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General Solutions of a Higher Order Impulsive Fractional Differential Equation Involving the Riemann-Liouville Fractional Derivatives

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Abstract We give general solutions (the explicit solutions) of a class of multi-term impulsive fractional differential equations involving the Riemann-Liouville fractional derivatives. This paper contributes within the domain of impulsive fractional differential equations. The author strongly believes that the article will highly be appreciated by the researchers working in the field of fractional calculus and on fractional differential models.

Keywords higher order fractional differential equation; piecewise continuous solution; Riemann-Liouville fractional derivative; impulse effect

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1. Introduction

Fractional differential equations have many applications [1–5]. Impulsive fractional differential equation is an important area of study [6]. In [7–9] and [10–16], authors studied the existence of solutions of the different initial or boundary value problems for impulsive fractional differential equations with the Caputo fractional derivatives or the Riemann-Liouville fractional derivatives.

However, we should point out that most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations in the last few years. In 1892, Hadamard introduced another kind of fractional derivatives, i.e., Hadamard type fractional differential equations, which differ from the preceding ones in the sense that the kernel of the integral and derivative contains logarithmic function of arbitrary exponent. The Hadamard and Riemann-Liouville fractional derivatives have one similar property, which is the fact that the derivative of a constant is not equal to zero. It is caused by the definitions of them containing the usual derivative outside the integrals. In 2012, Jarad et al. [12] presented the modifications of the Hadamard fractional derivative into a more suitable one having physically interpretable initial conditions similar to the Caputo sense. In 2014, Gambo et al. [17] proved the fundamental theorem of fractional calculus, some interesting results and also semigroup properties of

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Caputo-Hadamard operators.

In [18–23], the Cauchy problem or boundary value problems with Riemann-Liouville type Hadamard fractional derivative or Caputo type Hadamard fractional derivatives and impulsive effect were studied. We note that the fractional differential equations in known paper are single term ones, i.e., $D_{a^+}^{\alpha}x(t) = f(t, x(t))$. In applications such as biological models, a fractional differential equation involves the linear operator $D_{0^+}^{\alpha}x + \lambda D_{0^+}^{\beta}x$. This motivates us to consider the two-term fractional differential equation $D_{0^+}^{\alpha}x + \lambda D_{0^+}^{\beta}x = f(t, x(t))$ (see [1]). This model also comes from the standard Malthus population model $N' = -aN$ subjected to a perturbation $f(t, x(t))$. It is called two-term fractional differential equation. The fact forces us to study the solvability of initial or boundary value problems for impulsive two-term fractional differential equations.

Let $\alpha > \beta > 0$. To solve the two-term fractional differential equation $D_{0^+}^{\alpha}x - \lambda D_{0^+}^{\beta}x = h(t)$ by using the Laplace transform method, "it encounters very great difficulties except when $\alpha-\beta$ is an integer or half integer", for example, $\alpha = 2, \beta = \frac{3}{2}$ or $\beta = \alpha-1$, see page 139 in [3] or page 156 in [2]. It is interesting to find a new method to get the general solutions of the two-term equation $D_{0^+}^{\alpha}x - \lambda D_{0^+}^{\beta}x = h(t)$. Furthermore, one sees that $D_{0^+}^{\alpha}D_{0^+}^{\beta}x(t) \neq D_{0^+}^{\beta}D_{0^+}^{\alpha}x(t) \neq D_{0^+}^{\alpha+\beta}x(t)$ for $\alpha, \beta > 0$ (see [1]).

We provide new general solutions of the following multi-term fractional differential equation involving the Riemann-Liouville fractional derivatives

$$D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t) - \lambda D_{0^+}^{\gamma}x(t) = g(t), \quad \text{a.e., } t \in (0, 1], \tag{1.1}$$

$$D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t) - \lambda D_{0^+}^{\gamma}x(t) = g(t), \quad \text{a.e., } t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, \tag{1.2}$$

where

(i) n, l, p, k are positive integers, $\alpha \in (n-1, n)$, $\beta \in (l-1, l), \theta \in (p-1, p), \gamma \in (k-1, k)$ with $k \le \alpha + \beta + \theta$, $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ with a, b being nonnegative integers;

(ii) $\lambda \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{m+1} = 1$ are constants (impulse points);

(iii) $g \in C[0, 1]$;

(iv) $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $*$ with the starting point $t = 0$.

A continuous function $x : (0, 1] \to \mathbb{R}$ is called a continuous solution of (1.1) if $\lim_{t\to 0^+} t^{l-\beta}x(t)$ is finite, and x satisfies (1.1) for almost all $t \in (0, 1]$.

A function $x : (0, 1] \to \mathbb{R}$ is called a piecewise continuous solution of (1.2) if $x|_{(t_i,t_{i+1}]}$ ($i \in \mathbb{N}_0^m$) is continuous, $\lim_{t\to t_i^+}(t - t_i)^{l-\beta}x(t)$ is finite, and x satisfies (1.2) for almost all $t \in (0, 1]$.

The purpose of this paper is to give the explicit solutions (continuous solutions) of (1.1) and the explicit solutions (piecewise continuous solutions) of (1.2), respectively.

In Section 2, some definitions and the explicit solutions (continuous solutions) of (1.1) are given firstly (see Lemma 2.7). Then in Section 3, we establish the explicit expressions of general solutions of (1.2) (see Theorem 3.1). A conclusion section is given at the end of the paper.

2. Definitions and preliminary results

Let us recall some basic definitions of fractional calculus [1,17,22]. Let the Gamma function, the Beta function and the classical Mittag-Leffler special function be defined by

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}\mathrm{d}x,\ \ \mathbf{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x,\ \ E_{\delta,\sigma}(x)=\sum_{k=0}^{+\infty}\frac{x^k}{\Gamma(\delta k+\sigma)},$$

respectively, for $\alpha>0,p>0,q>0,\delta>0,\sigma>0$.

Definition 2.1 ([17,22]) *Let $a>0$ and $h:(a,+\infty)\to\mathbb{R}$ be a function. The left side Riemann-Liouville fractional integral of order $\alpha>0$ of h is given by $I_{a^+}^{\alpha}h(t)=\int_a^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s, t>a$ provided that the right-hand side exists.*

The left side Riemann-Liouville fractional derivative of order $\alpha\in(n-1,n)$ with n being a positive integer of h is given by $D_{a^+}^{\alpha}h(t)=[\int_a^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}h(s)\mathrm{d}s]^{(n)}, t>a$ provided that the right-hand side exists.

Suppose that n,l,p,k are positive integers, $\lambda\in\mathbb{R}$, $\alpha\in(n-1,n)$, $\beta\in(l-1,l)$, $\theta\in(p-1,p)$, $\gamma\in(k-1,k)$ with $k\le\alpha+\beta+\theta$ in this subsection. We seek continuous solutions of the linear Langevin fractional differential equation (LFDE for short)

$$D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t)-\lambda D_{0^+}^{\gamma}x(t)=g(t),\ \ \text{a.e., } t\in(0,1].$$

Let $x_i(i\in\mathbb{N}_1^{\max\{k,n\}}$ and $y_j(j\in\mathbb{N}_1^l)$ be fixed. We firstly give some claims and lemmas by using the Picard iterative methods. Consider the following initial value problem:

$$\begin{array}{l}
D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t)-\lambda D_{0^+}^{\gamma}x(t)=g(t),\ \text{a.e., } t\in(0,1],\\
I_{0^+}^{l-\beta}x(0)=y_l,\ D_{0^+}^{\beta-i}x(0)=y_i,\ \ i\in\mathbb{N}_1^{l-1},\\
I_{0^+}^{p-\theta}D_{0^+}^{\beta}x(0)=z_p,\ D_{0^+}^{\theta-j}D_{0^+}^{\beta}x(0)=z_j,\ \ j\in\mathbb{N}_1^{p-1},\\
\text{and}\left\{\begin{array}{ll}
(I_{0^+}^{n-\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda I_{0^+}^{k-\gamma})x(0)=x_n, & \text{if } k=n,\\
(D_{0^+}^{\alpha-j}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-j})x(0)=x_j,\ \ j\in\mathbb{N}_1^{k-1}, & \\
(D_{0^+}^{\alpha-k}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda I_{0^+}^{k-\gamma})x(0)=x_k,\ \ \text{or} & \\
(D_{0^+}^{\alpha-j}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-j})x(0)=x_j,\ \ j\in\mathbb{N}_1^{k-1}, & \text{if } k<n,\\
D_{0^+}^{\alpha-j}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0)=x_j, j\in\mathbb{N}_{k+1}^{n}, & \\
(I_{0^+}^{n-\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-n})x(0)=x_n,\ \ \text{or} & \\
(D_{0^+}^{\alpha-j}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-j})x(0)=x_j,\ \ j\in\mathbb{N}_1^{n-1}, & \text{if } k>n\\
-\lambda D_{0^+}^{\gamma-j}x(0)=x_j, j\in\mathbb{N}_{n+1}^{k}. &
\end{array}\right.
\end{array}\tag{2.1}$$

Remark 2.2 The initial conditions in (2.1) are complicated. When $\alpha\in(2-1,2)$, $\theta\in(2-1,2)$, $\beta\in(2-1,2),\gamma\in(1-1,1)$, the initial conditions in (2.1) become

$$\left\{\begin{array}{l}
I_{0^+}^{2-\beta}x(0)=y_2,\ D_{0^+}^{\beta-1}x(0)=y_1,\\
I_{0^+}^{2-\theta}D_{0^+}^{\beta}x(0)=z_2,\ D_{0^+}^{\theta-1}D_{0^+}^{\beta}x(0)=z_1,\\
I_{0^+}^{2-\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0)=x_2,\ [D_{0^+}^{2-\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda I_{0^+}^{1-\gamma}]x(0)=x_1.
\end{array}\right.$$

