order	$\|\|\epsilon_h\|\|$	order
1/32	1.2234e-01		2.4090e-03		2.4689e-02	
1/48	6.8911e-02	1.4157	1.0314e-03	2.0920	1.5144e-02	1.2053
1/64	4.6184e-02	1.3910	5.6993e-04	2.0619	1.0921e-02	1.1363
1/80	3.4031e-02	1.3685	3.6119e-04	2.0440	8.5467e-03	1.0988
1/96	2.6614e-02	1.3483	2.4933e-04	2.0328	7.0237e-03	1.0764
1/120	1.9796e-02	1.3263	1.5871e-04	2.0241	5.5452e-03	1.0592

Table 4 Numerical errors and orders for Example 6.2 when $\mu = 0.01$

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