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The Number of Failed Components in a Conditional Coherent Operating System

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Abstract Coherent systems are very important in reliability, survival analysis and other life sciences. In this paper, we consider the number of failed components in an $(n-k+1)$-out-of-n system, given that at least m $(m < k \leq n)$ components have failed before time t, and the system is still working at time t. In this case, we compute the probability that there are exactly i working components. First the reliability and several stochastic properties are obtained. Furthermore, we extend the results to general coherent systems with absolutely continuous and exchangeable components.

Keywords coherent system; stochastic order; IFR; mean; signature

MR(2010) Subject Classification 90B25; 62N05; 60E15

1. Introduction

In reliability theory and reliability engineering, building redundancy is a method of increasing the reliability of a system. Redundancy refers to the reconfiguration of critical components of a system to reduce system downtime. An important system is the $(n-k+1)$-out-of-n system $(k \leq n)$, which is widely used in weapons manufacturing, aerospace industry and electrical engineering as an important redundant structure. The $(n-k+1)$-out-of-n system works if and only if at least $(n-k+1)$ of the n components work. The $(n-k+1)$-out-of-n systems are special types of coherent system. A system with n components is said to be a coherent system if it has no irrelevant components and the structure function of the system is an increase function in every component. In recent years, some discussions of coherent systems appear in Asadi and Bayramoglu [1], Khaledi and Shaked [2], Li and Zhang [3, 4], Navarro, Samaniego and Balakrishnan [5, 6], Mahmoudi and Asadi [7], Asadi and Berred [8], Ling and Li [9], Eryilmaz [10]. In 1985, Samaniego [11] proposed a concept of "signature" of coherent systems, which can represent the lifetime distribution function of a coherent system with independent identically distributed component lifetimes. Assume that a coherent system has independent and identically distributed component lifetimes $X_1, X_2, ..., X_n$, the $X_i's$ are distributed according to the continuous distribution F. Let $T = T(X_1, ..., X_n)$ be the lifetime of the system. Then the signature of the system

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is defined as a probability vector $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ with

$$s_i = P\{T = X_{i:n}\}, \quad i = 1, 2, \dots, n,$$

where $X_{i:n}$ denotes the i-th ordered lifetime of the components lifetimes $X_1, X_2, \dots, X_n$, and $\sum_{i=1}^n s_i = 1$. The s_i also can be written as $s_i = \frac{A_i}{n!}$, which represents the probability that the ith failure of components causes the system failure, where A_i is the number of all possible permutations of $X_1, X_2, \dots, X_n$ causing system failure for the ith failure of components. As the signature vector $\boldsymbol{s}$ does not depend on the common distribution function of $X_i's$, the reliability function of T, denoted by $\bar{F}_T(t)$ (see [12]), can be represented as a mixture of the survival functions of $X_{i:n}$ with weights $s_1, \dots, s_n$. That is

$$\bar{F}_T(t) = \sum_{i=1}^n s_i P(X_{i:n} > t).$$

The concept of "signature" has been used by several authors in recent years to study the reliability of coherent systems. Da, Zheng and Hu [13], Eryilmaz and Zuo [14], and Eryilmaz [15, 16] used the signature of a system with exchangeable components to study the number of working components in consecutive k-out-of-n system while it is working. Eryilmaz [17] has also studied the number of failed components in a coherent system with exchangeable components.

In this paper, we study the properties of the number of failed components in a conditional coherent operating system. The paper is organized as follows. In Section 2, we consider an $(n-k+1)$-out-of-n system with n independent and identically distributed component lifetimes, given that at least m $(m < k \leq n)$ components have failed before time t, and the system is still working at time t. We compute the probabilities

$$p_t(i, m, k, n) = P(N_t = i | X_{m:n} \leq t, X_{k:n} > t), \quad i = m, m+1, \dots, k-1, \tag{1.1}$$

where $N_t = i$ denotes the number of failed components in the system. Some properties of $p_t(i, m, k, n)$ are studied. In Section 3, we extend the results of Section 2 to general coherent systems with signature vector $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$, and the coherent systems have independent and identically distributed lifetimes. Some different results and examples are provided. Furthermore, we compute the probability and mean of the number of failed components for a conditional coherent system with absolutely continuous and exchangeable components.