Choose $\theta=0$, we have $D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x=D_{0^+}^{\alpha}D_{0^+}^{\beta}x$. The initial conditions in (2.1) are complicated. When $\alpha\in(2-1,2)$, $\beta\in(2-1,2),\gamma\in(1-1,1)$, the initial conditions in (2.1) become

$$I_{0^+}^{2-\beta}x(0)=y_2,\ D_{0^+}^{\beta-1}x(0)=y_1,\ (D_{0^+}^{\alpha-1}D_{0^+}^{\beta}-\lambda I_{0^+}^{1-\gamma})x(0)=x_1,\ I_{0^+}^{2-\alpha}D_{0^+}^{\beta}x(0)=x_2$$

which are defined by some published papers when $\lambda=0$. When $\alpha\in(1-1,1)$, $\beta\in(1-1,1),\gamma\in(1-1,1)$, the initial conditions in (2.1) become

$$I_{0^+}^{1-\beta}x(0)=y_1,\ (I_{0^+}^{1-\alpha}D_{0^+}^{\beta}-\lambda I_{0^+}^{1-\gamma})x(0)=x_1$$

which are defined by some published papers when $\lambda=0$ (see [1]).

Denote $N=\max\{k,n\}$. Choose Picard function sequence as

$$\begin{aligned}\phi_0(t)=&\sum_{j=1}^{l}\frac{y_jt^{\beta-j}}{\Gamma(\beta-j+1)}+\sum_{j=1}^{p}\frac{z_jt^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)}+\sum_{i=1}^{N}\frac{x_it^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)}+\\&\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(s)\mathrm{d}s,\quad t\in(0,1],\\\phi_i(t)=&\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}\phi_{i-1}(s)\mathrm{d}s,\quad t\in(0,1],\ i=1,2,\dots.\end{aligned}$$

Claim 2.3 $\phi_\tau\in C(0,1]$ and $\lim\limits_{t\to0^+}t^{l-\beta}\phi_\tau(t)$ is finite.

Proof In fact, we have

$$\begin{aligned}t^{l-\beta}\Big|\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}g(s)\mathrm{d}s\Big|&\le t^{l-\beta}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}||g||_0\mathrm{d}s\\&\le\|g\|_0t^{l-\beta}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}\mathrm{d}s=\frac{t^{l+\alpha+\theta-\gamma}}{\alpha+\theta+\beta-\gamma}\to0\text{ as }t\to0^+.\end{aligned}$$

One sees $\phi_0\in C(0,1]$ and $\lim_{t\to0^+}t^{l-\beta}\phi_0(t)$ exists. So $\|\phi_0\|=\sup_{t\in(0,1]}|t^{l-\beta}\phi_0(t)|$ is finite.

It is easy to see $\phi_1\in C(0,1]$ and

$$\begin{aligned}&t^{l-\beta}\Big|\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}\phi_0(s)\mathrm{d}s\Big|\\&\le t^{l-\beta}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}s^{\beta-l}|s^{l-\beta}\phi_0(s)|\mathrm{d}s\\&\le\|\phi_0\|t^{l-\beta}\int_0^t(t-s)^{\alpha+\theta+\theta+\beta-\gamma-1}s^{\beta-l}\mathrm{d}s\text{ (by using }\frac{s}{t}=w)\\&=\|\phi_0\|t^{l-\beta}t^{\alpha+\theta+2\beta-\gamma-l}\int_0^1(1-w)^{\alpha+\theta+\beta-\gamma-1}w^{\beta-l}\mathrm{d}s\\&=\|\phi_0\|t^{\alpha+\theta+\beta-\gamma}\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)\to0\text{ as }t\to0^+.\end{aligned}$$

Then $\lim_{t\to0^+}t^{l-\beta}\phi_1(t)$ exists. By mathematical induction method, we see that $\phi_i\in C(0,1]$ and $\lim_{t\to0^+}t^{l-\beta}\phi_i(t)$ exists. □

Claim 2.4 $\{t\to t^{l-\beta}\phi_i(t)\}$ is convergent uniformly on $(0,1]$.

Proof In fact, from Claim 2.3, we know $\|\phi_0\|=\sup_{t\in(0,1]}|t^{l-\beta}\phi_0(t)|$ is finite. Then, we have for $t\in(0,1]$ that

$$\begin{aligned}t^{l-\beta}|\phi_1(t)-\phi_0(t)|&=\Big|\frac{t^{l-\beta}}{\Gamma(\alpha+\theta+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\theta+\beta-\gamma-1}\phi_0(s)\mathrm{d}s\Big|\\&\le\frac{\|\phi_0\|}{\Gamma(\alpha+\theta+\beta-\gamma)}t^{l-\beta}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}s^{\beta-l}\mathrm{d}s\\&=\frac{\|\phi_0\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}t^{\alpha+\theta+\beta-\gamma}.\end{aligned}$$

So

$$\begin{aligned}t^{l-\beta}|\phi_2(t)-\phi_1(t)|&=\Big|\frac{t^{l-\beta}}{\Gamma(\alpha+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}[\phi_1(s)-\phi_0(s)]\mathrm{d}s\Big|\\&\le\frac{t^{l-\beta}}{\Gamma(\alpha+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}s^{\beta-l}\frac{\|\phi_0\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}s^{\alpha+\theta+\beta-\gamma}\mathrm{d}s\\&=\frac{\|\phi_0\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-\gamma+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}t^{2(\alpha+\theta+\beta-\gamma)}.\end{aligned}$$

Similarly, by the mathematical induction method, we get for every $\tau=3,4,\dots$ that

$$\begin{aligned}t^{l-\beta}|\phi_\tau(t)-\phi_{\tau-1}(t)|&=|\frac{t^{l-\beta}}{\Gamma(\alpha+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}[\phi_{\tau-1}(s)-\phi_{\tau-2}(s)]\mathrm{d}s|\\&\le\frac{\|\phi_0\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\times\\&\quad\Big(\prod_{j=2}^{\tau}\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,(j-1)[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big)t^{\tau(\alpha+\theta+\beta-\gamma)}\\&\le\frac{\|\phi_0\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\times\\&\quad\Big(\prod_{j=2}^{\tau}\frac{\mathbf{B}(\alpha+\beta-\gamma,(j-1)[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big),\quad t\in(0,1].\end{aligned}$$

We consider

$$\begin{aligned}\sum_{\nu=1}^{+\infty}u_\nu=:\sum_{\nu=1}^{+\infty}&\frac{\|\phi_0\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\times\\&\Big(\prod_{j=2}^{\nu}\frac{\mathbf{B}(\alpha+\beta-\gamma,(j-1)[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big).\end{aligned}$$

Since

$$\begin{aligned}\frac{u_{\nu+1}}{u_\nu}&=\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,\nu[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\\&=\frac{\int_0^1 s^{\alpha+\theta+\beta-\gamma-1}(1-s)^{\nu[\alpha+\theta+\beta-\gamma]+\beta-l+1}\mathrm{d}s}{\Gamma(\alpha+\theta+\beta-\gamma)}\to0\text{ as }\nu\to\infty,\end{aligned}$$

we know that $\sum_{\nu=1}^{+\infty}u_\nu$ is convergent. Hence

$$t^{l-\beta}\phi_0(t)+t^{l-\beta}[\phi_1(t)-\phi_0(t)]+\cdots+t^{l-\beta}[\phi_\nu(t)-\phi_{\nu-1}(t)]+\cdots,\quad t\in(0,1]$$

is uniformly convergent. Then $\{t^{l-\beta}\phi_\nu(t)\}$ is convergent uniformly on $(0,1]$. □

Claim 2.5 $\phi(t)=\lim_{\nu\to+\infty}\phi_\nu(t)$ defined on $(0,1]$ is a unique continuous solution of the integral equation

$$x(t)=\phi_0(t)+\frac{\lambda}{\Gamma(\alpha+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}x(s)\mathrm{d}s. \tag{2.2}$$

Proof By Claim 2.4, we know that $\lim_{\nu\to+\infty}\phi_\nu(t)=\phi(t)$ is uniformly convergent on $(0,1]$. We see that $\phi(t)$ is continuous on $(0,1]$. We know that

$$\begin{aligned}\phi(t)=\lim_{\nu\to\infty}\phi_\nu(t)=&\lim_{\nu\to+\infty}\Big[\phi_0(t)+\frac{\lambda}{\Gamma(\alpha+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}\phi_{i-1}(s)\mathrm{d}s\Big]\\=&\phi_0(t)+\frac{\lambda}{\Gamma(\alpha+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}\phi(s)\mathrm{d}s.\end{aligned}$$

Then ϕ is a continuous solution of (2.2) defined on $(0,1]$.

