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Extensions of Some Sufficient Conditions for Starlikeness and Convexity of Order β

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Abstract The aim of the present paper is to investigate some sufficient conditions for starlikeness and convexity of two new operators $\mathcal{E}^{\gamma}_{\alpha,\lambda}$ and $H^l_m(\alpha_1)$ related to the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\lambda}$ and the generalized hypergeometric function, defined in the unit disk, respectively. The results presented here make connections to some of the earlier known developments.

Keywords Analytic functions; starlike function; convexity function; generalized Mitttag-Leffler function; generalized hypergeometric function

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1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathbb{U}$. Let $\mathcal{S}^*(\beta), K(\beta)$ denote the subclass of $\mathcal{A}$ consisting of f which satisfy

$$\Re(\frac{zf'(z)}{f(z)}) > \beta \text{ and } \Re(1 + \frac{zf''(z)}{f'(z)}) > \beta$$

for some real β ($0 \le \beta < 1$), respectively. A function $f \in K(\beta)$ is said to be convex of order β in $\mathbb{U}$ and $f \in K(\beta)$ is said to be starlike of order β in $\mathbb{U}$. Note that $f \in K(\beta)$ if and only if $zf'(z) \in \mathcal{S}^*(\beta)$, and that $f \in \mathcal{S}^*(\beta)$ if and only if $\int_0^z \frac{f(t)}{t}\mathrm{d}t \in K(\beta)$. We say that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $K(0) \equiv K$.

1.1. The generalized Mittag-Leffler

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A two-parameter Mittag-Leffler function (see [1,2]) is defined as

$$E_{\alpha,\lambda}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\lambda)},\quad z\in\mathbb{U},\alpha,\lambda\in\mathbb{C},\Re\alpha>0, \tag{1.2}$$

where $\Gamma(\cdot)$ denotes the Gamma function. For $\lambda=1$, its one-parameter form [3] is shown as

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}=E_{\alpha,1}(z),\quad z\in\mathbb{U},\alpha\in\mathbb{C},\Re\alpha>0. \tag{1.3}$$

The three-parametric Mittag-Leffler function (or Prabhakar function)

$$E^{\gamma}_{\alpha,\lambda}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\lambda)}\frac{z^n}{n!},\quad z\in\mathbb{U},\alpha,\lambda,\gamma\in\mathbb{C},\Re\alpha>0,\Re\gamma>0, \tag{1.4}$$

introduced by Prabhakar [4] where

$$(\mu)_n=\begin{cases}1, & n=0\\ \mu(\mu+1)\cdots(\mu+n-1), & n\in\mathbb{N}\end{cases}$$

is the well-known Pochhammer symbol.

Note that

$$E^{1}_{\alpha,\lambda}(z)=E_{\alpha,\lambda}(z)\ \text{ and }\ E^{1}_{\alpha,1}(z)=E_{\alpha}(z).$$

In virtue of (1.4), the recurrence relations for the function $E^{\gamma}_{\alpha,\lambda}(z)$ can be easily established

$$\gamma E^{\gamma+1}_{\alpha,\lambda}(z)=\gamma E^{\gamma}_{\alpha,\lambda}(z)+z(E^{\gamma}_{\alpha,\lambda}(z))',\quad z\in\mathbb{U}. \tag{1.5}$$

Having a glimpse of the recent research works, we may find variety of vast potential of applications associated with Mittag-Leffler type functions. The applications can be related to applied problems, neural networks, probability, statistical distribution theory, etc. For a detailed account of properties, generalizations and applications of Mittag-Leffler function, one may refer to the works [5–13].

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function $E_{\alpha,\lambda}(z)$ were recently investigated by Bansal and Prajapat [14]. Certain results on partial sums of the function $E_{\alpha,\lambda}(z)$ were also obtained in [15]. Second-order and third-order differential subordination and superordination results for functions $E^{\gamma}_{\alpha,\lambda}(z)$ were recently investigated by [16,17].

However, the Mittag-Leffler defined by (1.4) does not belong to the class $\mathcal{A}$. Therefore, we consider the following modified Mittag-Leffler function

$$\mathbb{E}^{\gamma}_{\alpha,\lambda}(z)=\Gamma(\lambda)zE_{\alpha,\lambda}(z)=z+\sum_{n=1}^{\infty}\frac{\Gamma(\lambda)(\gamma)_n}{\Gamma(\alpha n+\lambda)}\frac{z^{n+1}}{n!},\quad \alpha,\lambda,\gamma\in\mathbb{C},z\in\mathbb{U},\Re\alpha>0,\Re\gamma>0. \tag{1.6}$$

Whilst the above mentioned formula defined by (1.4) holds for complex-valued parameters α,λ,γ and $z\in\mathbb{C}$, yet throughout the paper we shall restrict our attention to the case of real-valued parameters α,λ,γ with $\alpha,\lambda,\gamma>0$ and $z\in\mathbb{U}$.