2. The failed components in an $(n-k+1)$-out-of-n system

In $(n-k+1)$-out-of-n system, let $X_1, X_2, \dots, X_n$ be lifetimes of n i.i.d. components with distribution F. Denote $X_{i:n}$ as the i-th smallest order statistic of $X_1, X_2, \dots, X_n$. The N_t denotes the number of failed components in the system on $[0, t]$. Now assume that at least m $(m < k \leq n)$ components have failed before time t, and the system is still working at time t. Then, we consider the probability that there are exactly i failures

$$p_t(i, m, k, n) = P(N_t = i | X_{m:n} \leq t, X_{k:n} > t), \quad i = m, m+1, \dots, k-1. \tag{2.1}$$

To compute $p_t(i, m, k, n)$, note that the event $\{N_t = i\}$ is equivalent to the event $\{X_{i:n} \leq t <$

$X_{i+1:n}\}$. We can write

$$\begin{aligned}p_t(i,m,k,n) =& P(N_t = i | X_{m:n} \le t, X_{k:n} > t)\\ =& P(X_{i:n} < t < X_{i+1:n} | X_{m:n} \le t, X_{k:n} > t)\\ =& P(X_{i:n} < t | X_{m:n} \le t, X_{k:n} > t) - P(X_{i+1:n} < t | X_{m:n} \le t, X_{k:n} > t).\end{aligned}$$

Now we have

$$\begin{aligned}P(X_{i:n} < t | X_{m:n} \le t, X_{k:n} > t) &= \frac{P(X_{i:n} < t, X_{m:n} \le t, X_{k:n} > t)}{P(X_{m:n} \le t, X_{k:n} > t)}\\ = \frac{P(X_{i:n} < t, X_{k:n} > t)}{P(X_{m:n} \le t, X_{k:n} > t)} &= \frac{\sum_{j=i}^{k-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t)}{\sum_{j=m}^{k-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t)}\\ = \frac{\sum_{j=i}^{k-1} \binom{n}{j} \phi^j(t)}{\sum_{j=m}^{k-1} \binom{n}{j} \phi^j(t)},&\end{aligned}$$

where $\phi(t) = \frac{F(t)}{\bar{F}(t)}$. The $p_t(i,m,k,n)$ can be rewritten as

$$p_t(i,m,k,n) = \frac{\binom{n}{i} \phi^i(t)}{\sum_{j=m}^{k-1} \binom{n}{j} \phi^j(t)}, \quad i = m, m+1, \ldots, k-1. \tag{2.2}$$

Remark 2.1 Let v be the median of the distribution F. Then $\phi(v) = 1$ and

$$p_v(i,m,k,n) = \frac{\binom{n}{i}}{\sum_{j=m}^{k-1} \binom{n}{j}}, \quad i = m, m+1, \ldots, k-1,$$

which shows that $p_v(i,m,k,n)$ does not depend on v, scilicet, in an $(n-k+1)$-out-of-n system, given that at least m ($m < k \le n$) components have failed before time v, and the system is still working at time v, the probability of having i failures in the system does not depend on the distribution function F(or v).

Remark 2.2 For fixed $m > 1$, when $k = n$, i.e., the system is a parallel system, we have

$$\begin{aligned}p_v(i,m,n,n) &= \frac{\binom{n}{i} \phi^i(t)}{\sum_{j=m}^{n-1} \binom{n}{j} \phi^j(t)}\\ &= \frac{\binom{n}{i} \phi^i(t)}{(1+\phi(t))^n - \phi^n(t) - \sum_{j=0}^{m-1} \binom{n}{j} \phi^j(t)}\\ &= \frac{\binom{n}{i}}{2^n - 2 - \sum_{j=1}^{m-1} \binom{n}{j}}, \quad i = m, m+1, \ldots, n-1.\end{aligned}$$

Note that, $p_v(i,m,n,n)$ first starts to increase in i to achieve its maximum and starts to decrease.