Suppose that ψ defined on $(0,1]$ is also a solution of (2.2) satisfying that $\lim_{t\to1^+}t^{l-\beta}\psi(t)$ is finite. Then

$$\psi(t)=\phi_0(t)+\frac{\lambda}{\Gamma(\alpha+\theta+\beta-\gamma)}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}\psi(s)\mathrm{d}s,\quad t\in(0,1].$$

We need to prove that $\phi(t)\equiv\psi(t)$ on $(0,1]$. Now we have

$$\begin{aligned}t^{l-\beta}|\psi(t)-\phi_0(t)|&=\frac{t^{l-\beta}}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big|\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}\psi(s)\mathrm{d}s\Big|\\&\le\frac{\|\psi\|}{\Gamma(\alpha+\theta+\beta-\gamma)}t^{l-\beta}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}s^{\beta-l}\mathrm{d}s\\&=\frac{\|\psi\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}t^{\alpha+\theta+\beta-\gamma}.\end{aligned}$$

Now suppose for $\nu\ge0$ that

$$\begin{aligned}t^{l-\beta}|\psi(t)-\phi_\nu(t)|\le&\frac{\|\psi\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\times\\&\Big(\prod_{j=1}^{\nu}\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,j[\alpha+\theta+\beta-\gamma]+beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big)t^{(\nu+1)(\alpha+\theta+\beta-\gamma)}.\end{aligned}$$

Furthermore, we have by the Mathematical induction method that

$$\begin{aligned}t^{l-\beta}|\psi(t)-\phi_{\nu+1}(t)|=&\frac{t^{l-\beta}}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big|\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}[\psi(s)-\phi_{\nu-1}(s)]\mathrm{d}s\Big|\\\le&\frac{1}{\Gamma(\alpha+\theta+\beta-\gamma)}t^{l-\beta}\int_0^t(t-s)^{\alpha+\theta+\beta-\gamma-1}s^{\beta-l}\frac{\|\psi\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\times\\&\Big(\prod_{j=1}^{\nu}\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,j[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big)s^{(\nu+1)(\alpha+\theta+\beta-\gamma)}\mathrm{d}s\\=&\frac{\|\psi\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big(\prod_{j=1}^{\nu+1}\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,j[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big)\times\\&t^{(\nu+2)(\alpha+\theta+\beta-\gamma)}.\end{aligned}$$

So

$$t^{l-\beta}|\psi(t)-\phi_{\nu+1}(t)|\le\frac{\|\psi\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\times \Big(\prod_{j=1}^{\nu+1}\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,j[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big),\quad \nu=1,2,\ldots.$$

Similarly we know

$$\sum_{\nu=1}^{\infty}u_\nu=:\sum_{\nu=1}^{\infty}\frac{\|\psi\|\mathbf{B}(\alpha+\theta+\beta-\gamma,\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big(\prod_{j=1}^{\nu+1}\frac{\mathbf{B}(\alpha+\theta+\beta-\gamma,j[\alpha+\theta+\beta-\gamma]+\beta-l+1)}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big)$$

is convergent. Then $\lim_{\nu\to\infty}u_\nu=0$. Hence $\lim_{\nu\to+\infty}\phi_\nu(t)=\psi(t)$ on $(0,1]$. Then $\phi(t)\equiv\psi(t)$ on $(0,1]$. Then (2.2) has a unique solution ϕ. The proof is completed. □

Lemma 2.6 *Suppose that x is a solution of IVP(2.1). Then x is a solution of the integral equation (2.2).*

Proof Suppose that x is a solution of IVP(2.1). Then Theorem 2.3 ((2.7.48) on page 116 in [1]) implies that

$$D_{0^+}^{\theta}D_{0^+}^{\beta}x(t)=\sum_{i=1}^{n}\frac{D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0)t^{\alpha-i}}{\Gamma(\alpha-i+1)}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)\mathrm{d}s+ \lambda\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\gamma}x(s)\mathrm{d}s,\quad t\in(0,1].$$

Furthermore, we have similarly that

$$D_{0^+}^{\beta}x(t)=\sum_{j=1}^{p}\frac{D_{0^+}^{\theta-j}D_{0^+}^{\beta}x(0)t^{\theta-j}}{\Gamma(\theta-j+1)}+\sum_{i=1}^{n}\frac{D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0)t^{\alpha+\theta-i}}{\Gamma(\alpha+\theta-i+1)}+ \int_0^t\frac{(t-s)^{\alpha+\theta-1}}{\Gamma(\alpha+\theta)}g(s)\mathrm{d}s+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta-1}}{\Gamma(\alpha+\theta)}D_{0^+}^{\gamma}x(s)\mathrm{d}s,\quad t\in(0,1].$$

So

$$x(t)=\sum_{\sigma=1}^{l}\frac{D_{0^+}^{\beta-\sigma}x(0)t^{\beta-\sigma}}{\Gamma(\beta-\sigma+1)}+\sum_{j=1}^{p}\frac{D_{0^+}^{\theta-j}D_{0^+}^{\beta}x(0)t^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)}+ \sum_{i=1}^{n}\frac{D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)}+\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(s)\mathrm{d}s+ \lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}D_{0^+}^{\gamma}x(s)\mathrm{d}s,\quad t\in(0,1].$$

Now, we have

$$\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}D_{0^+}^{\gamma}x(s)\mathrm{d}s=\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}\Big(\int_0^s\frac{(s-u)^{l-\gamma-1}}{\Gamma(l-\gamma)}x(u)\mathrm{d}u\Big)^{(l)}\mathrm{d}s$$
$$=\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}\mathrm{d}\Big(\int_0^s\frac{(s-u)^{l-\gamma-1}}{\Gamma(l-\gamma)}x(u)\mathrm{d}u\Big)^{(l-1)}$$
$$=\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}\Big(\int_0^s\frac{(s-u)^{l-\gamma-1}}{\Gamma(l-\gamma)}x(u)\mathrm{d}u\Big)^{(l-1)}\Big|_0^t+$$

$$
\begin{aligned}
&\int_0^t \frac{(t-s)^{\alpha+\theta+\beta-2}}{\Gamma(\alpha+\theta+\beta-1)}\Big(\int_0^s \frac{(s-u)^{l-\gamma-1}}{\Gamma(l-\gamma)}x(u)\mathrm{d}u\Big)^{(l-1)}\mathrm{d}s \\
&= -\frac{D_{0+}^{\gamma-1}x(0)t^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)} + \int_0^t \frac{(t-s)^{\alpha+\theta+\beta-2}}{\Gamma(\alpha+\theta+\beta-1)}\Big(\int_0^s \frac{(s-u)^{l-\gamma-1}}{\Gamma(l-\gamma)}x(u)\mathrm{d}u\Big)^{(l-1)}\mathrm{d}s \\
&= \cdots \\
&= -\sum_{j=1}^{k} \frac{D_{0+}^{\gamma-j}x(0)t^{\alpha+\theta+\beta-j}}{\Gamma(\alpha+\theta+\beta-j+1)} + \int_0^t \frac{(t-s)^{\alpha+\theta+\beta-l-1}}{\Gamma(\alpha+\theta+\beta-l)}\int_0^s \frac{(s-u)^{l-\gamma-1}}{\Gamma(l-\gamma)}x(u)\mathrm{d}u\mathrm{d}s \\
&= -\sum_{j=1}^{k} \frac{D_{0+}^{\gamma-j}x(0)t^{\alpha+\theta+\beta-j}}{\Gamma(\alpha+\theta+\beta-j+1)} + \int_0^t \int_u^t \frac{(t-ss)^{\alpha+\theta+\beta-l-1}}{\Gamma(\alpha+\theta+\beta-l)}\frac{(s-u)^{l-\gamma-1}}{\Gamma(l-\gamma)}\mathrm{d}sx(u)\mathrm{d}u \\
&= -\sum_{j=1}^{k} \frac{D_{0+}^{\gamma-j}x(0)t^{\alpha+\theta+\beta-j}}{\Gamma(\alpha+\theta+\beta-j+1)} + \int_0^t \frac{(t-u)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}x(u)\mathrm{d}u.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
x(t) &= \sum_{\sigma=1}^{l} \frac{D_{0+}^{\beta-\sigma}x(0)t^{\beta-\sigma}}{\Gamma(\beta-\sigma+1)} + \sum_{j=1}^{p} \frac{D_{0+}^{\theta-j}D_{0+}^{\beta}x(0)t^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)} + \\
&\sum_{i=1}^{n} \frac{D_{0+}^{\alpha-i}D_{0+}^{\theta}D_{0+}^{\beta}x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)} + \int_0^t \frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(s)\mathrm{d}s - \\
&\lambda\sum_{j=1}^{k} \frac{D_{0+}^{\gamma-j}x(0)t^{\alpha+\theta+\beta-j}}{\Gamma(\alpha+\theta+\beta-j+1)} + \lambda\int_0^t \frac{(t-u)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}x(u)\mathrm{d}u \\
&= \sum_{\sigma=1}^{l} \frac{D_{0+}^{\beta-\sigma}x(0)t^{\beta-\sigma}}{\Gamma(\beta-\sigma+1)} + \sum_{j=1}^{p} \frac{D_{0+}^{\theta-j}D_{0+}^{\beta}x(0)t^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)} + \\
&\sum_{i=1}^{n} \frac{D_{0+}^{\alpha-i}D_{0+}^{\theta}D_{0+}^{\beta}x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)} - \lambda\sum_{j=1}^{k} \frac{D_{0+}^{\gamma-j}x(0)t^{\alpha+\theta+\beta-j}}{\Gamma(\alpha+\theta+\beta-j+1)} + \\
&\int_0^t \frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(s)\mathrm{d}s + \lambda\int_0^t \frac{(t-u)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}x(u)\mathrm{d}u
\end{aligned}
$$