Observing that the modified version Mittag-Leffler function $\mathbb{E}^{\gamma}_{\alpha,\lambda}$ contains such well-known functions as its special cases

$$\begin{cases} \mathbb{E}^1_{1,1}(z) = ze^z, & \mathbb{E}^2_{1,1}(z) = ze^z(z+1), \\ \mathbb{E}^1_{2,1}(z) = z\cosh(\sqrt{z}), & \mathbb{E}^2_{2,1}(z) = z\cosh(\sqrt{z}) + \frac{1}{2}z\sqrt{z}\sinh(\sqrt{z}), \\ \mathbb{E}^1_{2,2}(z) = \sqrt{z}\sinh(\sqrt{z}) - 2, & \mathbb{E}^2_{2,2}(z) = \frac{1}{2}(\sqrt{z}\sinh(\sqrt{z}) + z\cosh(\sqrt{z}), \\ \mathbb{E}^1_{2,3}(z) = 2\cosh(\sqrt{z}) - 2, & \mathbb{E}^1_{2,4}(z) = 6(\frac{\sinh(\sqrt{z})}{\sqrt{z}} - 1). \end{cases} \tag{1.7}$$

Making use of the function $\mathbb{E}^\gamma_{\alpha,\lambda}$ given by (1.6), a new operator $\mathcal{E}^{\ \gamma}_{\alpha,\lambda} : \mathcal{A} \to \mathcal{A}$ can be defined, in terms of Hadamard product, as follows

$$\mathcal{E}^\gamma_{\alpha,\lambda} f(z) = (\mathbb{E}^\gamma_{\alpha,\lambda} * f)(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\lambda)(\gamma)_n a_{n+1}}{\Gamma(\alpha n + \lambda)} \frac{z^{n+1}}{n!}, \quad z \in \mathbb{U}. \tag{1.8}$$

From definition (1.8) and the equalities (1.5), we obtain the following recurrence relations for the operator $\mathcal{E}^\gamma_{\alpha,\lambda} f(z)$

$$z(\mathcal{E}^\gamma_{\alpha,\lambda} f(z))' = \gamma \mathcal{E}^{\gamma+1}_{\alpha,\lambda} f(z) - (\gamma - 1)\mathcal{E}^\gamma_{\alpha,\lambda} f(z), \quad z \in \mathbb{U}. \tag{1.9}$$

We deduce that

$$\mathcal{E}^1_{0,1} f(z) = f(z);\ \mathcal{E}^2_{0,1} f(z) = zf'(z);\ \mathcal{E}^3_{0,1} f(z) = \frac{1}{2}z^2 f''(z) + zf'(z).$$

1.2. The generalized hypergeometric function

For

$$\alpha_j \in \mathbb{C},\ j = 1, \ldots, l \text{ and } \beta_j \in \mathbb{C} \setminus \mathbb{Z}^-_0,\ \mathbb{Z}^-_0 := \{0, -1, -2, \ldots\}; j = 1, \ldots, m,$$

the generalized hypergeometric function ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!},$$
$$l \le m + 1;\ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}.$$

Dziok and Srivastava [18] introduced the liner operator

$$H^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) :\ \ \mathcal{A} \longrightarrow \mathcal{A},$$

defined by

$$H^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := [z\,{}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)] * f(z),$$
$$l \le m + 1;\ l, m \in \mathbb{N}_0;\ z \in \mathbb{U}. \tag{1.10}$$

If $f \in \mathcal{A}$ given by (1.1), then

$$H^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} a_{n+1} \frac{z^{n+1}}{n!}, \quad n \in \mathbb{N}.$$

For convenience, write

$$H^l_m(\alpha_1) := H^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m), \quad l \le m + 1;\ l, m \in \mathbb{N}_0.$$

Remark 1.1 When $m=0$, the denominator $(\beta_1)_n\cdots(\beta_m)_n$ can be seen as no appearance. We can easily get

$$H_0^1(1)f(z)=f(z);\ H_0^1(2)f(z)=zf'(z);\ H_0^1(3)f(z)=\frac{1}{2}z^2f''(z)+zf'(z).$$

By view of (1.10), we can deduce that

$$z(H_m^l(\alpha_1)f)'(z)=\alpha_1H_m^l(\alpha_1+1)f(z)-(\alpha_1-1)H_m^l(\alpha_1)f(z). \tag{1.11}$$

In recent years, some researchers have obtained many interesting results related to the Dziok-Srivastava operator [19–24]. In fact, the Dziok-Srivastava linear operator includes some special cases of linear operator [25–29].

In the present paper, we obtain certain sufficient conditions for two kinds of operators $H_m^l(\alpha_1)f(z)\in\mathcal{S}^*(\beta)$, $H_m^l(\alpha_1)f(z)\in K(\beta)$ and $\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)\in\mathcal{S}^*(\beta)$, $\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)\in K(\beta)$. The results present here generalizes some earlier well-known interesting works.

To prove our main results, we will need the following lemmas.

Lemma 1.2 ([30]) *If $f\in\mathcal{A}$ and satisfies $|f'(z)-1|<\frac{2}{\sqrt{5}}$ $(z\in\mathbb{U})$, then $f\in S^*$.*

Lemma 1.3 ([30]) *If $f\in\mathcal{A}$ and satisfies $|\arg f'(z)|<\frac{\pi}{2}\delta$ $(z\in\mathbb{U})$, then $f\in S^*$, where $\delta=0.6165\cdots$ is the unique root of the equation $2\tan^{-1}(1-\delta)+(1-2\delta)\pi=0$.*

Lemma 1.4 ([31]) *If $f\in\mathcal{A}$ and satisfies $|f''(z)|<\frac{1}{\sqrt{5}}=0.4472\cdots$ $(z\in\mathbb{U})$, then $f\in K$.*

2. Main results

From an application of Lemma 1.2, we get Theorem 2.1.

