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The Asymptotic Attractor of the Damped Navier-Stokes System

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Abstract In this paper, the asymptotic attractor of the 2-D damped and driven Navier-Stokes equation is studied by constructing a finite-dimensional solution sequence, and it is proved that this solution sequence approximates the global attractor infinitely after a long time. The dimension estimate of the asymptotic attractor is obtained in the end.

Keywords asymptotic attractor; asymptotic solution sequence; damped navier-stokes equation

MR(2010) Subject Classification 35B40; 35B41

1. Introduction

In the process of studying the properties of infinite dimensional dynamical systems, global attractors are established to study the long-time behavior of solutions in [1, 2]. However, the structure of the global attractor is very complex, and it is very difficult to further study the reduced system. Therefore Temam et al. introduced some concepts such as approximate inertial manifold and exponential attractor and so on [3], but it is difficult to establish strict equivalence between the approximate system and the original system. Afterward Wang et al. introduced the concept of finite-dimensional asymptotic attractor, and proved that the obtained system is equivalent to the original system in [4]. In recent years, the asymptotic attractor of some systems are studied in [4–8].

The 2-D damped and driven Navier-Stokes systems attracted considerable attention over the last decades and were studied from different points of view [9–16]. In this paper, we consider the asymptotic attractor governed by the following 2-D damped and driven Navier-Stokes system with periodic boundary-initial condition

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla p + ru = \nu\Delta u + g(x), \nu > 0, \\ \nabla\cdot u = 0, \\ u(x,t) = u(x+2\pi,t), \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u(x,t) : Q\times \mathbf{R}^+ \to \mathbf{R}^2$ is velocity field of the fluid, $Q = (0,2\pi)\times(0,2\pi)$, $\nu > 0$ is the kinematic viscosity, $r > 0$ is the Rayleigh or Ekman friction coefficient. The damping term ru parameterizes the main dissipation occurring in the planetary boundary layer in [17].

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In this paper, our main purpose is to study the asymptotic attractor of the 2-D damped and driven Navier-Stokes system, that is to construct a finite-dimensional solution sequence by using orthogonal decomposition method. In Section 2, we present some preliminaries such as spaces, concepts and lemma. In Section 3, we will show that the asymptotic attractor tends to global attractor and give the dimension estimate of the asymptotic attractor.

2. Preliminaries

We define space

$$H = \Big\{u \in (L_p^2(Q))^2 : \int_Q u\mathrm{d}x = 0, \nabla \cdot u = 0\Big\},$$

$$V = \Big\{u \in (H_p^1(Q))^2 : \int_Q u\mathrm{d}x = 0, \nabla \cdot u = 0\Big\},$$

where the subscript p denotes that functions discussed here are periodic, and $|\cdot|$ is the norm in the space L^2.

The bilinear term is defined as follows

$$b(u, v, w) = \langle B(u, v), w\rangle = \int (u \cdot \bigtriangledown)v \cdot w\mathrm{d}x,$$

and for all $u, v, w \in H_p^1([0, 2\pi])$, we have $|b(u, v, w)| = C|u|_{L_\infty}|Dv||w|$. By the definition of the bilinear term, the system (1.1) is equivalent to

$$\begin{cases} u_t + B(u, u) + \nabla p + ru = \nu\Delta u + g(x), \nu > 0, \\ \nabla \cdot u = 0, \\ u(x, t) = u(x + 2\pi, t), \\ u(x, 0) = u_0(x). \end{cases} \tag{2.1}$$

Then we introduce some basic concept and lemma as follows.

Definition 2.1 ([4]) *Consider the evolution system*

$$\begin{cases} u_t + Au = g(u), \\ u(x, 0) = u_0(x). \end{cases} \tag{2.2}$$

Let H denote the phase space, $S(t)_{t\geq 0}$ be the semigroup of solution operators, B denote the absorbing set. Suppose that for all $u_0 \in B$, there exists an asymptotic solution sequence $\{u^k(t)\}_{k\geq 1}$ and $t^*(B)$, satisfying $|u^k(t) - S(t)u_0|_H \to 0, k \to \infty, t > t^*(B)$, then define $B^k = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s, u_0\in B} u^k(t)}$ as the asymptotic attractor for the system (2.2), where $|\cdot|_H$ is the norm in the phase space H, $u^k(t)$ depending on initial value u_0, and $t^*(B)$ depending only on the diameter of the absorbing set.