$$
= \begin{cases}
\sum\limits_{i=1}^{n-1} \frac{(D_{0+}^{\alpha-i}D_{0+}^{\theta}D_{0+}^{\beta} - \lambda D_{0+}^{\gamma-i})x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)} + \frac{(I_{0+}^{n-\alpha}D_{0+}^{\theta}D_{0+}^{\beta} - \lambda I_{0+}^{k-\gamma})x(0)t^{\alpha+\theta+\beta-k}}{\Gamma(\alpha+\theta+\beta-k+1)}, & k=n, \\
\sum\limits_{i=1}^{k-1} \frac{(D_{0+}^{\alpha-i}D_{0+}^{\theta}D_{0+}^{\beta} - \lambda D_{0+}^{\gamma-i})x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)} + \\
\frac{(D_{0+}^{\alpha-k}D_{0+}^{\theta}D_{0+}^{\beta} - \lambda I_{0+}^{k-\gamma})x(0)t^{\alpha+\theta+\beta-k}}{\Gamma(\alpha+\theta+\beta-k+1)} + \sum\limits_{i=k+1}^{n} \frac{D_{0+}^{\alpha-j}D_{0+}^{\theta}D_{0+}^{\beta}x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)}, & k<n, \\
\sum\limits_{i=1}^{n-1} \frac{(D_{0+}^{\alpha-i}D_{0+}^{\theta}D_{0+}^{\beta} - \lambda D_{0+}^{\gamma-i})x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)} + \\
\frac{(I_{0+}^{n-\alpha}D_{0+}^{\theta}D_{0+}^{\beta} - \lambda D_{0+}^{\gamma-n})x(0)t^{\alpha+\theta+\beta-n}}{\Gamma(\alpha+\theta+\beta-n+1)} - \lambda\sum\limits_{i=n+1}^{k} \frac{D_{0+}^{\gamma-j}x(0)t^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)}, & k>n
\end{cases}
$$

$$\sum_{\sigma=1}^{l}\frac{D_{0^+}^{\beta-\sigma}x(0)t^{\beta-\sigma}}{\Gamma(\beta-\sigma+1)}+\sum_{j=1}^{p}\frac{D_{0^+}^{\theta-j}D_{0^+}^{\beta}x(0)t^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)}+$$
$$\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(s)\mathrm{d}s+\lambda\int_0^t\frac{(t-u)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}x(u)\mathrm{d}u$$
$$=\sum_{j=1}^{l}\frac{y_jt^{\beta-j}}{\Gamma(\beta-j+1)}+\sum_{j=1}^{p}\frac{z_jt^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)}+\sum_{i=1}^{\max\{k,n\}}\frac{x_it^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)}+$$
$$\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(s)\mathrm{d}s+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}x(s)\mathrm{d}s$$
$$=\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}x(s)\mathrm{d}s,\quad t\in(0,1].$$

Then $x\in C(0,1]$ is a solution of (2.2). The proof is completed. □

Lemma 2.7 *x is a solution of (2.1) if and only if x satisfies*

$$\begin{aligned}x(t)=&\sum_{j=1}^{l}y_jt^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+\\&\sum_{j=1}^{p}z_jt^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+\\&\sum_{i=1}^{N}x_it^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+\\&\int_0^t(t-s)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(t-s)^{\alpha+\theta+\beta-\gamma})g(s)\mathrm{d}s,\ t\in(0,1].\end{aligned}\tag{2.3}$$

Proof Suppose that x is a solution of (2.1). From Lemma 2.6, we know x is a solution of (2.2). By Claim 2.5, (2.2) has a unique solution given by $x(t)=\lim_{\nu\to\infty}\phi_\nu(t)$. We see that

$$\begin{aligned}\phi_i(t)=&\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}\phi_{i-1}(s)\mathrm{d}s\\=&\phi_0(s)+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}\Big[\phi_0(s)+\lambda\int_0^s\frac{(s-u)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}\phi_{i-2}(u)\mathrm{d}u\Big]\mathrm{d}s\\=&\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}\phi_0(s)\mathrm{d}s+\frac{\lambda}{\Gamma(\alpha+\theta+\beta-\gamma)}\frac{\lambda}{\Gamma(\alpha+\theta+\beta-\gamma)}\times\\&\int_0^t\int_u^t(t-s)^{\alpha+\theta+\beta-\gamma-1}(s-u)^{\alpha+\theta+\beta-\gamma-1}\mathrm{d}s\phi_{i-2}(u)\mathrm{d}u\\=&\phi_0(t)+\lambda\int_0^t\frac{(t-s)^{\alpha+\theta+\beta-\gamma-1}}{\Gamma(\alpha+\theta+\beta-\gamma)}\phi_0(s)\mathrm{d}s+\lambda^2\int_0^t\frac{(t-u)^{2(\alpha+\theta+\beta-\gamma)-1}}{\Gamma(2(\alpha+\theta+\beta-\gamma))}\phi_{i-2}(u)\mathrm{d}u\\=&\cdots\\=&\phi_0(t)+\sum_{j=1}^{i}\lambda^j\int_0^t\frac{(t-s)^{j(\alpha+\theta+\beta-\gamma)-1}}{\Gamma(j(\alpha+\theta+\beta-\gamma))}\phi_0(s)\mathrm{d}s\\=&\sum_{j=1}^{l}\frac{y_jt^{\beta-j}}{\Gamma(\beta-j+1)}+\sum_{j=1}^{p}\frac{z_jt^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)}+\sum_{i=1}^{N}\frac{x_it^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)}+\end{aligned}$$

$$\int_0^t \frac{(t-s)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(s)\mathrm{d}s+$$

$$\sum_{j=1}^{i}\lambda^j\int_0^t \frac{(t-s)^{j(\alpha+\theta+\beta-\gamma)-1}}{\Gamma(j(\alpha+\theta+\beta-\gamma))}\Big(\sum_{j=1}^{l}\frac{y_j s^{\beta-j}}{\Gamma(\beta-j+1)}+\sum_{j=1}^{p}\frac{z_j s^{\theta+\beta-j}}{\Gamma(\theta+\beta-j+1)}+$$

$$\sum_{i=1}^{N}\frac{x_i s^{\alpha+\theta+\beta-i}}{\Gamma(\alpha+\theta+\beta-i+1)}+\int_0^s \frac{(s-u)^{\alpha+\theta+\beta-1}}{\Gamma(\alpha+\theta+\beta)}g(u)\mathrm{d}u\Big)\mathrm{d}s$$

$$=\sum_{\chi=0}^{i}\sum_{j=1}^{l}\frac{y_j t^{\chi(\alpha+\theta+\beta-\gamma)+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-j+1)}+$$

$$\sum_{\chi=0}^{i}\sum_{j=1}^{p}\frac{z_j t^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j+1)}+$$

$$\sum_{\chi=0}^{i}\sum_{i=1}^{N}\frac{x_i t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-i}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-i+1)}+$$

$$\sum_{\chi=0}^{i}\int_0^t \frac{(t-s)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta)}g(s)\mathrm{d}s.$$

Hence

$$x(t)=\lim_{i\to\infty}\phi_i(t)=\sum_{j=1}^{l}y_j t^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+$$

$$\sum_{j=1}^{p}z_j t^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+$$

$$\sum_{i=1}^{N}x_i t^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+$$

$$\int_0^t (t-s)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(t-s)^{\alpha+\theta+\beta-\gamma})g(s)\mathrm{d}s.$$

We get (2.3).

Now, suppose that x satisfies (2.3). We prove that x is a solution of (2.1). By computation, firstly we get by using (2.3) for $i\in\mathbb{N}_0^l$ that

$$D_{0^+}^{\beta-i}x(t)=\Big[\int_0^t \frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}x(s)\mathrm{d}s\Big]^{(l-i)}$$

$$=\Big[\int_0^t \frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}\Big(\sum_{j=1}^{l}y_j s^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda s^{\alpha+\theta+\beta-\gamma})+$$

$$\sum_{j=1}^{p}z_j s^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda s^{\alpha+\theta+\beta-\gamma})+$$

$$\sum_{j=1}^{N}x_j s^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-j+1}(\lambda s^{\alpha+\theta+\beta-\gamma})+$$

$$\int_0^s (s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\Big)\mathrm{d}s\Big]^{(l-i)}$$

$$=\Big[\int_0^t \frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}\Big(\sum_{j=1}^{l} y_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-j+1)}+$$

$$\sum_{j=1}^{p} z_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j+1)}+$$

$$\sum_{j=1}^{N} x_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-j+1)}+$$

$$\int_0^s\sum_{\chi=0}^{\infty}\lambda^\chi\frac{(s-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta)}g(u)\mathrm{d}u\Big)\mathrm{d}s\Big]^{(l-i)}$$

$$=\Big[\sum_{j=1}^{l} y_j\sum_{\chi=0}^{\infty}\lambda^\chi\int_0^t\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}\frac{s^{\chi(\alpha+\theta+\beta-\gamma)+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-j+1)}\mathrm{d}s+$$

$$\sum_{j=1}^{p} z_j\sum_{\chi=0}^{\infty}\lambda^\chi\int_0^t\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}\frac{s^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j+1)}\mathrm{d}s+$$

$$\sum_{j=1}^{N} x_j\sum_{\chi=0}^{\infty}\lambda^\chi\int_0^t\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}\frac{s^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-j+1)}\mathrm{d}s+$$

$$\sum_{\chi=0}^{\infty}\lambda^\chi\int_0^t\int_u^t\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}\frac{(s-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta)}\mathrm{d}s g(u)\mathrm{d}u\Big]^{(l-i)}$$

$$=\Big[\sum_{j=1}^{l} y_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+l-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+l-j+1)}+$$

$$\sum_{j=1}^{p} z_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\theta+l-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+l-j+1)}+$$

$$\sum_{j=1}^{N} x_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+l-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+l-i+1)}+$$

$$\sum_{\chi=0}^{\infty}\lambda^\chi\int_0^t\frac{(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+l-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+l)}g(u)\mathrm{d}u\Big]^{(l-i)}$$

$$=\sum_{j=i+1}^{l} y_j\sum_{\chi=1}^{\infty}\frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$

$$\sum_{j=1}^{i} y_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$

$$\sum_{j=1}^{p} z_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\theta+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+i-j+1)}+$$

$$\sum_{j=1}^{N} x_j\sum_{\chi=0}^{\infty}\frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+i-j+1)}+$$

$$\int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+i-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+i)} g(u)\mathrm{d}u, \quad i \in \mathbb{N}_0^l.$$