Theorem 2.1 *If $f\in\mathcal{A}$ and satisfies*

$$\Big|\Big(\frac{\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)}{z}\Big)^{\frac{1}{1-\beta}}\Big(\frac{\gamma\mathcal{E}_{\alpha,\lambda}^{\gamma+1}f(z)}{\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)}-(\gamma-1)-\beta\Big)-1+\beta\Big| \tag{2.1}$$

$$<\frac{2}{\sqrt{5}}(1-\beta),\quad z\in\mathbb{U}, \tag{2.2}$$

where $0\le\beta<1$, then $\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)\in\mathcal{S}^*(\beta)$.

Proof Let $f\in\mathcal{A}$. Define the function $p(z)$ by

$$p(z)=\Big(\frac{\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)}{z^{\beta}}\Big)^{\frac{1}{1-\beta}}=z+\frac{\Gamma(\lambda)\gamma}{\Gamma(\alpha+\lambda)}\frac{a_2}{1-\beta}z^2+\cdots, \tag{2.3}$$

then $p(z)\in\mathcal{A}$, and

$$p'(z)=\frac{1}{1-\beta}\Big(\frac{\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)}{z}\Big)^{\frac{1}{1-\beta}}\times\Big(\frac{z(\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z))'}{\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)}-\beta\Big).$$

Combining (1.9), we get

$$p'(z)=\frac{1}{1-\beta}\Big(\frac{\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)}{z}\Big)^{\frac{1}{1-\beta}}\times\Big(\frac{\gamma(\mathcal{E}_{\alpha,\lambda}^{\gamma+1}f(z))'}{\mathcal{E}_{\alpha,\lambda}^{\gamma}f(z)}-(\gamma-1)-\beta\Big), \tag{2.4}$$

by means of the condition of (2.1), we get

$$|p'(z)-1| = \frac{1}{1-\beta}\Big|\Big(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z}\Big)^{\frac{1}{1-\beta}}\Big(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - (\gamma-1)-\beta\Big) - 1 + \beta\Big| < \frac{2}{\sqrt{5}}.$$

Applying Lemma 1.2 gives $p(z) \in \mathcal{S}^*$. Since

$$\frac{zp'(z)}{p(z)} = \frac{1}{1-\beta}\Big(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - (\gamma-1)-\beta\Big),$$

we have $\Re\big(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - (\gamma-1)\big) > \beta$ $(z \in \mathbb{U})$, that is $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z) \in \mathcal{S}^*(\beta)$. □

Let $\beta = \frac{1}{2}$, $\beta = 0$ in Theorem 2.1, respectively. We get following results.

Corollary 2.2 *If $f \in \mathcal{A}$ and satisfies*

$$\Big|\Big(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z}\Big)^2\Big(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - \gamma + \frac{1}{2}\Big) - \frac{1}{2}\Big| < \frac{1}{\sqrt{5}} = 0.4472\cdots, \quad z \in \mathbb{U},$$

then $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z) \in \mathcal{S}^*(\frac{1}{2})$.

Corollary 2.3 *If $f \in \mathcal{A}$ satisfies*

$$\Big|\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{z} - \frac{(\gamma-1)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z} - 1\Big| < \frac{2}{\sqrt{5}}, \quad z \in \mathbb{U},$$

then $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z) \in \mathcal{S}^*$.

Let $\alpha = 0, \lambda = \gamma = 1$ in Corollary 2.3. We get

Corollary 2.4 *If $f \in \mathcal{A}$ and satisfies*

$$\Big|\frac{\mathcal{E}^{1}_{0,1}f(z)}{z} - 1\Big| < \frac{2}{\sqrt{5}}, \quad z \in \mathbb{U},$$

then $f \in \mathcal{S}^*$.

Remark 2.5 Corollary 2.4 can be seen as Lemma 1.2, since $\mathcal{E}^{2}_{0,1}f(z) = zf'(z)$.

Remark 2.6 Let $\alpha = 0, \lambda = \gamma = 1$ in Theorem 2.1. We get [31, Theorem 2.1].

Applying Lemma 1.3, we show

Theorem 2.7 *If $f \in \mathcal{A}$ and satisfies*

$$\Big|\arg\Big(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z}\Big) + (1-\beta)\arg\Big(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - (\gamma-1)-\beta\Big)\Big| < \frac{\pi}{2}\delta(1-\beta), \quad z \in \mathbb{U}, \tag{2.5}$$

for some real β $(0 \le \beta < 1)$, then $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z) \in \mathcal{S}^*(\beta)$, where $\delta = 0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta) + (1-2\delta)\pi = 0$.

Proof Suppose that $p(z)$ is given by (2.3), then, from (2.4) we know that

$$|\arg p'(z)| = \Big|\frac{1}{1-\beta}\arg\Big(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z}\Big) + \arg\Big(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - (\gamma-1)-\beta\Big)\Big|$$

$$=\frac{1}{1-\beta}\big|\arg(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z})+(1-\beta)\arg(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}-(\gamma-1)-\beta)\big|$$

By virtue of Lemma 1.3, if the condition

$$\frac{1}{1-\beta}\big|\arg(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z})+(1-\beta)\arg(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}-(\gamma-1)-\beta)\big|<\frac{\pi}{2}\delta$$

holds, then $p(z)\in\mathcal{S}^*$, which implies that $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)\in\mathcal{S}^*(\beta)$. This completes the proof of the theorem. □

Let $\beta=\frac{1}{2},\beta=0$ in Theorem 2.7, respectively. We get following results.