In the following, we derive some stochastic order comparisons and properties of $p_t(i,m,k,n)$, we need the following definition.

Definition 2.3 *For two discrete distributions* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$, $\boldsymbol{p}$ *is said to be smaller than* $\boldsymbol{q}$ *in the*

(a) *Usual stochastic order (denoted by* $\boldsymbol{p} \le_{st} \boldsymbol{q}$*) if* $\sum_{j=i}^n p_j \le \sum_{j=i}^n q_j$ *for all* $i = 1, 2, \ldots, n$;

(b) *Hazard rate order (denoted by* $\boldsymbol{p} \le_{hr} \boldsymbol{q}$*) if* $\sum_{j=i}^n p_j / \sum_{j=i}^n q_j$ *is decreasing in* i;

(c) Reversed hazard rate order (denoted by $\mathbf{p} \leq_{rh} \mathbf{q}$) if $\sum_{j=1}^{i} p_j / \sum_{j=1}^{i} q_j$ is decreasing in i;

(d) Likelihood rate order (denoted by $\mathbf{p} \leq_{lr} \mathbf{q}$) if p_i/q_i is decreasing in i.

The following relation of implications can be established:

$$X \leq_{lr} Y \Longrightarrow X \leq_{hr} (\leq_{rh}) Y \Longrightarrow X \leq_{st} Y.$$

For further study about the properties of these stochastic orders, the reader can refer to [18–26].

Theorem 2.4 (a) *For $m \leq a \leq k-1$, $a \in N$, we have*

$$(N_t | X_{m:n} \leq t, X_{k:n} > t) \leq_{st} (N_t | X_{m:n} \leq t, X_{k-1:n} > t);$$

(b) For $m \leq b \leq k-1, b \in N$, we have

$$(N_t | X_{m-1:n} \leq t, X_{k:n} > t) \leq_{st} (N_t | X_{m:n} \leq t, X_{k:n} > t);$$

(c) Assume that there are two $(n-k+1)$-out-of-n systems of order n with the same structures, and the components have independent lifetimes. Let $\bar{F}$ denote the survival function of a random variable X, and $\bar{G}$ denote the survival function of a random variable Y. If $X \leq_{st} Y$, then

$$(N_t | Y_{m:n} \leq t, Y_{k:n} > t) \leq_{lr} (N_t | X_{m:n} \leq t, X_{k:n} > t).$$

Proof We prove part (a), part (b) can be established similarly. For $m \leq a \leq k-1$, $a \in N$, we have

$$\begin{aligned}
&\frac{\sum_{i=a}^{k-1} P(N_t = i | X_{m:n} \leq t, X_{k:n} > t)}{\sum_{i=a}^{k-1} P(N_t = i | X_{m:n} \leq t, X_{k-1:n} > t)} \\
&\quad = \frac{\sum_{i=a}^{k-1} \binom{n}{i} \phi^i(t)}{\sum_{j=m}^{k-1} \binom{n}{j} \phi^j(t)} \Big/ \frac{\sum_{i=a}^{k-1} \binom{n}{i} \phi^i(t)}{\sum_{j=m}^{k-2} \binom{n}{j} \phi^j(t)} \\
&\quad = \frac{\sum_{j=m}^{k-2} \binom{n}{j} \phi^j(t)}{\sum_{j=m}^{k-1} \binom{n}{j} \phi^j(t)} < 1.
\end{aligned}$$

Based on Definition 2.3, this completes the proof.

To prove the result of (c), we have to discuss the monotonicity in t of the following formula

$$\begin{aligned}
&\frac{P(N_t = i | X_{m:n} \leq t, X_{k:n} > t)}{P(N_t = i | Y_{m:n} \leq t, Y_{k:n} > t)} \\
&\quad = \frac{\binom{n}{i} F^i(t) \bar{F}^{n-i}(t)}{\sum_{j=m}^{k-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t)} \Big/ \frac{\binom{n}{i} G^i(t) \bar{G}^{n-i}(t)}{\sum_{j=m}^{k-1} \binom{n}{j} G^j(t) \bar{G}^{n-j}(t)} \\
&\quad \propto (\frac{F(t)}{G(t)})^i (\frac{\bar{F}(t)}{\bar{G}(t)})^{n-i}.
\end{aligned}$$