It follows that

$$\begin{aligned} D_{0^+}^{\beta} x(t) = & \sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)} + \\ & \sum_{j=1}^{p} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta-j+1)} + \\ & \sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j+1)} + \\ & \int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta)} g(u)\mathrm{d}u \end{aligned} \tag{2.4}$$

and

$$I_{0^+}^{l-\beta} x(0) = y_l, \ D_{0^+}^{\beta-i} x(0) = y_i, \quad i \in \mathbb{N}_1^{l-1}. \tag{2.5}$$

Secondly, we have for $i \in \mathbb{N}_0^p$ by using (2.4) that

$$\begin{aligned} & D_{0^+}^{\theta-i} D_{0^+}^{\beta} x(t) = \Big[\int_0^t \frac{(t-s)^{p-\theta-1}}{\Gamma(p-\theta)} D_{0^+}^{\beta} x(s)\mathrm{d}s\Big]^{(p-i)} \\ & = \Big[\int_0^t \frac{(t-s)^{p-\theta-1}}{\Gamma(p-\theta)} \Big(\sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)} + \\ & \sum_{j=1}^{p} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta-j+1)} + \\ & \sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j+1)} + \\ & \int_0^s \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (s-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta)} g(u)\mathrm{d}u\Big)\mathrm{d}s\Big]^{(p-i)} \\ & = \Big[\sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)-\theta+p-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\theta+p-j+1)} + \\ & \sum_{j=1}^{p} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+p-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+p-j+1)} + \\ & \sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+p-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+p-j+1)} + \\ & \int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+p-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+p)} g(u)\mathrm{d}u\Big]^{(p-i)} \\ & = \sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)-\theta+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\theta+i-j+1)} + \end{aligned}$$

$$\sum_{j=i+1}^{p} z_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$
$$\sum_{j=1}^{i} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$
$$\sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+i-j+1)}+$$
$$\int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+i-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+i)} g(u)\mathrm{d}u.$$

It follows that

$$D_{0^+}^{\theta} D_{0^+}^{\beta} x(t) = \sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\theta-j+1)}+$$
$$\sum_{j=1}^{p} z_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+$$
$$\sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha-j+1)}+$$
$$\int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha)} g(u)\mathrm{d}u, \tag{2.6}$$

$$I_{0^+}^{p-\theta} D_{0^+}^{\beta} x(0) = z_p,\ D_{0^+}^{\theta-i} D_{0^+}^{\beta} x(0) = z_i, \quad i \in \mathbb{N}_1^{p-1}. \tag{2.7}$$

Thirdly, we have similarly for $i \in \mathbb{N}_0^n$ by using (2.6) that

$$D_{0^+}^{\alpha-i} D_{0^+}^{\theta} D_{0^+}^{\beta} x(t) = \Big[\int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} D_{0^+}^{\theta} D_{0^+}^{\beta} x(s)\mathrm{d}s\Big]^{(n-i)}$$
$$= \Big[\int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} \Big(\sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\theta-j+1)}+$$
$$\sum_{j=1}^{p} z_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+$$
$$\sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^\chi s^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha-j+1)}+$$
$$\int_0^s \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (s-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha)} g(u)\mathrm{d}u\Big)\mathrm{d}s\Big]^{(n-i)}$$
$$= \Big[\sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+n-\alpha-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+n-\alpha-\theta-j+1)}+$$
$$\sum_{j=1}^{p} z_j \sum_{\chi=1}^{\infty} \frac{\lambda^\chi t^{\chi(\alpha+\theta+\beta-\gamma)+n-\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+n-\alpha-j+1)}+$$

$$\sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+n-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+n-j+1)}+$$
$$\int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+n-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+n)} g(u)\mathrm{d}u\Big]^{(n-i)}$$
$$=\sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-\theta-j+1)}+$$
$$\sum_{j=1}^{p} z_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-j+1)}+$$
$$\sum_{j=i+1}^{N} x_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$
$$\sum_{j=1}^{i} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$
$$\begin{cases} \int_0^t \sum\limits_{\chi=0}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+i-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i)} g(u)\mathrm{d}u, & i \in \mathbb{N}_1^n, \\ g(t)+\int_0^t \sum\limits_{\chi=1}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma))} g(u)\mathrm{d}u, & i=0. \end{cases}$$

It follows that

$$\begin{aligned} D_{0^+}^{\alpha} D_{0^+}^{\theta} D_{0^+}^{\beta} x(t) = & \sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)-\alpha-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\alpha-\theta-j+1)}+ \\ & \sum_{j=1}^{p} z_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)-\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\alpha-j+1)}+ \\ & \sum_{j=1}^{N} x_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+ \\ & g(t)+\int_0^t \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma))} g(u)\mathrm{d}u, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} D_{0^+}^{\alpha-i} D_{0^+}^{\theta} D_{0^+}^{\beta} x(t) = & \sum_{j=1}^{l} y_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-\theta-j+1)}+ \\ & \sum_{j=1}^{p} z_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-j+1)}+ \\ & \sum_{j=i+1}^{N} x_j \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+ \\ & \sum_{j=1}^{i} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+ \end{aligned}$$

$$\int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+i-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i)} g(u)\mathrm{d}u, \quad i \in \mathbb{N}_1^n. \tag{2.9}$$

Fourthly, by using (2.3), we have for $i \in \mathbb{N}_0^k$ similarly that

$$\begin{aligned}
D_{0^+}^{\gamma-i}x(t) &= \Big[\int_0^t \frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s)\mathrm{d}s\Big]^{(k-i)} \\
&= \Big[\int_0^t \frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\Big(\sum_{j=1}^{l} y_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} s^{\chi(\alpha+\theta+\beta-\gamma)+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-j+1)} + \\
&\quad \sum_{j=1}^{p} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} s^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j+1)} + \\
&\quad \sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} s^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-j+1)} + \\
&\quad \int_0^s \sum_{\chi=0}^{\infty} \lambda^{\chi} \frac{(s-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta)} g(u)\mathrm{d}u\Big)\mathrm{d}s\Big]^{(k-i)} \\
&= \Big[\sum_{j=1}^{l} y_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma+k-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma+k-j+1)} + \\
&\quad \sum_{j=1}^{p} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma+k-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma+k-j+1)} + \\
&\quad \sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k-j+1)} + \\
&\quad \int_0^t \sum_{\chi=0}^{\infty} \lambda^{\chi} \frac{(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k)} g(u)\mathrm{d}u\mathrm{d}s\Big]^{(k-i)} \\
&= \sum_{j=1}^{l} y_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma+i-j+1)} + \\
&\quad \sum_{j=1}^{p} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma+i-j+1)} + \\
&\quad \sum_{j=1}^{N} x_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i-j+1)} + \\
&\quad \int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i)} g(u)\mathrm{d}u.
\end{aligned}$$

It follows that

$$\begin{aligned}
D_{0^+}^{\gamma}x(t) = &\sum_{j=1}^{l} y_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma-j+1)} + \\
&\sum_{j=1}^{p} z_j \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi} t^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma-j+1)} +
\end{aligned}$$

$$\sum_{j=1}^{N}x_j\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-j+1)}+$$
$$\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma)}g(u)\mathrm{d}u, \qquad (2.10)$$

and

$$[D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-i}]x(t)=\sum_{j=1}^{l}y_j\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-\theta-j+1)}+$$
$$\sum_{j=1}^{p}z_j\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-\alpha-j+1)}+$$
$$\sum_{j=i+1}^{N}x_j\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$
$$\sum_{j=1}^{i}x_j\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i-j+1)}+$$
$$\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+i-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+i)}g(u)\mathrm{d}u-$$
$$\lambda\Big[\sum_{j=1}^{l}y_j\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma+i-j+1)}+$$
$$\sum_{j=1}^{p}z_j\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma+i-j+1)}+$$
$$\sum_{j=1}^{N}x_j\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}t^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i-j+1)}+$$
$$\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+i)}g(u)\mathrm{d}u\Big]$$
$$=\sum_{j=1}^{i}\frac{x_jt^{i-j}}{\Gamma(i-j+1)}+\int_0^t\frac{(t-u)^{i-1}}{\Gamma(i)}g(u)\mathrm{d}u,\quad i\in\mathbb{N}_1^{\min\{k,n\}}. \qquad (2.11)$$

It follows from (2.8) and (2.10) that

$$D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t)-\lambda D_{0^+}^{\gamma}x(t)=g(t),\quad \text{a.e.}, t\in(0,1].$$

The fractional differential equation in (2.1) is satisfied. We note

$$[D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-i}]x(t)=\begin{cases}D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0),\ i\in\mathbb{N}_{k+1}^{n}, n>k,\\ [D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-i}]x(0),\quad i\in\mathbb{N}_1^{n}, n=k,\\ -\lambda D_{0^+}^{\gamma-i}x(0),\quad i\in\mathbb{N}_{n+1}^{k}, k>n.\end{cases}$$