Corollary 2.8 *If $f\in\mathcal{A}$ and satisfies the condition*

$$\big|\arg(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z})+\frac{1}{2}\arg(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}-\gamma+\frac{1}{2})\big|<\frac{\pi}{4}\delta,\quad z\in\mathbb{U},$$

then $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)\in\mathcal{S}^*(\frac{1}{2})$, where $\delta=0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi=0$.

Corollary 2.9 *If $f\in\mathcal{A}$ and satisfies*

$$\big|\arg(\frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z})+\arg(\frac{\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}-(\gamma-1))\big|<\frac{\pi}{2}\delta,\quad z\in\mathbb{U},$$

then $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)\in\mathcal{S}^*$, where $\delta=0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi=0$.

Let $\alpha=0,\lambda=\gamma=1$ in Corollary 2.9. We get following corollary.

Corollary 2.10 *If $f\in\mathcal{A}$ and satisfies the condition*

$$\big|\arg(\frac{\mathcal{E}^{1}_{0,1}f(z)}{z})+\arg(\frac{\mathcal{E}^{2}_{0,1}f(z)}{\mathcal{E}^{1}_{0,1}f(z)})\big|<\frac{\pi}{2}\delta,\quad z\in\mathbb{U},$$

then $f\in\mathcal{S}^*$, where $\delta=0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi=0$.

Remark 2.11 Corollary 2.10 can be seen as Lemma 1.3, since $\mathcal{E}^{1}_{0,1}f(z)=f(z)$, $\mathcal{E}^{2}_{0,1}f(z)=zf'(z)$.

Remark 2.12 Let $\alpha=0,\lambda=\gamma=1$ in Theorem 2.7. We get [31, Theorem 2.3].

Theorem 2.13 *If $f\in\mathcal{A}$ and satisfies*

$$\begin{aligned}&\big|(\frac{[\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)-(\gamma-1)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)]^{\beta}}{z})^{\frac{1}{1-\beta}}(\gamma(\gamma+1)\mathcal{E}^{\gamma+2}_{\alpha,\lambda}f(z)+\\&\quad\gamma(1-\beta-2\gamma)\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)-(\gamma-1)(1-\beta-\gamma)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z))-1+\beta\big|\\&<\frac{2}{\sqrt{5}}(1-\beta),\quad z\in\mathbb{U},\end{aligned}\tag{2.6}$$

where $0\le\beta<1$, then $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)\in K(\beta)$.

Proof Let $f \in \mathcal{A}$. Suppose that $p(z)$ has the following form

$$p(z) = \int_0^z \big((\mathcal{E}^{\gamma}_{\alpha,\lambda} f(t))'\big)^{\frac{1}{1-\beta}} \mathrm{d}t = z + \frac{\Gamma(\lambda)\gamma}{\Gamma(\alpha+\lambda)} \frac{a_2}{1-\beta} z^2 + \cdots . \tag{2.7}$$

Let

$$g(z) = zp'(z) = z\big((\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))'\big)^{\frac{1}{1-\beta}} = z + \frac{\Gamma(\lambda)\gamma}{\Gamma(\alpha+\lambda)} \frac{2a_2}{1-\beta} z^2 + \cdots ,$$

then $p(z) \in \mathcal{A}, g(z) \in \mathcal{A}$ and

$$\begin{aligned} g'(z) &= \big((\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))'\big)^{\frac{\beta}{1-\beta}} \big((\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))' + \frac{1}{1-\beta} z\big(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z)\big)''\big) \\ &= \frac{1}{1-\beta} \Big(\frac{(z(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))')^{\frac{\beta}{1-\beta}}}{z^{\frac{1}{1-\beta}}}\Big)\big((1-\beta)z(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))' + z^2(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))''\big). \end{aligned}$$

Combining (1.3), (1.10) and (1.11), we get

$$\begin{aligned} g'(z) =& \frac{1}{1-\beta} \Big(\frac{[\gamma \mathcal{E}^{\gamma+1}_{\alpha,\lambda} f(z) - (\gamma-1)\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z)]^{\beta}}{z}\Big)^{\frac{1}{1-\beta}} \times \\ & \big(\gamma(\gamma+1)\mathcal{E}^{\gamma+2}_{\alpha,\lambda} f(z) + \gamma(1-\beta-2\gamma)\mathcal{E}^{\gamma+1}_{\alpha,\lambda} f(z) - \\ & (\gamma-1)(1-\beta-\gamma)\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z)\big). \end{aligned} \tag{2.8}$$

By the condition (2.6), we derive that

$$\begin{aligned} |g'(z) - 1| =& \frac{1}{1-\beta} \Big| \Big(\frac{[\gamma \mathcal{E}^{\gamma+1}_{\alpha,\lambda} f(z) - (\gamma-1)\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z)]^{\beta}}{z}\Big)^{\frac{1}{1-\beta}} \times \\ & \big(\gamma(\gamma+1)\mathcal{E}^{\gamma+2}_{\alpha,\lambda} f(z) + \gamma(1-\beta-2\gamma)\mathcal{E}^{\gamma+1}_{\alpha,\lambda} f(z) - \\ & (\gamma-1)(1-\beta-\gamma)\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z)\big) - 1 + \beta\Big| < \frac{2}{\sqrt{5}}, \end{aligned} \tag{2.9}$$

so that $g(z) = zp'(z) \in \mathcal{S}^*$, which is equivalent to $p(z) \in K$.

