Journal of Mathematical Research with Applications
May, 2020, Vol. 40, No. 3, pp. 297–304
DOI:10.3770/j.issn:2095-2651.2020.03.006
Http://jmre.dlut.edu.cn

More Asymptotic Expansions for the Harmonic Numbers

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Abstract In this paper, by exponential complete Bell polynomials, we establish a (general) harmonic number asymptotic expansion, and give the corresponding recurrence of the coefficient sequence in the expansion. By the methods of the generating functions and summation transformations, we also present an explicit expression for the coefficient sequence of the expansion. Moreover, we establish two (general) lacunary harmonic number asymptotic expansions, which contain only even or odd power terms in the logarithmic term.

Keywords asymptotic expansions; harmonic numbers; Bell polynomials

MR(2010) Subject Classification 41A60; 05A19; 11B83

1. Introduction

Ramanujan (see [1, p.276] and [2, p.531]) proposed the following asymptotic expansion for the nth harmonic number:

$$\begin{aligned}H_n :=&\sum_{k=1}^{n}\frac{1}{k}\sim\frac{1}{2}\ln(2m)+\gamma+\sum_{k=1}^{\infty}\frac{R_k}{m^k}\\=&\frac{1}{2}\ln(2m)+\gamma+\frac{1}{12m}-\frac{1}{120m^2}+\frac{1}{630m^3}-\frac{1}{1680m^4}+\frac{1}{2310m^5}-\\&\frac{191}{360360m^6}+\frac{29}{30030m^7}-\frac{2833}{1166880m^8}+\frac{140051}{17459442m^9}-\cdots,\end{aligned}\tag{1.1}$$

as $n\to\infty$, where γ is the Euler-Mascheroni constant and $m=\frac{1}{2}n(n+1)$ is the nth triangular number. In [1], Ramanujan did not give the formula's proof and the explicit expression for the coefficient sequence. Until 2008, Villarino [3] gave the complete proof of the expansion (1.1) and established an explicit expression for the coefficient sequence (R_k) as

$$R_k=\frac{(-1)^{k-1}}{2k\cdot 8^k}\sum_{j=0}^{k}\binom{k}{j}(-4)^jB_{2j}(\frac{1}{2}),\tag{1.2}$$

where $B_k(x)$ are the Bernoulli polynomials. In 2015, Chen and Cheng [4] gave a recurrence for (R_k):

$$R_1=\frac{1}{12},\quad R_k=\frac{1}{2^k}\Big\{\frac{1}{4k}-\frac{B_{2k}}{2k}-\sum_{j=1}^{k-1}2^jR_j\binom{2k-j-1}{j-1}\Big\},\quad k\ge 2,\tag{1.3}$$

Received May 24, 2019; Accepted March 17, 2020
Supported by the National Natural Science Foundation of China (Grant No. 11501081).
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where B_k are the classical Bernoulli numbers. In 2016, Wang and Feng [5] verified directly that the sequences defined by (1.2) and (1.3) are indeed the same by using Riordan arrays.

Recently, many researches have been devoted to the harmonic number expansions of the Ramanujan type. For example, in 1991, DeTemple and Wang [6] established the half integer expansion

$$H_n \sim \ln(n+\frac{1}{2})+\gamma+\sum_{k=1}^{\infty}\frac{\alpha_k}{(n+\frac{1}{2})^{2k}},\quad n\to\infty,$$

which can be written as

$$H_n \sim \frac{1}{2}\ln(2m+\frac{1}{4})+\gamma+\sum_{k=1}^{\infty}\frac{\alpha_k}{(2m+\frac{1}{4})^{k}},\quad n\to\infty. \tag{1.4}$$

Further, in 2015, Mortici and Villarino [7] proposed the following expansion of the Ramanujan type:

$$H_n \sim \frac{1}{2}\ln(2m+\frac{1}{3})+\gamma+\sum_{k=2}^{\infty}\frac{d_k}{(m+\frac{1}{6})^{k}},\quad n\to\infty.$$

It is equivalent to

$$H_n \sim \frac{1}{2}\ln(2m+\frac{1}{3})+\gamma+\sum_{k=2}^{\infty}\frac{c_k}{(2m+\frac{1}{3})^{k}},\quad n\to\infty, \tag{1.5}$$

with $c_k = 2^k d_k$, which is also one result given in [8]. Very recently, motivated by these works, Wang [9] gave a general harmonic number expansion of the Ramanujan type, which is of the form

$$H_n \sim \frac{1}{2}\ln(2m+h)+\gamma+\sum_{k=1}^{\infty}\frac{\alpha_k(h)}{(2m+h)^{k}},\quad n\to\infty,$$

where h is a parameter. The classical Ramanujan formula (1.1), the DeTemple-Wang formula (1.4) and the Chen-Mortici-Villarino formula (1.5) are all special cases of this formula.