From (2.11), we have

$$[D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}-\lambda D_{0^+}^{\gamma-i}]x(0)=x_i, i\in\mathbb{N}_1^{\min\{k,n\}}. \qquad (2.12)$$

If $k > n$, we have $N = k$. Then

$$-\lambda I_{0^+}^{k-\gamma}x(0) = x_k,\ -\lambda D_{0^+}^{\gamma-i}x(0) = x_i, i \in \mathbb{N}_{n+1}^{k}. \tag{2.13}$$

If $k < n$, we have $N = n$. we get

$$I_{0^+}^{n-\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0) = x_n,\ D_{0^+}^{\alpha-i}D_{0^+}^{\theta}D_{0^+}^{\beta}x(0) = x_i, i \in \mathbb{N}_{k+1}^{n-1}. \tag{2.14}$$

It is easy to see from (2.5), (2.7), (2.12), (2.13) and (2.14) that all initial conditions in (2.1) are satisfied. Then we get (2.1). The proof is completed. □

3. Piecewise continuous solutions of IFDE

Suppose that n, l, p, k are positive integers, $\lambda \in \mathbb{R}$, $\alpha \in (n-1, n)$, $\beta \in (l-1, l)$, $\theta \in (p-1, p)$, $\gamma \in (k-1, k)$ with $k \le \alpha + \theta + \beta$ in this subsection. We seek piecewise continuous solutions of the linear Langevin fractional differential equation with impulse effects (ILFDE for short)

$$D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t) - \lambda D_{0^+}^{\gamma}x(t) = g(t), \quad \text{a.e., } t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0^m. \tag{3.1}$$

Theorem 3.1 *Suppose that (i)-(iv) hold. Then x is a solution of (3.1) if and only if there exist constants $c_{\nu,j} \in \mathbb{R}(\nu \in \mathbb{N}_0^m, j \in \mathbb{N}_1^m), b_{\nu,i}(\nu \in \mathbb{N}_0^m, i \in \mathbb{N}_1^p), d_{\nu,i} \in \mathbb{R}(\nu \in \mathbb{N}_0^m, i \in \mathbb{N}_1^{\max\{n,l\}})$ such that*

$$\begin{aligned}x(t) = &\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}(t-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\ &\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}(t-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\ &\sum_{\nu=0}^{v}\sum_{i=1}^{N} d_{\nu,i}(t-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\ &\int_0^t (t-s)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(t-s)^{\alpha+\theta+\beta-\gamma})g(s)\mathrm{d}s, \quad t \in (t_v, t_{v+1}],\ v \in \mathbb{N}_0^m.\end{aligned} \tag{3.2}$$

Proof Step 1. Suppose that x is a solution of (3.2). We prove that x is a piecewise continuous solution of (3.1).

Since $g \in C[0,1]$, we know for $t \in (0,1]$ and $j \in \mathbb{N}_0^m$ that

$$\begin{aligned}&(t-t_j)^{l-\beta}\Big|\int_0^t (t-s)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(t-s)^{\alpha+\theta+\beta-\gamma})g(s)\mathrm{d}s\Big|\\ &\le (t-t_j)^{l-\beta}\int_0^t (t-s)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(|\lambda|)\|g\|\mathrm{d}s\\ &\le \|g\|(t-t_j)^{l-\beta}\int_0^t (t-s)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(|\lambda|)\mathrm{d}s\\ &= \|g\|(t-t_j)^{l-\beta}\frac{t^{\alpha+\theta+\beta}}{\alpha+\theta+\beta+1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(|\lambda|).\end{aligned}$$

Then $x|_{(t_j,t_{j+1}]} \in C(t_j, t_{j+1}]$ and $\lim_{t\to t_j^+}(t-t_j)^{l-\beta}x(t)$ is finite.

From (3.2), for $t\in(t_v,t_{v+1}](v\in\mathbb{N}_0^m)$ and Remark 2.2, we have

$$
\begin{aligned}
D_{0+}^{\gamma}x(t)&=\Big[\int_0^t\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}x(s)\mathrm{d}s\Big]^{(k)}\\
&=\Big[\sum_{o=0}^{v-1}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}x(s)\mathrm{d}s+\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}x(s)\mathrm{d}s\Big]^{(k)}\\
&=\Big[\sum_{o=0}^{v-1}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\Big(\sum_{\nu=0}^{o}\sum_{j=1}^{l}c_{\nu,j}(s-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\quad\sum_{\nu=0}^{o}\sum_{j=1}^{p}b_{\nu,j}(s-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\quad\sum_{\nu=0}^{o}\sum_{i=1}^{N}d_{\nu,i}(s-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\quad\int_0^s(s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\Big)\mathrm{d}s\Big]^{(k)}+\\
&\quad\Big[\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\Big(\sum_{\nu=0}^{v}\sum_{j=1}^{l}c_{\nu,j}(s-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\quad\sum_{\nu=0}^{v}\sum_{j=1}^{p}b_{\nu,j}(s-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\quad\sum_{\nu=0}^{v}\sum_{i=1}^{N}d_{\nu,i}(s-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\quad\int_0^s(s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\Big)\mathrm{d}s\Big]^{(k)}\\
&=\Big[\sum_{\nu=0}^{v-1}\sum_{o=\nu}^{v-1}\sum_{j=1}^{l}c_{\nu,j}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\quad\sum_{\nu=0}^{v-1}\sum_{o=\nu}^{v-1}\sum_{j=1}^{p}b_{\nu,j}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\quad\sum_{\nu=0}^{v-1}\sum_{o=\nu}^{v-1}\sum_{i=1}^{N}d_{\nu,i}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\quad\sum_{o=0}^{v-1}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\int_0^s(s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\mathrm{d}s\Big]^{(k)}+\\
&\quad\Big[\sum_{\nu=0}^{v}\sum_{j=1}^{l}c_{\nu,j}\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\quad\sum_{\nu=0}^{v}\sum_{j=1}^{p}b_{\nu,j}\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\quad\sum_{\nu=0}^{v}\sum_{i=1}^{N}d_{\nu,i}\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\quad\int_{t_v}^{t}\int_0^s\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\mathrm{d}s\Big]^{(k)}\\
&=\Big[\sum_{\nu=0}^{v-1}\sum_{j=1}^{l}c_{\nu,j}\int_{t_\nu}^{t_v}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\nu=0}^{v-1}\sum_{j=1}^{p} b_{\nu,j}\int_{t_\nu}^{t_v}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\sum_{\nu=0}^{v-1}\sum_{i=1}^{N} d_{\nu,i}\int_{t_\nu}^{t_v}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\int_0^{t_v}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\int_0^s(s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\mathrm{d}s\Big]^{(k)}+\\
&\Big[\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\sum_{\nu=0}^{v}\sum_{i=1}^{N} d_{\nu,i}\int_{t_v}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\int_{t_v}^{t}\int_0^s\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\mathrm{d}s\Big]^{(k)}\\
=&\Big[\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}\int_{t_\nu}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}\int_{t_\nu}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\sum_{\nu=0}^{v}\sum_{i=1}^{N} d_{\nu,i}\int_{t_\nu}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-t_\nu)^{\alpha+\theta+\beta-i}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-i+1}(\lambda(s-t_\nu)^{\alpha+\theta+\beta-\gamma})\mathrm{d}s+\\
&\int_0^{t}\int_0^s\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}(s-u)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(s-u)^{\alpha+\theta+\beta-\gamma})g(u)\mathrm{d}u\mathrm{d}s\Big]^{(k)}\\
=&\Big[\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}\sum_{\chi=0}^{\infty}\lambda^\chi\int_{t_\nu}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\frac{(s-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-j+1)}\mathrm{d}s+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}\sum_{\chi=0}^{\infty}\lambda^\chi\int_{t_\nu}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\frac{(s-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j+1)}\mathrm{d}s+\\
&\sum_{\nu=0}^{v}\sum_{i=1}^{N} d_{\nu,i}\sum_{\chi=0}^{\infty}\lambda^\chi\int_{t_\nu}^{t}\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\frac{(s-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-i}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-i+1)}\mathrm{d}s+\\
&\sum_{\chi=0}^{\infty}\lambda^\chi\int_0^t\int_s^t\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}\frac{(s-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta)}\mathrm{d}sg(u)\mathrm{d}u\Big]^{(k)}\\
=&\Big[\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}\sum_{\chi=0}^{\infty}\lambda^\chi\frac{(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma+k-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}\sum_{\chi=0}^{\infty}\lambda^\chi\frac{(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma+k-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{i=1}^{N} d_{\nu,i}\sum_{\chi=0}^{\infty}\lambda^\chi\frac{(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k-i}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k-i+1)}+\\
&\sum_{\chi=0}^{\infty}\lambda^\chi\int_0^t\frac{(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma+k)}g(u)\mathrm{d}u\Big]^{(k)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma-j+1)}+\\
&+\sum_{\nu=0}^{v}\sum_{j=1}^{N} d_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-j+1)}+\\
&\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma)}g(u)\mathrm{d}u
\end{aligned}
$$

and for $t\in(t_v,t_{v+1}]$ that

$$
\begin{aligned}
D_{0^+}^{\beta}x(t) =&\Big[\int_0^t\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}x(s)\mathrm{d}s\Big]^{(l)}\\
=&\Big[\sum_{o=0}^{v-1}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}x(s)\mathrm{d}s+\int_{t_v}^{t}\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}x(s)\mathrm{d}s\Big]^{(l)}\\
=&\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{N} d_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j+1)}+\\
&\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta)}g(u)\mathrm{d}u.
\end{aligned}
$$