Noting that

$$\frac{zp''(z)}{p'(z)} = \frac{1}{1-\beta} \frac{z(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))''}{(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))'},$$

we know

$$\Re\Big(1 + \frac{zp''(z)}{p'(z)}\Big) = \Re\Big(1 + \frac{1}{1-\beta} \frac{z(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))''}{(\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z))'}\Big) > 0, \quad z \in \mathbb{U},$$

thus, $\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z) \in K(\beta)$. Theorem 2.13 is completely proved. □

Let $\beta = 0$ in Theorem 2.13. We get following corollary.

Corollary 2.14 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\Big|\frac{\gamma(\gamma+1)\mathcal{E}^{\gamma+2}_{\alpha,\lambda} f(z) + \gamma(1-2\gamma)\mathcal{E}^{\gamma+1}_{\alpha,\lambda} f(z) + (\gamma-1)^2 \mathcal{E}^{\gamma}_{\alpha,\lambda} f(z)}{z} - 1\Big| < \frac{2}{\sqrt{5}}, \quad z \in \mathbb{U},$$

then $\mathcal{E}^{\gamma}_{\alpha,\lambda} f(z) \in K$.

If we let $\alpha = 0, \lambda = \gamma = 1$ in Corollary 2.14, we get following corollary.

Corollary 2.15 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\Big|\frac{2\mathcal{E}^{3}_{0,1} f(z) - \mathcal{E}^{2}_{0,1} f(z)}{z} - 1\Big| < \frac{2}{\sqrt{5}}, \quad z \in \mathbb{U},$$

then $f \in K$.

Remark 2.16 Let $\alpha = 0, \lambda = \gamma = 1$ in Theorem 2.13. The result coincides with [31, Theorem 3.1].

Theorem 2.17 *If* $f \in \mathcal{A}$ *and satisfies*

$$\begin{aligned}&\Big|\arg\Big(\frac{[\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z) - (\gamma-1)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)]^{\beta}}{z}\Big) + \\ &\quad (1-\beta)\arg\big(\gamma(\gamma+1)\mathcal{E}^{\gamma+2}_{\alpha,\lambda}f(z) + \gamma(1-\beta-2\gamma)\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z) - \\ &\quad (\gamma-1)(1-\beta-\gamma)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)\big)\Big| < \frac{\pi}{2}(1-\beta), \quad z \in \mathbb{U},\end{aligned}$$

where $0 \le \beta < 1$, *then* $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z) \in K(\beta)$, *where* $\delta = 0.6165\cdots$ *is the unique root of the equation of* $2\tan^{-1}(1-\delta) + (1-2\delta)\pi = 0$.

Proof Suppose that $p(z)$ is given by (2.7), and $g(z) = zp'(z)$. Then, from (2.8) we can obtain

$$\begin{aligned}g'(z) =& \frac{1}{1-\beta}\Big(\frac{[\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z) - (\gamma-1)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)]^{\beta}}{z}\Big)^{\frac{1}{1-\beta}} \times \\ & \big(\gamma(\gamma+1)\mathcal{E}^{\gamma+2}_{\alpha,\lambda}f(z) + \gamma(1-\beta-2\gamma)\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z) - (\gamma-1)(1-\beta-\gamma)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)\big).\end{aligned}$$

From an application of Lemma 1.3, we get

$$\begin{aligned}|\arg g'(z)| =& \Big|\frac{1}{1-\beta}\arg\Big(\frac{[\gamma\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z) - (\gamma-1)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)]^{\beta}}{z}\Big) + \\ & \arg\big(\gamma(\gamma+1)\mathcal{E}^{\gamma+2}_{\alpha,\lambda}f(z) + \gamma(1-\beta-2\gamma)\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z) - \\ & (\gamma-1)(1-\beta-\gamma)\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)\big)\Big| < \frac{\pi}{2}\delta,\end{aligned}$$

where $z \in \mathbb{U}$, thus $g(z) \in \mathcal{S}^*$, which equivalent to $p(z) \in K$, implies that $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z) \in K(\beta)$. $\square$

Remark 2.18 Let $\alpha = 0, \lambda = \gamma = 1$ in Theorem 2.17. The result coincides with [31, Theorem 3.3].

Theorem 2.19 *If* $f \in \mathcal{A}$ *and satisfies*

$$\Big|\frac{\gamma\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z^2}\Big(\frac{\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - 1\Big)\Big| < \frac{1}{\sqrt{5}} = 0.4472\cdots, \quad z \in \mathbb{U},$$

then $\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z) \in \mathcal{S}^*$.