In 2016, Chen [8] established another harmonic number expansion form:

$$H_n \sim \frac{1}{2}\ln\Big(2m+\frac{1}{3}+\sum_{k=1}^{\infty}\frac{b_k}{(2m)^{k}}\Big)+\gamma,\quad n\to\infty. \tag{1.6}$$

By changing the logarithmic term, many researchers, for example, DeTemple [10], Negoi [11], Chen-Srivastava-Li-Manyama [12], Chen-Mortici [13] and Yang [14] also gave some faster and faster asymptotic expansions for the harmonic number and the Euler-Mascheroni constant. For other recent works on the asymptotic expansions of the harmonic numbers, the readers are referred to, for example, the papers [15–19].

Inspired by these works, in this paper, by using exponential complete Bell polynomials, we establish a general harmonic number asymptotic expansion, which is of the form

$$H_n \sim \frac{1}{2}\ln\Big(2m+\frac{1}{3}+\sum_{k=1}^{\infty}\frac{b_k(h)}{(2m+h)^{k}}\Big)+\gamma,\quad n\to\infty.$$

When $h=0$, it reduces to the asymptotic expansion (1.6). Then, similar to the lacunary expansions in [20], we give another two asymptotic expansions, which have only even or odd power terms in the logarithmic term:

$$H_n \sim \frac{1}{2}\ln\Big(2m+\frac{1}{3}-\frac{1}{180m}+\sum_{k=1}^{\infty}\frac{u_k}{(2m+v_k)^{2k}}\Big)+\gamma,\ \ n\to\infty,$$
$$H_n \sim \frac{1}{2}\ln\Big(2m+\frac{1}{3}+\sum_{k=1}^{\infty}\frac{p_k}{(2m+q_k)^{2k-1}}\Big)+\gamma,\ \ n\to\infty.$$

2. A general harmonic number asymptotic expansion

The exponential complete Bell polynomials Y_n are defined by [21, Section 3.3]

$$\exp\Big(\sum_{k=1}^{\infty}x_k\frac{t^k}{k!}\Big)=\sum_{n=0}^{\infty}Y_n(x_1,x_2,\ldots,x_n)\frac{t^n}{n!}. \tag{2.1}$$

Then $Y_0=1$ and

$$Y_n(x_1,x_2,\ldots,x_n)=\sum_{c_1+2c_2+\cdots+nc_n=n}\frac{n!}{c_1!c_2!\cdots c_n!}\Big(\frac{x_1}{1!}\Big)^{c_1}\Big(\frac{x_2}{2!}\Big)^{c_2}\cdots\Big(\frac{x_n}{n!}\Big)^{c_n},\ \ n\geq 1.$$

Additionally, based on [22, Eq. (2.44)], the polynomials Y_n satisfy the following recurrence:

$$Y_n(x_1,x_2,\ldots,x_n)=\sum_{j=0}^{n-1}\binom{n-1}{j}x_{n-j}Y_j(x_1,x_2,\ldots,x_j),\ \ n\geq 1,$$

from which, we can obtain the explicit expressions of the sequence $(Y_n)_{n\geq 0}$ immediately. The first few terms are

$$Y_0=1,\ \ Y_1(x_1)=x_1,\ \ Y_2(x_1,x_2)=x_2+x_1^2,\ \ Y_3(x_1,x_2,x_3)=x_3+3x_2x_1+x_1^3,$$
$$Y_4(x_1,x_2,x_3,x_4)=x_4+4x_3x_1+3x_2^2+6x_2x_1^2+x_1^4,$$
$$Y_5(x_1,x_2,x_3,x_4,x_5)=x_5+5x_4x_1+10x_3x_2+10x_3x_1^2+15x_2^2x_1+10x_2x_1^3+x_1^5.$$

Recently, the Bell polynomials are used to establish asymptotic expansions for the gamma function, the hyperfactorial function and the Barnes G-function by Wang, Liu and Xu [23–25]. In this section, we use the Bell polynomials to deduce a general asymptotic expansion for the harmonic numbers.