Similarly we have for $t\in(t_v,t_{v+1}]$ ($v\in\mathbb{N}_0^m$) that

$$
\begin{aligned}
D_{0^+}^{\theta}D_{0^+}^{\beta}x(t) =& \Big[\int_0^t\frac{(t-s)^{p-\theta-1}}{\Gamma(p-\theta)}D_{0^+}^{\beta}x(s)\mathrm{d}s\Big]^{(p)}\\
=&\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\theta-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{N} d_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha-j+1)}+\\
&\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha)}g(u)\mathrm{d}u.
\end{aligned}
$$

Finally, we have for $t\in(t_v,t_{v+1}]$ that

$$
D_{0^+}^{\alpha}D_{0^+}^{\theta}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t) = \Big[\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}D_{0^+}^{\theta}D_{0^+}^{\beta}x(s)\mathrm{d}s\Big]^{(n)}
$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j} \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-\alpha-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\alpha-\theta-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j} \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\alpha-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{N} d_{\nu,j} \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi}(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+\\
&g(t)+\int_0^t \sum_{\chi=1}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi(\alpha+\theta+\beta-\gamma)-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma))} g(u)\mathrm{d}u.
\end{aligned}
$$

Then

$$
D_{0^+}^{\alpha} D_{0^+}^{\theta} D_{0^+}^{\beta} x(t) - \lambda D_{0^+}^{\gamma} x(t) = g(t), \quad t \in (t_v, t_{v+1}],\ v \in \mathbb{N}_0^m.
$$

It follows that x is a piecewise continuous solution of (3.1).

Step 2. We prove that x satisfies (3.2) if x is a piecewise continuous solution of (3.1).

For $t \in (t_0, t_1]$, we have from Lemma 2.7 that there exist constants $c_{0,j}, b_{0,j}, d_{0,i}$ such that

$$
\begin{aligned}
x(t) =&\sum_{j=1}^{l} c_{0,j} t^{\beta-j} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda t^{\alpha+\theta+\beta-\gamma})\\
&\sum_{j=1}^{p} b_{0,j} t^{\theta+\beta-j} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+\\
&\sum_{j=1}^{N} d_{0,j} t^{\alpha+\theta+\beta-j} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-j+1}(\lambda t^{\alpha+\theta+\beta-\gamma})+\\
&\int_0^t (t-s)^{\alpha+\theta+\beta-1} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(t-s)^{\alpha+\theta+\beta-\gamma}) g(s)\mathrm{d}s, \quad t \in (t_0, t_1].
\end{aligned}
$$

We know (3.2) holds for $v = 0$. Suppose that (3.2) holds for $v = 0, 1, 2, \ldots, \mu$, i.e.,

$$
\begin{aligned}
x(t) =&\sum_{\nu=0}^{v}\sum_{j=1}^{l} c_{\nu,j} (t-t_\nu)^{\beta-j} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p} b_{\nu,j} (t-t_\nu)^{\theta+\beta-j} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{N} d_{\nu,j} (t-t_\nu)^{\alpha+\theta+\beta-j} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+\\
&\int_0^t (t-s)^{\alpha+\theta+\beta-1} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(t-s)^{\alpha+\theta+\beta-\gamma}) g(s)\mathrm{d}s, \quad t \in (t_v, t_{v+1}],\ v \in \mathbb{Z}_0^{\mu}.
\end{aligned}
$$

We will prove that (3.2) holds for $v = \mu + 1$. Then by mathematical induction method, we see that (3.2) holds for all $v \in \mathbb{N}_0^m$.

In fact, we suppose that

$$
x(t) = \Phi(t) + \sum_{\nu=0}^{\mu}\sum_{j=1}^{l} c_{\nu,j} (t-t_\nu)^{\beta-j} \mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+
$$

$$\sum_{\nu=0}^{\mu}\sum_{j=1}^{p}b_{\nu,j}(t-t_\nu)^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+$$
$$\sum_{\nu=0}^{\mu}\sum_{j=1}^{N}d_{\nu,j}(t-t_\nu)^{\alpha+\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-j+1}(\lambda(t-t_\nu)^{\alpha+\theta+\beta-\gamma})+$$
$$\int_0^t(t-s)^{\alpha+\theta+\beta-1}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta}(\lambda(t-s)^{\alpha+\theta+\beta-\gamma})g(s)\mathrm{d}s,\quad t\in(t_{\mu+1},t_{\mu+2}]. \tag{3.3}$$

We seek $\Phi(t), t\in(t_{\mu+1},t_{\mu+2}]$. Then for $t\in(t_{\mu+1},t_{\mu+2}]$ we have

$$\begin{aligned}D_{0^+}^{\gamma}x(t)=&\Big[\int_0^t\frac{(t-s)^{k-\gamma-1}}{\Gamma(k-\gamma)}x(s)\mathrm{d}s\Big]^{(k)}\\
=&\sum_{\nu=0}^{\mu+1}\sum_{j=1}^{l}c_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\beta-\gamma-j+1)}+\\
&\sum_{\nu=0}^{\mu+1}\sum_{j=1}^{p}b_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta+\beta-\gamma-j+1)}+\\
&\sum_{\nu=0}^{\mu+1}\sum_{j=1}^{N}d_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-j+1)}+\\
&\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta+\beta-\gamma)}g(u)\mathrm{d}u+D_{t_{\mu+1}^+}^{\gamma}\Phi(t)\end{aligned}$$

and for $t\in(t_v,t_{v+1}]$ $(v\in\mathbb{N}_0^\mu)$ that

$$\begin{aligned}D_{0^+}^{\beta}x(t)=&\Big[\int_0^t\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}x(s)\mathrm{d}s\Big]^{(l)}\\
=&\Big[\sum_{o=0}^{v-1}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}x(s)\mathrm{d}s+\int_{t_v}^{t}\frac{(t-s)^{l-\beta-1}}{\Gamma(l-\beta)}x(s)\mathrm{d}s\Big]^{(l)}\\
=&\sum_{\nu=0}^{v}\sum_{j=1}^{l}c_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{p}b_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\theta-j+1)}+\\
&\sum_{\nu=0}^{v}\sum_{j=1}^{N}d_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-j+1)}+\\
&\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha+\theta)}g(u)\mathrm{d}u+D_{t_{\mu+1}^+}^{\beta}\Phi(t).\end{aligned}$$

Then for $t\in(t_v,t_{v+1}]$ $(v\in\mathbb{N}_0^\mu)$ we have

$$\begin{aligned}D_{0^+}^{\theta}D_{0^+}^{\beta}x(t)=&\Big[\int_0^t\frac{(t-s)^{p-\theta-1}}{\Gamma(p-\theta)}D_{0^+}^{\beta}x(s)\mathrm{d}s\Big]^{(p)}\\
=&\sum_{\nu=0}^{v}\sum_{j=1}^{l}c_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\theta-j+1)}+\end{aligned}$$

$$\sum_{\nu=0}^{v}\sum_{j=1}^{p}b_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+$$
$$\sum_{\nu=0}^{v}\sum_{j=1}^{N}d_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha-j+1)}+$$
$$\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-u)^{\chi(\alpha+\theta+\beta-\gamma)+\alpha-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)+\alpha)}g(u)\mathrm{d}u+D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\nu+1}^+}\Phi(t).$$

Finally, we have for $t\in(t_{\mu+1},t_{\mu+2}]$ that

$$D^\alpha_{0^+}D^\theta_{0^+}D^\beta_{0^+}x(t)=\Big[\int_0^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}D^\theta_{0^+}D^\beta_{0^+}x(s)\mathrm{d}s\Big]^{(n)}$$
$$=\Big[\sum_{o=0}^{\mu}\int_{t_o}^{t_{o+1}}\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}D_{0^+}\theta D^\beta_{0^+}x(s)\mathrm{d}s+\int_{t_{\mu+1}}^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}D^\theta_{0^+}D^\beta_{0^+}x(s)\mathrm{d}s\Big]^{(n)}$$
$$=\sum_{\nu=0}^{\mu+1}\sum_{j=1}^{l}c_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-\alpha-\theta-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\alpha-\theta-j+1)}+$$
$$\sum_{\nu=0}^{\mu+1}\sum_{j=1}^{p}b_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-\alpha-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-\alpha-j+1)}+$$
$$\sum_{\nu=0}^{\mu+1}\sum_{j=1}^{N}d_{\nu,j}\sum_{\chi=1}^{\infty}\frac{\lambda^\chi(t-t_\nu)^{\chi(\alpha+\theta+\beta-\gamma)-j}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma)-j+1)}+$$
$$g(t)+\int_0^t\sum_{\chi=1}^{\infty}\frac{\lambda^\chi(t-u)^{\chi(\alpha+\theta+\beta-\gamma)-1}}{\Gamma(\chi(\alpha+\theta+\beta-\gamma))}g(u)\mathrm{d}u+D^\alpha_{t_{\mu+1}^+}D^\beta_{t_{\mu+1}^+}\Phi(t).$$

Then (3.1) implies that

$$\begin{aligned}g(t)=&D^\alpha_{0^+}D^\theta_{0^+}D^\beta_{0^+}x(t)-\lambda D^\gamma_{0^+}x(t)\\=&g(t)+D^\alpha_{t_{\mu+1}^+}D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\mu+1}^+}\Phi(t)-\lambda D^\gamma_{t_{\mu+1}^+}\Phi(t),\quad t\in(t_{\mu+1},t_{\mu+2}].\end{aligned}$$