Proof Let $f \in \mathcal{A}$. If $g(z)$ has the form

$$g(z) = \int_0^z \frac{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(t)}{t}\mathrm{d}t = z + \sum_{n=1}^{\infty}\frac{\Gamma(\lambda)\gamma}{2\Gamma(\alpha+\lambda)}\frac{a_2}{1-\beta}z^2 + \cdots.$$

A simple calculation yields

$$g''(z) = \frac{\gamma\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z^2}\Big(\frac{\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - 1\Big).$$

Combining with (1.11), we have

$$|g''(z)| = \Big|\frac{\gamma\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)}{z^2}\Big(\frac{\mathcal{E}^{\gamma+1}_{\alpha,\lambda}f(z)}{\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z)} - 1\Big)\Big| < \frac{1}{\sqrt{5}} = 0.4472\cdots, \quad z \in \mathbb{U}.$$

From an application of Lemma 1.4 comes out $g(z) \in K$. Since

$$\Re\Big(1 + \frac{zg''(z)}{g'(z)}\Big) = \Re\Big(\frac{z(\mathcal{E}^{\gamma}_{\alpha,\lambda}f(z))'}{\mathcal{E}^{\gamma}_{\alpha,\lambda}}\Big) > 0, \quad z \in \mathbb{U},$$

the proof of Theorem 2.19 is completed. □

Remark 2.20 Let $m = 0, l = \alpha_1 = 1$ in Theorem 2.17. The result coincides with [31, Theorem 4.1].

In the following we will give several results associated with the generalized hypergeometric function. The method and the process of the proofs that we use are the same as above mentioned and we choose to omit here.

By application of Lemma 1.2, we get Theorem 2.21.

Theorem 2.21 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\Big|\Big(\frac{H^l_m(\alpha_1)f(z)}{z}\Big)^{\frac{1}{1-\beta}}\Big(\frac{\alpha_1 H^l_m(\alpha_1+1)f(z)}{H^l_m(\alpha_1)f(z)} - (\alpha_1 - 1) - \beta\Big) - 1 + \beta\Big| \tag{2.10}$$

$$< \frac{2}{\sqrt{5}}(1-\beta), \quad z \in \mathbb{U}, \tag{2.11}$$

where $0 \le \beta < 1$, then $H^l_m(\alpha_1)f(z) \in \mathcal{S}^*(\beta)$.

Let $\beta = \frac{1}{2}, \beta = 0$ in Theorem 2.21, respectively. We get following results.

Corollary 2.22 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\Big|\Big(\frac{H^l_m(\alpha_1)f(z)}{z}\Big)^2\Big(\frac{\alpha_1 H^l_m(\alpha_1+1)f(z)}{H^l_m(\alpha_1)f(z)} - \alpha_1 + \frac{1}{2}\Big) - \frac{1}{2}\Big| < \frac{1}{\sqrt{5}} = 0.4472\cdots, \quad z \in \mathbb{U},$$

then $H^l_m(\alpha_1)f(z) \in \mathcal{S}^*(\frac{1}{2})$.

Corollary 2.23 *If $f \in \mathcal{A}$ satisfies the condition*

$$\Big|\frac{\alpha_1 H^l_m(\alpha_1+1)f(z)}{z} - \frac{(\alpha_1-1)H^l_m(\alpha_1)f(z)}{z} - 1\Big| < \frac{2}{\sqrt{5}}, \quad z \in \mathbb{U},$$

then $H^l_m(\alpha_1)f(z) \in \mathcal{S}^*$.

Let $m = 0, l = \alpha_1 = 1$ in Corollary 2.23. We get

Corollary 2.24 *If $f \in \mathcal{A}$ and satisfies the condition $|\frac{H^1_0(2)f(z)}{z} - 1| < \frac{2}{\sqrt{5}}$ $(z \in \mathbb{U})$, then $f \in \mathcal{S}^*$.*

Remark 2.25 Corollary 2.24 can be seen as Lemma 1.2, since $H^1_0(2)f(z) = zf'(z)$.

Remark 2.26 Let $m = 0, l = \alpha_1 = 1$ in Theorem 2.21. We get [31, Theorem 2.1].

Applying Lemma 1.3, we show

Theorem 2.27 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\begin{aligned}&\big|\arg(\frac{H_m^l(\alpha_1)f(z)}{z}) + (1-\beta)\arg(\frac{\alpha_1 H_m^l(\alpha_1+1)f(z)}{H_m^l(\alpha_1)f(z)} - (\alpha_1-1)-\beta)\big| \\ &< \frac{\pi}{2}\delta(1-\beta), \quad z \in \mathbb{U},\end{aligned} \tag{2.12}$$

for some real β ($0 \le \beta < 1$), then $H_m^l(\alpha_1)f(z) \in \mathcal{S}^*(\beta)$, where $\delta = 0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi = 0$.

Let $\beta = \frac{1}{2}, \beta = 0$ in Theorem 2.27, respectively. We get the following results.

Corollary 2.28 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\big|\arg(\frac{H_m^l(\alpha_1)f(z)}{z}) + \frac{1}{2}\arg(\frac{\alpha_1 H_m^l(\alpha_1+1)f(z)}{H_m^l(\alpha_1)f(z)} - \alpha_1 + \frac{1}{2})\big| < \frac{\pi}{4}\delta, \quad z \in \mathbb{U},$$

then $H_m^l(\alpha_1)f(z) \in \mathcal{S}^*(\frac{1}{2})$, where $\delta = 0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi = 0$.

Corollary 2.29 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\big|\arg(\frac{H_m^l(\alpha_1)f(z)}{z}) + \arg(\frac{\alpha_1 H_m^l(\alpha_1+1)f(z)}{H_m^l(\alpha_1)f(z)} - (\alpha_1-1))\big| < \frac{\pi}{2}\delta, \quad z \in \mathbb{U},$$

then $H_m^l(\alpha_1)f(z) \in \mathcal{S}^*$, where $\delta = 0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi = 0$.