Theorem 2.1 *Let h be a real number. Then we have*

$$H_n \sim \frac{1}{2}\ln\Big(2m+\frac{1}{3}+\sum_{k=1}^{\infty}\frac{b_k(h)}{(2m+h)^k}\Big)+\gamma, \tag{2.2}$$

as $n\to\infty$, where the sequence $(b_k(h))_{k\geq 1}$ can be determined by the recurrence

$$b_1(h)=-\frac{1}{90},$$

$$b_k(h) = \frac{2^{k+1}}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}) - \sum_{j=1}^{k-1} b_j(h) \binom{k-1}{j-1} (-h)^{k-j},\ k \geq 2, \tag{2.3}$$

where R_k are the coefficients in Ramanujan's formula (1.1).

Proof The Ramanujan's expansion (1.1) can be rewritten as

$$\exp(2H_n - 2\gamma) \sim 2m \cdot \exp\Big(\sum_{k=1}^{\infty} \frac{2R_k}{m^k}\Big), \quad n \to \infty.$$

By the definition of the Bell polynomial (2.1), we have

$$\exp(2H_n - 2\gamma) \sim 2m \sum_{k=0}^{\infty} Y_k(2R_1, 2\cdot 2!R_2, \ldots, 2\cdot k!R_k) \frac{(\frac{1}{m})^k}{k!}, \quad n \to \infty. \tag{2.4}$$

The asymptotic expansion of the form (2.2) can also be expressed as

$$\begin{aligned}\exp(2H_n - 2\gamma) \sim & 2m + \frac{1}{3} + \sum_{k=1}^{\infty} \frac{b_k(h)}{(2m+h)^k} \\ = & 2m\Big(1 + \frac{1}{6m} + \frac{1}{2m} \sum_{k=1}^{\infty} \frac{b_k(h)}{(2m+h)^k}\Big), \quad n \to \infty.\end{aligned}$$

To establish the expansion formula (2.2), it is sufficient to show that

$$1 + \frac{1}{6m} + \frac{1}{2m} \sum_{k=1}^{\infty} \frac{b_k(h)}{(2m+h)^k} = \sum_{k=0}^{\infty} Y_k(2R_1, 2\cdot 2!R_2, \ldots, 2\cdot k!R_k) \frac{(\frac{1}{m})^k}{k!}.$$

In view of $Y_0 = 1$ and $Y_1(2R_1) = \frac{1}{6}$, we only make

$$\frac{1}{2m} \sum_{k=1}^{\infty} \frac{b_k(h)}{(2m+h)^k} = \sum_{k=2}^{\infty} Y_k(2R_1, 2\cdot 2!R_2, \ldots, 2\cdot k!R_k) \frac{(\frac{1}{m})^k}{k!}. \tag{2.5}$$

The left side of the formula (2.5) can be expanded as

$$\begin{aligned}\frac{1}{2m} \sum_{j=1}^{\infty} \frac{b_j(h)}{(2m+h)^j} = & \frac{1}{2m} \sum_{j=1}^{\infty} b_j(h) \sum_{i=0}^{\infty} \binom{-j}{i} h^i (2m)^{-j-i} \\ = & \sum_{j=1}^{\infty} b_j(h) \sum_{i=0}^{\infty} \binom{j+i-1}{j-1} (-h)^i (2m)^{-j-i-1} \\ = & \sum_{k=1}^{\infty} \frac{k!}{2^{k+1}} \sum_{j=1}^{k} b_j(h) \binom{k-1}{j-1} (-h)^{k-j} \frac{(\frac{1}{m})^{k+1}}{k!} \\ = & \sum_{k=2}^{\infty} \frac{k!}{2^k} \sum_{j=1}^{k-1} b_j(h) \binom{k-2}{j-1} (-h)^{k-j-1} \frac{(\frac{1}{m})^k}{k!}.\end{aligned}$$

Equating the coefficients of $\left(\frac{1}{m}\right)^k/k!$ of the both sides in the formula (2.5), we obtain the result in this theorem. $\Box$

3. An explicit expression of the coefficient sequence

In this section, we present an explicit expression of the coefficient sequence $(b_k(h))$ in the asymptotic expansion (2.2).