Lemma 2.2 ([18]) *For any $u_0 \in H$ satisfying $|u_0| \leq s < \infty$, then there exists $t^* = t^*(s) > 0$, ρ_0, ρ_1, $\rho_2 > 0$, such that*

$$|u(\cdot, t)| \leq \rho_0, |u(\cdot, t)|_V \leq \rho_1, |u(\cdot, t)|_{H_p^2} \leq \rho_2, \quad \forall t \geq t^*,$$

where $|\cdot|$ *is the norm in the space* H*. Namely, the following set is an absorbing set of the system (2.1) on* H*,* $B = \{u \in V : |u(\cdot,t)| \le \rho_0, |u(\cdot,t)|_V \le \rho_1, |u(\cdot,t)|_{H_p^2} \le \rho_2\}$*.*

3. Asymptotic attractor

The set

$$\left\{ \begin{bmatrix} \sin kx_1 & \sin kx_2 \\ \cos kx_1 & \cos kx_2 \end{bmatrix}, \begin{bmatrix} -\sin kx_1 & \cos kx_2 \\ \cos kx_1 & \sin kx_2 \end{bmatrix}, \begin{bmatrix} -\cos kx_1 & \sin kx_2 \\ \sin kx_1 & \cos kx_2 \end{bmatrix}, \begin{bmatrix} \cos kx_1 & \cos kx_2 \\ \sin kx_1 & \sin kx_2 \end{bmatrix} \right\}$$

is a complete orthogonal basis of H, where $k = 1, 2, \ldots$. For any positive integer N, let H_N be a $4N$-dimensional subspace of H. P_N is the orthogonal projection from H to H_N. For all $u \in H$, we have $p = P_N u$ and $q = Q_N u$, then $u = p + q$ and $I = P_N + Q_N$.

Then we decompose the system (2.1) into two parts by orthogonal projection

$$p_t - \nu\Delta p + rp + P_N B(u,u) = P_N g, \tag{3.1}$$

$$q_t - \nu\Delta q + rq + Q_N B(u,u) = Q_N g. \tag{3.2}$$

For the solution $u = p+q$ determined by the initial value $u_0 \in B$, we construct the asymptotic solution sequence by iteration method

$$\begin{cases} q_t^0 - \nu\Delta q^0 + rq^0 + Q_N B(p,p) = Q_N g, \quad k = 0, \\ q^0(x,0) = Q_N u_0, \end{cases} \tag{3.3}$$

$$\begin{cases} q_t^k - \nu\Delta q^k + rq^k + Q_N B(u^{k-1},u^{k-1}) = Q_N g, \quad k = 1, 2, \ldots, \\ q^k(x,0) = Q_N^k u_0, \end{cases} \tag{3.4}$$

where $u^k = p + q^k$. The existence and uniqueness of the solution and the existence of the global attractor for the system (3.3) and (3.4) are similar to the damped and driven Navier-Stokes system, so we do not present the proof here.

Next we take two steps to study the asymptotic approximation $u^k(x,t)$ of $u(x,t)$. Firstly, we will prove that for all $u_0 \in B$, the asymptotic solution sequence obtained above will not go away from the absorbing set B. Secondly, we will prove $q^k(x,t)$ converges to $q(x,t)$ in the space V.

Theorem 3.1 *If* $u(x,t)$ *is the solution of system (2.1) corresponding to* $u_0 \in B$*,* $g \in L^2(Q)$*,* q^k $(k = 0, 1, 2, \ldots)$ *are given by (3.3) and (3.4), then there exists a positive integer* N_0 *and a positive constant* $t_0(B) > 0$*, such that*

$$|u^k| \le 2\rho_0, \ |\nabla u^k| \le 2\rho_1, \ t \ge t_0(B), \ N \ge N_0, \quad k = 0, 1, 2, \ldots.$$

Proof Because of the positive invariance of the absorbing set, we have $u \in B$ for all $u_0 \in B$. Thus $|p| \le \rho_0$, $|p|_V \le \rho_1$. Since $u^k = p + q^k$, we just need to prove

$$|q^k| \le \rho_0, \ |\nabla q^k| \le \rho_1, \quad k = 0, 1, 2, \ldots. \tag{3.5}$$

We use induction to prove.