Since $X \leq_{st} Y$, then $\bar{F}(t) \leq \bar{G}(t), F(t) \geq G(t)$. It is increase in i. $\Box$

Definition 2.5 *Let $\{p_i, i = 0, 1, \ldots\}$ be the probability mass function of a random variable X taking values on $\{x_i, i = 0, 1, \ldots\}$. This sequence $\{p_i\}$ is said to be a log-concave function if and only if*

$$p_i^2 \geq p_{i-1} p_{i+1}, \quad i = 0, 1, \ldots,$$

where $p_{-1} = 0$ by convention. It is known that the log-concavity of $\{p_i\}$ implies that its distribution function is IFR. A probability mass function $\{p_i, i = 0, 1, \ldots\}$ with survival function $\bar{P}_i$ is said to be IFR if $\frac{\bar{P}_i}{\bar{P}_{i-1}}$ is decreasing in $i, i = 0, 1, \ldots, n-1$.

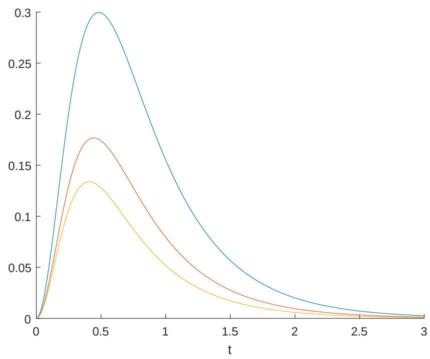


Figure 1 The plot displays $p_t(i, m, k, n)$, for $i = 3, m = 1, k = 6, n = 8, 9, 10$ and the components of system have exponential distribution with mean 1.

Theorem 2.6 *For fixed values of m, k and n, the function $p_t(i, m, k, n)$ is a log-concave function.*

Proof From the fact that for $i = m, m+1, \ldots, k-1$,

$$\binom{n}{i}^2 \geq \binom{n}{i+1}\binom{n}{i-1}$$

and from the expression of $p_t(i, m, k, n)$ in (2.2), we have $p_t^2(i, m, k, n) \geq p_t(i-1, m, k, n)p_t(i+1, m, k, n)$. □

Theorem 2.7 *The function F is uniquely determined by $p_t(i, m, k, n)$ as follows:*

$$F(t) = \frac{(i+1)p_t(i+1, m, k, n)}{(i+1)p_t(i+1, m, k, n) + (n-i)p_t(i, m, k, n)}, \quad i = m, m+1, \ldots, k-1. \tag{2.3}$$

Proof From (2.2), we can write for $i = m, m+1, \ldots, k-1$,

$$\frac{p_t(i+1, m, k, n)}{p_t(i, m, k, n)} = \frac{\binom{n}{i+1}}{\binom{n}{i}}\phi(t).$$

This in turn, produces a representation (2.3) after calculation. □

Theorem 2.8 *Denote by v the median of the distribution function F. Then*

(a) For $t \geq v$ and $i \leq \frac{n-1}{2}$, $p_t(i+1, m, k, n) \geq p_t(i, m, k, n)$;

(b) For $t \leq v$ and $i \geq \frac{n-1}{2}$, $p_t(i+1, m, k, n) \leq p_t(i, m, k, n)$.

Proof We prove part (a), part (b) can be established similarly. For $t \geq v$, we have $F(t) \geq \frac{1}{2}$, which in turn implies that $\phi(t) \geq 1$. On the other hand for $i \leq \frac{n-1}{2}$, we have

$$\binom{n}{i+1} \geq \binom{n}{i}.$$

Hence, we have

$$\binom{n}{i+1}\phi^{i+1}(t) \geq \binom{n}{i}\phi^{i}(t). \quad \Box$$

We need to show the following lemma to prove the next result.