Theorem 3.1 *The coefficient sequence $(b_k(h))$ in the asymptotic expansion (2.2) is expressed by*

$$b_k(h) = \sum_{i=1}^{k} \frac{2^{i+1}}{(i+1)!} \binom{k-1}{i-1} h^{k-i} Y_{i+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(i+1)!R_{i+1}). \tag{3.1}$$

Proof Let the generating function of the sequence $(b_k(h))$ be

$$f(t) = \sum_{k=1}^{\infty} b_k(h) t^k.$$

By the recurrence relation (2.3), we have

$$\begin{aligned} f(t) =& \sum_{k=1}^{\infty} \frac{2^{k+1}}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}) t^k - \\ & \sum_{k=1}^{\infty}\sum_{i=1}^{k-1} b_i(h)(-h)^{k-i} \binom{k-1}{k-i} t^k. \end{aligned}$$

The second term of the right side of the above equation can be computed as

$$\begin{aligned} & \sum_{k=1}^{\infty}\sum_{i=1}^{k-1} b_i(h)(-h)^{k-i} \binom{k-1}{k-i} t^k \\ &= \sum_{i=1}^{\infty}\sum_{k=i+1}^{\infty} b_i(h)(-h)^{k-i} \binom{k-1}{k-i} t^k \\ &= \sum_{i=1}^{\infty} b_i(h) t^i \sum_{j=1}^{\infty} \binom{-i}{j} (ht)^j \\ &= \sum_{i=1}^{\infty} b_i(h) t^i [(1+ht)^{-i} - 1] = f(\frac{t}{1+ht}) - f(t). \end{aligned}$$

So, we have

$$f(t) = \sum_{k=1}^{\infty} \frac{2^{k+1}}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}) t^k - f(\frac{t}{1+ht}) + f(t).$$

Then

$$f(\frac{t}{1+ht}) = \sum_{k=1}^{\infty} \frac{2^{k+1}}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}) t^k.$$

Setting $T = \frac{t}{1+ht}$, we have

$$\begin{aligned} f(T) =& \sum_{k=1}^{\infty} \frac{2^{k+1}}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}) \Big(\frac{T}{1-hT}\Big)^k \\ =& \sum_{k=1}^{\infty} \frac{2^{k+1}}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}) \sum_{i=0}^{\infty} \binom{k+i-1}{i} h^i T^{k+i} \end{aligned}$$

$$=\sum_{N=1}^{\infty}\Big\{\sum_{k=1}^{N}\binom{N-1}{k-1}h^{N-k}\frac{2^{k+1}}{(k+1)!}Y_{k+1}(2R_1,2\cdot 2!R_2,\ldots,2(k+1)!R_{k+1})\Big\}T^N.$$

Equating the coefficients of T^N in both sides of the above formula, we obtain the desired result. □

By the formula (3.1), we can compute the coefficients $b_k(h)$ immediately. The first few are

$$b_2(h)=\frac{53}{5670}-\frac{1}{90}h,\quad b_3(h)=-\frac{3929}{340200}+\frac{53}{2835}h-\frac{1}{90}h^2,$$
$$b_4(h)=\frac{9817}{449064}-\frac{3929}{113400}h+\frac{53}{1890}h^2-\frac{1}{90}h^3.$$

When $h=0$ in Theorem 2.1, the expansion formula (1.6) is deduced. Setting $h=\frac{1}{3}$ gives the following asymptotic expansion:

$$H_n\sim\gamma+\frac{1}{2}\ln\big(2m+\frac{1}{3}-\frac{1}{90}\frac{1}{2m+\frac{1}{3}}+\frac{16}{2835}\frac{1}{(2m+\frac{1}{3})^2}-\frac{743}{113400}\frac{1}{(2m+\frac{1}{3})^3}+$$
$$\frac{18266}{1403325}\frac{1}{(2m+\frac{1}{3})^4}+\cdots\big),$$

as $n\to\infty$, which can be seen an asymptotic expansion into powers of $\frac{1}{2m+\frac{1}{3}}$ for the harmonic numbers and Euler-Mascheroni constant.

4. Lacunary expansions with only even or odd power terms

In this section, we establish two asymptotic expansions with only even power terms or odd power terms for the harmonic numbers and Euler-Mascheroni constant.

Theorem 4.1 *For the nth harmonic number, we have the following asymptotic expansion:*

$$H_n\sim\frac{1}{2}\ln\Big(2m+\frac{1}{3}-\frac{1}{180m}+\sum_{k=1}^{\infty}\frac{u_k}{(2m+v_k)^{2k}}\Big)+\gamma \tag{4.1}$$
$$=\frac{1}{2}\ln\big(2m+\frac{1}{3}-\frac{1}{180m}+\frac{\frac{53}{5670}}{(2m+\frac{3929}{6360})^2}+\frac{\frac{511031167}{47600784000}}{(2m+\frac{4763506280738402628 0841}{35764757856821081088 0000000})^4}+\cdots\big)+$$
$$\gamma,$$

as $n\to\infty$, where the sequences $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ can be determined by the recurrences

$$u_l=\frac{2^{2l+1}}{(2l+1)!}Y_{2l+1}(2R_1,2\cdot 2!R_2,\ldots,2(2l+1)!R_{2l+1})-$$
$$\sum_{j=1}^{l-1}\binom{2l-1}{2j-1}u_jv_j^{2l-2j},\quad l\geq 1, \tag{4.2}$$
$$2lu_lv_l=-\frac{2^{2l+2}}{(2l+2)!}Y_{2l+2}(2R_1,2\cdot 2!R_2,\ldots,2(2l+2)!R_{2l+2})-$$
$$\sum_{j=1}^{l-1}\binom{2l}{2j-1}u_jv_j^{2l+1-2j},\quad l\geq 1. \tag{4.3}$$