So

$$D^\alpha_{t_{\nu+1}^+}D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\nu+1}^+}\Phi(t)-\lambda D^\gamma_{t_{\nu+1}^+}x(t)=0,\quad t\in(t_{\nu+1},t_{\nu+2}].\tag{3.4}$$

By computing the following derivatives and integrals directly, we also find

$$I^{l-\beta}_{t_{\mu+1}^+}\Phi(t_{\nu+1}),\ D^{\beta-i}_{t_{\mu+1}^+}\Phi(t_{\nu+1}),i\in\mathbb{N}_1^{l-1},$$

$$I^{p-\theta}_{t_{\mu+1}^+}D^\beta_{t_{\nu+1}^+}\Phi(t_{\nu+1}),\ D^{\beta-i}_{t_{\mu+1}^+}D^\beta_{t_{\nu+1}^+}\Phi(t_{\nu+1}),\quad i\in\mathbb{N}_1^{p-1},$$

$$\begin{cases}(I^{n-\alpha}_{t_{\mu+1}^+}D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\mu+1}^+}-\lambda I^{k-\gamma}_{t_{\mu+1}^+})\Phi(t_{\nu+1}),\\(D^{\alpha-j}_{t_{\mu+1}^+}D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\mu+1}^+}-\lambda D^{\gamma-j}_{t_{\mu+1}^+})\Phi(t_{\nu+1}),j\in\mathbb{N}_1^{k-1},\end{cases}\quad k=n,\tag{3.5}$$

$$\begin{cases}(D^{\alpha-k}_{t_{\mu+1}^+}D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\mu+1}^+}-\lambda I^{k-\gamma}_{t_{\mu+1}^+})\Phi(t_{\nu+1}),\\(D^{\alpha-j}_{t_{\mu+1}^+}D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\mu+1}^+}-\lambda D^{\gamma-j}_{t_{\mu+1}^+})\Phi(t_{\nu+1}),j\in\mathbb{N}_1^{k-1},\\D^{\alpha-j}_{t_{\mu+1}^+}D^\theta_{t_{\nu+1}^+}D^\beta_{t_{\mu+1}^+}\Phi(t_{\nu+1}),j\in\mathbb{N}_{k+1}^{n},\end{cases}\quad k<n,$$

$$\left\{\begin{array}{ll} (I^{n-\alpha}_{t^+_{\mu+1}} D^{\theta}_{t^+_{\nu+1}} D^{\beta}_{t^+_{\mu+1}} - \lambda I^{n-\gamma}_{t^+_{\mu+1}})\Phi(t_{\nu+1}), & \\ (D^{\alpha-j}_{t^+_{\mu+1}} D^{\theta}_{t^+_{\nu+1}} D^{\beta}_{t^+_{\mu+1}} - \lambda D^{\gamma-j}_{t^+_{\mu+1}})\Phi(t_{\nu+1}), j\in\mathbb{N}_1^{n-1}, & k>n \\ -\lambda D^{\gamma-j}_{t^+_{\mu+1}}\Phi(t_{\nu+1}), j\in\mathbb{N}_{n+1}^{k}, & \end{array}\right.$$

are finite.

By using the similar method in the proof of Lemma 2.7 and (3.4), (3.5), there exist constants $c_{\mu+1j}$, $b_{\mu+1,j}$, $d_{\mu+1,i}\in\mathbb{R}$ such that

$$\begin{aligned}\Phi(t) =& \sum_{j=1}^{l} c_{\mu+1,j}(t-t_{\mu+1})^{\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\beta-j+1}(\lambda(t-t_{\mu+1})^{\alpha+\beta-\gamma})+ \\ & \sum_{j=1}^{p} b_{\mu+1,j}(t-t_{\mu+1})^{\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\theta+\beta-j+1}(\lambda(t-t_{\mu+1})^{\alpha+\theta+\beta-\gamma})+ \\ & \sum_{j=1}^{N} d_{\mu+1,j}(t-t_{\nu})^{\alpha+\theta+\beta-j}\mathbf{E}_{\alpha+\theta+\beta-\gamma,\alpha+\theta+\beta-j+1}(\lambda(t-t_{\mu+1})^{\alpha+\theta+\beta-\gamma}),\quad t\in(t_{\mu+1},t_{\mu+2}].\end{aligned}$$

Substituting Φ into (3.3), we get that (3.2) holds for $v=\nu+1$. By mathematical induction method, we know that (3.2) holds for all $v\in\mathbb{N}_0^m$. So x satisfies (3.2) if x is a piecewise continuous solution of (3.1). The proof is completed. □

4. Conclusions

We obtain new general solutions of a class of singular impulsive multi-order fractional differential equations involving the Riemann-Liouville fractional derivatives. The methods used are standard, however their exposition in the framework of such kind of problems is new and skillful.

The author strongly believes that the article will highly be appreciated by the researchers working in the field of impulsive fractional calculus and be helpful for study on the boundary value problems for impulsive fractional differential equations involving Riemann-Liouville fractional derivatives and in the nonlinear area and the numerical simulation, especially for study in the solvability of boundary value problems, initial value problems or numerical solutions of boundary value problems for impulsive fractional differential equation involving the Rienmann-Liouville fractional derivatives.

References

[1] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[2] K. B. OLDHAM, C. G. ZOSKI. *The Fractional Calculus*. Academic Press, New York, London, 1974.

[3] I. PODLUBNY. *Fractional Differential Equations*. Academic Press, New York London Toroto, 1999.

[4] J. SABATIER, O. P. AGRAWAL, J. A. T. MACHADO. *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*. Springer, Dordrecht, 2007.

[5] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV. *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach Science Publishers, Yverdon, 1993.

[6] R. P. AGARWAL, S. HIRSTOVA, D. O'REGAN. *Stability of solutions to impulsive Caputo fractional differential equations*. Electron. J. Differential Equations, 2016, **58**: 1–22.

[7] R. P. AGARWAL, M. BENCHOHRA, S. HAMANI. *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*. Acta Appl. Math., 2010, **109**(3): 973–1033.

[8] R. P. AGARWAL, M. BENCHOHRA, B. A. SLIMANI. *Existence results for differential equations with fractional order and impulses*. Mem. Differential Equations Math. Phys., 2008, **44**: 1–21.

[9] Yuji LIU. *On piecewise continuous solutions of higher order impulsive fractional differential equations and applications*. Appl. Math. Comput., 2016, **287**: 38–49.

[10] M. FEČKAN, Yong ZHOU, Jinrong WANG. *On the concept and existence of solution for impulsive fractional differential equations*. Commun. Nonlinear Sci. Numer. Simul., 2012, **17**(7): 3050–3060.

[11] M. FEČKAN, Yong ZHOU, Jinrong WANG. *Response to "Comments on the concept of existence of solution for impulsive fractional differential equations"*. Commun. Nonlinear Sci. Numer. Simul., 2014, **19**(12): 4213–4215.

[12] F. JARAD, T. ABDELJAWAD, D. BALEANU. *Caputo type modification of the Hadamard fractional derivatives*. Adv. Difference Equ., 2012, **142**: 1–8.

[13] Yuji LIU, Shimin LI. *Periodic boundary value problems of singular fractional differential equations with impulse effects*. Malaya Journal of Matematik, 2015, **3**(4): 423–490.

[14] Guotao WANG, B. AHMAD, Lihong ZHANG, et al. *Comments on the concept of existence of solution for impulsive fractional differential equations*. Commun. Nonlinear Sci. Numer. Simul., 2014, **19**(3): 401–403.

[15] Jinrong WANG, Yong ZHOU, M. FECKAN. *On recent developments in the theory of boundary value problems for impulsive fractional differential equations*. Comput. Math. Appl., 2012, **64**(10): 3008–3020.

[16] Guotao WANG, Lihong ZHANG, Guangxing SONG. *Systems of first order impulsive functional differential equations with deviating arguments and nonlinear boundary conditions*. Nonlinear Anal., 2011, **74**(3): 974–982.

[17] Y. GAMBO, F. JARAD, D. BALEANU, et al. *On Caputo modification of the Hadamard fractional derivatives*. Adv. Difference Equ., 2014, **2014**(10): 1–12.

[18] Yuji LIU. *Existence of solutions of BVPs for a class of IFDSs on half line involving Hadamard fractional derivatives*. J. Nonlinear Funct. Anal., 2016, **2016**: Article ID 26.

[19] Guotao WANG, Sanyang LIU, D. BALEANU, et al. *A new impulsive multi-orders fractional differential equationinvolving multi point fractional integral boundary conditions*. Abstr. Appl. Anal. 2014, **2014**: Article ID932747, 10 pages.

[20] Jinrong WANG, Yong ZHOU, M. FEČKAN. *On recent developments in the theory of boundary value problems for impulsive fractional differential equations*. Comput. Math. Appl., 2012, **64**(10): 3008–3020.

[21] Jinrong WANG, Yong ZHOU. *On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives*. Appl. Math. Lett., 2015, **39**: 85–90.

[22] W. YUKUNTHORN, B. AHMAD, S. K. NTOUYAS, et al. *On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions*. Nonlinear Anal. Hybrid Syst., 2016, **19**: 77–92.

[23] W. YUKUNTHORN, S. SUANTAI, S. K. NTOUYAS, et al. *Boundary value problems for impulsive multi-order Hadamard fractional differential equations*. Bound. Value Probl., 2015, **2015**(148): 1–13.