Lemma 2.9 *For fixed j, m and n, the function $\gamma^j_{m,k,n}$ defined as*

$$\gamma^j_{m,k,n} = \frac{\sum_{l=j}^{k-1}\binom{n}{l}t^l}{\sum_{i=m}^{k-1}\binom{n}{i}t^i}$$

is an increase function of t.

Proof We need to show that for $0 < t_1 < t_2$,

$$\frac{\sum_{l=j}^{k-1}\binom{n}{l}t_1^l}{\sum_{i=m}^{k-1}\binom{n}{i}t_1^i} \leq \frac{\sum_{l=j}^{k-1}\binom{n}{l}t_2^l}{\sum_{i=m}^{k-1}\binom{n}{i}t_2^i}. \tag{2.4}$$

The inequality in (2.4) holds if and only if

$$\sum_{l=j}^{k-1}\sum_{i=m}^{j-1}\binom{n}{l}\binom{n}{i}(t_1^l t_2^i - t_1^i t_2^l) \leq 0, \tag{2.5}$$

as

$$\sum_{l=j}^{k-1}\sum_{i=j}^{k-1}\binom{n}{l}\binom{n}{i}(t_1^l t_2^i - t_1^i t_2^l) = 0.$$

Since $0 \leq t_1 \leq t_2$, it is easy to see that (2.5) holds. This proof is completed. $\Box$

In the following we assume that $\bar{P}_t(j,m,k,n)$ denotes the survival function of $p_t(i,m,k,n)$, i.e. $\bar{P}_t(j,m,k,n) = \sum_{i=j}^{k-1} p_t(i,m,k,n)$.

Theorem 2.10 *Assume there are two coherent systems of order n with the same structures and the components have independent lifetimes, with survival functions $\bar{F}$ and $\bar{G}$, respectively. If for all t, $\bar{F} \leq \bar{G}$, then $\bar{P}^F_t(j,m,k,n) \geq \bar{P}^G_t(j,m,k,n)$.*

Proof The $\bar{F} \leq \bar{G}$ implies that $\phi^F(t) \leq \phi^G(t)$. Now, the result follows from the fact that $\bar{P}^F_t(j,m,k,n) = \gamma^j_{m,k,n}(\phi^F(t))$ and $\bar{P}^G_t(j,m,k,n) = \gamma^j_{m,k,n}(\phi^G(t))$ and that, based on Lemma 2.9, $\gamma^j_{m,k,n}$ is increase in t. $\Box$

Figure 1 These plots display the $p_t(3,1,6,8), p_t(3,1,6,9)$ and $p_t(3,1,6,10)$. For fixed i, m and k, it clearly shows that for n, the $p_t(i,m,k,n)$ is not a monotone function. It first starts to increase in t to achieve its maximum and then starts to decrease. Assume that components of the system have exponential distribution with mean 1.

3. The failed components in a coherent system

In this section, we consider a coherent system with signature vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$. Assume that $\bar{S}_j = \sum_{i=j+1}^{n} s_i$, $0 \leq j \leq n-1$. If $p^c_t(i,m,n)$ denotes the probability of having i failures in the system, given that at least m ($m < n$) components have failed before time t, and the system is still working at time t, we have

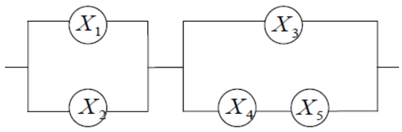
$$\begin{aligned}
p_t^c(i,m,n) &= P(X_{i:n} < t < X_{i+1:n} | X_{m:n} \le t, T > t) \\
&= \frac{P(X_{i:n} < t < X_{i+1:n}, X_{m:n} \le t, T > t)}{P(X_{m:n} \le t, T > t)} \\
&= \frac{\sum_{k=i+1}^{n} P(X_{i:n} < t < X_{i+1:n}, X_{m:n} \le t, X_{k:n} > t, T = X_{k:n})}{\sum_{l=m+1}^{n} P(X_{m:n} \le t, X_{l:n} > t, T = X_{l:n})} \\
&= \frac{\sum_{k=i+1}^{n} P(X_{i:n} < t < X_{i+1:n}, T = X_{k:n})}{\sum_{l=m+1}^{n} P(X_{m:n} \le t, X_{l:n} > t, T = X_{l:n})} \\
&= \frac{\sum_{k=i+1}^{n} P(X_{i:n} < t < X_{i+1:n}) P(T = X_{k:n})}{\sum_{l=m+1}^{n} P(X_{m:n} \le t, X_{l:n} > t) P(T = X_{l:n})} \\
&= \frac{\sum_{k=i+1}^{n} s_k \binom{n}{i} F^i(t) \bar{F}^{n-i}(t)}{\sum_{l=m+1}^{n} s_l \sum_{j=m}^{l-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t)}, \quad i = m, m+1, \ldots, n-1,
\end{aligned}$$