Let $m = 0$, $l = \alpha_1 = 1$ in Corollary 2.29. We get the following corollary.

Corollary 2.30 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\big|\arg(\frac{H_0^1(1)f(z)}{z}) + \arg(\frac{H_0^1(2)f(z)}{H_0^1(1)f(z)})\big| < \frac{\pi}{2}\delta, \quad z \in \mathbb{U},$$

then $f \in \mathcal{S}^*$, where $\delta = 0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi = 0$.

Remark 2.31 Corollary 2.30 can be seen as Lemma 1.2, since $H_0^1(1)f(z) = f(z)$, $H_0^1(2)f(z) = zf'(z)$.

Remark 2.32 Let $m = 0, l = \alpha_1 = 1$ in Theorem 2.27. We get [31, Theorem 2.3].

Theorem 2.33 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\begin{aligned}&\big|\big(\frac{[\alpha_1 H_m^l(\alpha_1+1)f(z) - (\alpha_1-1)H_m^l(\alpha_1)f(z)]^\beta}{z}\big)^{\frac{1}{1-\beta}}\big(\alpha_1(\alpha_1+1)H_m^l(\alpha_1+2)f(z)+ \\ &\quad \alpha_1(1-\beta-2\alpha_1)H_m^l(\alpha_1+1)f(z) - (\alpha_1-1)(1-\beta-\alpha_1)H_m^l(\alpha_1)f(z)\big) - 1 + \beta\big| \\ &< \frac{2}{\sqrt{5}}(1-\beta), \quad z \in \mathbb{U},\end{aligned} \tag{2.13}$$

where $0 \le \beta < 1$, then $H_m^l(\alpha_1)f(z) \in K(\beta)$.

Let $\beta = 0$ in Theorem 2.13. We get the following corollary.

Corollary 2.34 *If $f \in \mathcal{A}$ and satisfies the condition*

$$|\frac{\alpha_1(\alpha_1+1)H_m^l(\alpha_1+2)f(z)+\alpha_1(1-2\alpha_1)H_m^l(\alpha_1+1)f(z)+(\alpha_1-1)^2H_m^l(\alpha_1)f(z)}{z}-1| < \frac{2}{\sqrt{5}}, \quad z \in \mathbb{U},$$

then $H_m^l(\alpha_1)f(z) \in K$.

Let $m=0$, $l=\alpha_1=1$ in Corollary 2.14. We get the following corollary.

Corollary 2.35 *If $f \in \mathcal{A}$ and satisfies the condition*

$$|\frac{2H_0^1(3)f(z)-H_0^1(2)f(z)}{z}-1| < \frac{2}{\sqrt{5}}, \quad z \in \mathbb{U},$$

then $f \in K$.

Remark 2.36 Let $m=0, l=\alpha_1=1$ in Theorem 2.33. The result coincides with [31, Theorem 3.1].

Theorem 2.37 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\begin{aligned}&\Big|\arg\big(\frac{[\alpha_1 H_m^l(\alpha_1+1)f(z)-(\alpha_1-1)H_m^l(\alpha_1)f(z)]^\beta}{z}\big)+\\ &(1-\beta)\arg\big(\alpha_1(\alpha_1+1)H_m^l(\alpha_1+2)f(z)+\alpha_1(1-\beta-2\alpha_1)H_m^l(\alpha_1+1)f(z)-\\ &(\alpha_1-1)(1-\beta-\alpha_1)H_m^l(\alpha_1)f(z)\big)\Big| < \frac{\pi}{2}(1-\beta), \quad z \in \mathbb{U},\end{aligned}$$

where $0 \le \beta < 1$, then $H_m^l(\alpha_1)f(z) \in K(\beta)$, where $\delta = 0.6165\cdots$ is the unique root of the equation of $2\tan^{-1}(1-\delta)+(1-2\delta)\pi = 0$.

Remark 2.38 Let $m=0, l=\alpha_1=1$ in Theorem 2.37. The result coincides with [31, Theorem 3.3].

Theorem 2.39 *If $f \in \mathcal{A}$ and satisfies the condition*

$$\Big|\frac{\alpha_1 H_m^l(\alpha_1)f(z)}{z^2}\big(\frac{H_m^l(\alpha_1+1)f(z)}{H_m^l(\alpha_1)f(z)}-1\big)\Big| < \frac{1}{\sqrt{5}} = 0.4472\cdots, \quad z \in \mathbb{U},$$

then $H_m^l(\alpha_1)f(z) \in \mathcal{S}^*$.

Remark 2.40 Let $m=0, l=\alpha_1=1$ in Theorem 2.37. The result coincides with [31, Theorem 4.1].

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References

[1] A. WIMAN. *Über den Fundamentalsatz in der Theorie der Funcktionen $E_\alpha(x)$*. Acta Math., 1905, **29**(1): 191–201.

[2] A. WIMAN. *Über die Nullstellun der Funcktionen $E_\alpha(x)$*. Acta Math., 1905, **29**: 217–234.

[3] G. M. MITTAG-LEFFLER. *Sur la nouvelle fonction $E_\alpha(x)$*. C. R. Acad. Sci. Paris., 1903, **137**(2): 554–558.