Proof We expand the following sum in powers of $1/m$:

$$\sum_{k=1}^{\infty} \frac{u_k}{(2m+v_k)^{2k}} = \sum_{k=2}^{\infty} \Big\{ \frac{(-1)^k}{2^k} \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k-1}{k-2j} u_j v_j^{k-2j} \Big\} \frac{1}{m^k}. \tag{4.4}$$

When $k=1$, the coefficient of $1/m^k$ in (2.4) is equal to $Y_2(2R_1, 4R_2) = -\frac{1}{180}$, which is just the coefficient of $1/m$ in the logarithmic term of (4.1). To establish the expansion (4.1), the coefficient of $1/m^k$ in (2.4) should be equal to the coefficient of $1/m^k$ in (4.4) for $k \geq 2$, that is

$$\frac{(-1)^k}{2^k} \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k-1}{k-2j} u_j v_j^{k-2j} = \frac{2}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}). \tag{4.5}$$

Then by setting $k = 2l$ in (4.5), for $l = 1, 2, 3, \ldots$, we have

$$\frac{1}{2^{2l}} \sum_{j=1}^{l} \binom{2l-1}{2l-2j} u_j v_j^{2l-2j} = \frac{2}{(2l+1)!} Y_{2l+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(2l+1)!R_{2l+1}).$$

Then the recurrence (4.2) is obtained. Setting $k = 2l+1$ in (4.5), we establish the recurrence (4.3). □

Theorem 4.2 *For the nth harmonic number, we have the following asymptotic expansion:*

$$\begin{aligned} H_n &\sim \frac{1}{2} \ln \Big(2m + \frac{1}{3} + \sum_{k=1}^{\infty} \frac{p_k}{(2m+q_k)^{2k-1}} \Big) + \gamma \\ &= \frac{1}{2} \ln \big(2m + \frac{1}{3} + \frac{-\frac{1}{90}}{2m + \frac{53}{63}} + \frac{-\frac{26329}{7144200}}{(2m + \frac{73383853}{54737991})^3} + \cdots \big) + \gamma, \end{aligned} \tag{4.6}$$

as $n \to \infty$, where the sequences $(p_k)_{k\geq 1}$ and $(q_k)_{k \geq 1}$ can be determined by the recurrences

$$p_l = \frac{2^{2l}}{(2l)!} Y_{2l}(2R_1, 2\cdot 2!R_2, \ldots, 2(2l)!R_{2l}) - \sum_{i=1}^{l-1} \binom{2l-2}{2i-2} p_i q_i^{2l-2i}, \quad l \geq 1, \tag{4.7}$$

$$\begin{aligned} (2l-1)p_l q_l = &- \frac{2^{2l+1}}{(2l+1)!} Y_{2l+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(2l+1)!R_{2l+1}) - \\ &\sum_{i=1}^{l-1} \binom{2l-1}{2i-2} p_i q_i^{2l+1-2i}, \quad l \geq 1. \end{aligned} \tag{4.8}$$

Proof Similarly, we also expand the following sum in powers of $1/m$:

$$\sum_{k=1}^{\infty} \frac{p_k}{(2m+q_k)^{2k-1}} = \sum_{k=1}^{\infty} \Big\{ \frac{(-1)^{k+1}}{2^k} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-1}{2i-2} p_i q_i^{k+1-2i} \Big\} \frac{1}{m^k}.$$

By (2.4), to establish the expansion (4.6), it suffices to show that, for $k \geq 1$,

$$\frac{(-1)^{k+1}}{2^k} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-1}{2i-2} p_i q_i^{k+1-2i} = \frac{2}{(k+1)!} Y_{k+1}(2R_1, 2\cdot 2!R_2, \ldots, 2(k+1)!R_{k+1}).$$

By setting $k = 2l-1$ and $k = 2l$ in the above equation, for $l = 1, 2, 3, \ldots$, we obtain the recurrences (4.7) and (4.8). □

Acknowledgements We thank the referees for their time and comments.

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