where the fourth equation comes from the fact that the event $\{X_{i:n} < t < X_{i+1:n}, X_{m:n} \le t, X_{k:n} > t\} (\equiv \{X_{i:n} < t < X_{i+1:n}\})$ and $\{T = X_{k:n}\}$ are independent, and $p_t^c(i,m,n)$ can be written as

$$p_t^c(i,m,n) = \frac{\bar{S}_i \binom{n}{i} \phi^i(t)}{\sum_{l=m+1}^{n} s_l \sum_{j=m}^{l-1} \binom{n}{j} \phi^j(t)}, \quad i = m, m+1, \ldots, n-1. \tag{3.1}$$

Example 3.1 The structure function of a coherent system is

$$T = \min\{\max(X_1, X_2), \max(X_3, \min(X_4, X_5))\},$$



where the $X_i's, i = 1,2,3,4,5$ are assumed to be independent and identical exponential distribution with mean 1. Let $m = 1$. Then $\phi(t) = e^t - 1$ and

$$\begin{aligned}
p_t^c(1,1,5) &= \frac{5}{5 + 7(e^t - 1) + 2(e^t - 1)^2}, \\
p_t^c(2,1,5) &= \frac{7(e^t - 1)}{5 + 7(e^t - 1) + 2(e^t - 1)^2}, \\
p_t^c(3,1,5) &= \frac{2(e^t -)^2}{5 + 7(e^t - 1) + 2(e^t - 1)^2}.
\end{aligned}$$

Note that the $p_t^c(4,1,5) = 0$ in this system, because, $P(T = X_{4:5}) = 0$, the $X_{4:5}$ is the structure of system for the component's lifetime, i.e. the lifetime $X_{4:5}$ of component will never cause the system failure.

In the following, we derive some properties of $p_t^c(i,m,n)$ that can be extended by the results of Section 2. As $p_t^c(i,m,n)$ and $p_t(i,m,k,n)$ have very similar forms, we do not give the details of the proofs.

Theorem 3.2 *(a) For $m \le a \le n-1, a \in N$, we have*

$$(N_t|X_{m-1:n} \le t, T > t) \le_{st} (N_t|X_{m:n} \le t, T > t);$$

(b) Assume that there are two coherent systems with lifetimes T_1 and T_2 of order n with the same structures, the components have independent lifetimes, with survival function $\bar{F}$ and $\bar{G}$, respectively. If $X \le_{st} Y$. Then

$$(N_t|Y_{m:n} \le t, T_2 > t) \le_{lr} (N_t|X_{m:n} \le t, T_1 > t).$$

Proof (a) For $m \le a \le n-1, a \in N$, we have

$$\begin{aligned}&\frac{\sum_{i=a}^{n-1} P(N_t = i|X_{m:n} \le t, T > t)}{\sum_{i=a}^{n-1} P(N_t = i|X_{m-1:n} \le t, T > t)}\\ &= \frac{\sum_{l=m}^{n} s_l \sum_{j=m-1}^{l-1} \binom{n}{j}\phi^j(t)}{\sum_{l=m+1}^{n} s_l \sum_{j=m}^{l-1} \binom{n}{j}\phi^j(t)} > 1.\end{aligned}$$

Based on Definition 2.3. This proof is completed.