[4] T. R. PRABHAKAR. *A singular integral equation with a generalized Mittag-Leffler function in the kernel*. Yokohama Math. J., 1971, **19**(1): 7–15.

[5] R. GORENFLO, A. A. KILBAS, T. MAINARDI, et al. *Mittag-Leffler Functions*. Springer, New York, 2014.

[6] I. S. GUPTA, L. DEBNATH. *Some properties of the Mittag-Leffler functions*. Integral Transform. Spec. Funct., 2007, **18**(5): 329–336.

[7] A. A. KILBAS, M. SAIGO, R. K. SAXENA. *Generalized Mittag-Leffler function and generalized fractional calculus operators*. Integral Transform. Spec. Funct., 2004, **15**(1): 31–49.

[8] Xiao PENG, Huaiqin WU, Ka SONG, et al. *Global synchronization in finite time for fractional-order neural networks with discontinuous activations and time delays*. Neural Networks, 2017, **94**: 46–54.

[9] Xiao PENG, Huaiqin WU, *Robust Mittag-Leffler synchronization for uncertain fractional-order discontinuous neural networks via non-fragile control strategy*. Neural Processing Letters, 2018, **48**(3): 1521–1542.

[10] T. O. SALIM. *Some properties relating to the generalized Mittag-Leffler function*. Adv. Appl. Math. Anal., 2009, **4**(1): 21–30.

[11] A. K. SHUKLA, J. C. PRAJAPATI. *On a generalization of Mittag-Leffler function and its properties*. J. Math. Anal. Appl., 2007, **336**(2): 797–811.

[12] H. M. SRIVASTAVA, Z. YOMOVSKI. *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*. Appl. Math. Comput., 2009, **211**(1): 198–210.

[13] H. M. SRIVASTAVA, B. A. FRASIN, V. PESCAR. *Univalence of integral operators involving Mittag-Leffler functions*. Appl. Math. Inf. Sci., 2017, **11**(3): 635–641.

[14] D. BANSAL, J. K. PRAJAPAT. *Certain geometric properties of the Mittag-Leffler functions*. Complex Var. Elliptic Equ., 2016, **61**(3): 338–350.

[15] D. RĂDUCANU. *On partial sums of normalized Mittag-Leffler functions*. Analele Universitatii "Ovidius" Constanta-Seria Matematica, 2017, **25**(2): 123–133.

[16] D. RĂDUCANU. *Differential subordination associated with generalized Mittag-Leffler function*. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., 2019, **113**(2): 435–452.

[17] D. RĂDUCANU. *Third-order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions*. Mediterr. J. Math., 2017, **14**(4): 167.

[18] J. DZIOK, H. M. SRIVASTAVA. *Classes of analytic functions associated with the generalized hypergeometric function*. Appl. Math. Comput., 1999, **103**(1): 1–13.

[19] R. AGHALARY, S. B. JOSHI, R. N. MOHAPATRA, et al. *Subordinations for analytic functions defined by the Dziok-Srivastava linear operator*. Appl. Math. Comput., 2007, **187**(1): 13–19.

[20] Yiling CANG, Jinlin LIU. *Some subclasses of meromorphically multivalent functions associated with the Dziok-Srivastava operator*. Filomat, 2017, **31**(8): 2449–2458.

[21] J. DZIOK. *On the convex combination of the Dziok-Srivastava operator*. Appl. Math. Comput., 2007, **188**(2): 1214–1220.

[22] Jinlin LIU, *Strongly starlike functions associated with the Dziok-Srivastava operator*. Tamkang J. Math., 2004, **35**(1): 37–42.

[23] G. MURUGUSUNARAMOORTHY, N. MAGESH. *Starlike and convex functions of complex order involving the Dziok-Srivastava operator*. Integral Transforms Spec. Funct., 2007, **18**(6): 419–425.

[24] Zhigang WANG, Yueping JIANG, H. M. SRIVASTAVA. *Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function*. Comput. Math. Appl., 2009, **57**(4): 571–586.

[25] Ş. ALTINKAYA, S. YALÇIN. *Sibel General properties of multivalent concave functions involving linear operator of Carlson-Shaffer type*. C. R. Acad. Bulgare Sci., 2016, **69**(12): 1533–1540.

[26] S. D. BERNARDI. *Convex and starlike univalent functions*. Trans. Am. Math. Soc., 1969, **135**: 429–446.

[27] B. C. CARLSON, D. B. SHAFFER. *Starlike and prestarlike hypergeometric functions*. SIAM J. Math. Anal., 1984, **15**(4): 737–745.

[28] Y. E. HOHLOV. *Operators and operations on the class of univalent functions*. Izv. Vyssh. Uchebn. Zaved. Mat., 1978, **10**(197): 83–89.

[29] S. OWA, H. M. SRIVASTAVA. *Univalent and starlike generalized hypergeometric functions*. Canad. J. Math., 1987, **39**(5): 1057–1077.

[30] P. T. MOCANU. *Some starlikeness conditions for analytic functions*. Rev. Roumaine Math. Pur. Appl., 1988, **33**(1): 117–124.

[31] N. UYANIK, M. AYDOGAN, S. OWA. *Extensions of sufficient conditions for starlikeness and convexity of order β*. Appl. Math. Lett., 2011, **24**(8): 1393–1399.