As for (b), one can use the same steps as in the proof of Theorem 2.4 (c) to establish the result. □

Theorem 3.3 *If the structure vector $\boldsymbol{s}$ is IFR, then the function $p_t^c(i, m, n)$ is a log-concave function and hence IFR.*

Proof To prove this result, we have to show that

$$(p_t^c(i, m, n))^2 \ge p_t^c(i-1, m, n)p_t^c(i+1, m, n), \quad i = m, m+1, \ldots, n-1,$$

where by convention $p_t^c(n, m, n) = 0$. This is equivalent to showing that

$$(\bar{S}_i \binom{n}{i})^2 \ge \bar{S}_{i-1}\bar{S}_{i+1}\binom{n}{i-1}\binom{n}{i+1}.$$

The result follows from the condition that $\boldsymbol{s}$ is IFR and Theorem 2.6. □

The signature of a system with exchangeable components can be computed by finding the number of path sets of the system containing exactly i working components. Let $r_i(n)$ be the number of path sets of the system containing exactly i working components. Define

$$a_i(n) = \frac{r_i(n)}{\binom{n}{i}}, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

Then the quantities

$$s_i(n) = a_{n-i+1}(n) - a_{n-i}(n), \quad i = 1, 2, \ldots, n, \tag{3.3}$$

or equivalently

$$a_i(n) = \sum_{j=n-i+1}^{n} s_j(n), \quad i = 1, 2, \ldots, n, \tag{3.4}$$

give the signature of a system, where $a_0(n) = 0$.

Definition 3.4 ([27]) *The sequence of lifetime $T_1, \ldots, T_n$ is exchangeable if for each n,*

$$P\{T_1 \le t_1, \ldots, T_n \le t_n\} = P\{T_{\pi(1)} \le t_1, \ldots, T_{\pi(n)} \le t_n\},$$

for any permutation $\pi = (\pi(1), \ldots, \pi(n))$ *of* $\{1, 2, \ldots, n\}$.

Theorem 3.5 *Let* $T = \phi(T_1, T_2, \ldots, T_n)$ *be the lifetime of a coherent system with signature* $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$. *If* $T_1, T_2, \ldots, T_n$ *have an absolutely continuous exchangeable joint distribution, then for* $m \le j \le n-1$

$$P\{N_t = j | T_{m:n} \le t, T > t\} = \frac{P\{T_{j+1:n} > t\} - P\{T_{j:n} > t\}}{P\{T_{m:n} \le t, T > t\}} \sum_{i=j+1}^{n} s_i,$$

where $T_{i:n}$ *is the* i*-th smallest among* $T_1, T_2, \ldots, T_n$.

Proof We can write

$$P\{N_t = j | T_{m:n} \le t, T > t\} = \frac{P\{T > t | T_{m:n} \le t, N_t = j\} P\{T_{m:n} \le t, N_t = j\}}{P\{T_{m:n} \le t, T > t\}}.$$

It is clear that

$$P\{T > t | T_{m:n} \le t, N_t = j\} = P\{T > t | N_t = j\} = \frac{r_{n-j}(n)}{\binom{n}{n-j}}.$$

We obtain

$$P\{N_t = j | T_{m:n} \le t, T > t\} = \frac{r_{n-j}(n)}{\binom{n}{n-j}} \frac{P\{T_{j+1:n} > t\} - P\{T_{j:n} > t\}}{P\{T_{m:n} \le t, T > t\}}.$$

With (3.2) and (3.3), we have

$$\frac{r_{n-j}(n)}{\binom{n}{n-j}} = a_{n-j}(n) = \sum_{i=j+1}^{n} s_i.$$

This completes the proof. □

Remark 3.6 The conditional probability given in Theorem 3.5 can also be written as

$$P\{N_t = j | T_{m:n} \le t, T > t\} = \frac{P\{N_t = j\}}{P\{T_{m:n} \le t, T > t\}} \sum_{i=j+1}^{n} s_i.$$

Corollary 3.7 *As a direct consequence of Remark 3.6 we have*

$$E\{N_t | T_{m:n} \le t, T > t\} = \frac{E(g(N_t))}{P\{T_{m:n} \le t, T > t\}},$$

where $g(j) = j \sum_{i=j+1}^{n} s_i$.

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