Let $k=0$. Considering the inner product of (3.3) with q^0, and using Sobolev embedding inequality $|u|_{L^\infty}\le C_0|u|_{H_p^2},\forall u\in H_p^2(Q)$, we get

$$\begin{aligned}\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|q^0|^2+\nu|\nabla q^0|^2+r|q^0|^2&\le(|B(p,p)|+|g|)\cdot|q^0|\\ &\le(|p|_{L^\infty}|\nabla p|+|g|)\cdot|q^0|\le(C_0\rho_2\rho_1+|g|)\cdot|q^0|.\end{aligned}$$

From the complete orthogonal basis of H, we have $|\nabla q|\ge(N+1)|q|$, and get

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|q^0|^2+[\nu(N+1)^2+r]|q^0|^2\le(C_0\rho_2\rho_1+|g|)\cdot|q^0|.$$

By the Gronwall's inequality, we obtain

$$|q^0(t)|\le\rho_0\cdot e^{-[\nu(N+1)^2+r]t}+\frac{C_0\rho_2\rho_1+|g|}{\nu(N+1)^2+r}(1-e^{-[\nu(N+1)^2+r]t}).$$

So there exists $t_1(B)=\frac{1}{\nu(N+1)^2+r}\ln\frac{[\nu(N+1)^2+r]\rho_0}{C_0\rho_2\rho_1+|g|}$, such that

$$|q^0(t)|\le\frac{2(C_0\rho_2\rho_1+|g|)}{\nu(N+1)^2+r}.$$

When $N_1\in\mathbf{N}$ is large enough, satisfying $\frac{2(C_0\rho_2\rho_1+|g|)}{\nu(N+1)^2+r}\le\rho_0$, we get $|q^0(t)|\le\rho_0$, $t\ge t_1(B)$.

Considering the inner product of (3.3) with $-\Delta q^0$, we have

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|\nabla q^0|^2+\nu|\Delta q^0|^2+r|\nabla q^0|^2\le(|B(p,p)|+|g|)\cdot|\Delta q^0|\le(C_0\rho_2\rho_1+|g|)\cdot|\Delta q^0|.$$

From Cauchy inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla q^0|^2+[\nu(N+1)^2+2r]|\nabla q^0|^2\le\frac{(C_0\rho_2\rho_1+|g|)^2}{\nu}.$$

By the Gronwall's inequality, we obtain

$$|\nabla q^0|^2\le\rho_1^2\cdot e^{-[\nu(N+1)^2+2r]t}+\frac{(C_0\rho_2\rho_1+|g|)^2}{\nu[\nu(N+1)^2+2r]}(1-e^{-[\nu(N+1)^2+2r]t}).$$

So there exists $t_2(B)=\frac{1}{\nu(N+1)^2+2r}\ln\frac{\nu[\nu(N+1)^2+2r]\rho_1^2}{(C_0\rho_2\rho_1+|g|)^2}$, such that

$$|\nabla q^0(t)|^2\le\frac{2(C_0\rho_2\rho_1+|g|)^2}{\nu[\nu(N+1)^2+2r]}.$$

When $N_2\in\mathbf{N}$ is large enough, satisfying $\frac{2(C_0\rho_2\rho_1+|g|)^2}{\nu[\nu(N+1)^2+2r]}\le\rho_1^2$, we get $|\nabla q^0(t)|^2\le\rho_1^2$, $t\ge t_2(B)$. So (3.5) holds for $k=0$.

Suppose (3.5) holds for $k-1$, such that $|q^{k-1}|\le\rho_0$, $|\nabla q^{k-1}|\le\rho_1$.

Since $u^{k-1}=p+q^{k-1}$, we have

$$|B(u^{k-1},u^{k-1})|\le|u^{k-1}|_{L^\infty}|\nabla u^{k-1}|=|p+q^{k-1}|_{L^\infty}|\nabla p+\nabla q^{k-1}|\le4C_0\rho_2\rho_1.$$

Considering the inner product of (3.4) with q^k, we get

$$\begin{aligned}\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|q^k|^2+[\nu(N+1)^2+r]|q^k|^2&\le(|B(u^{k-1},u^{k-1})|+|g|)\cdot|q^k|\\ &\le(4C_0\rho_2\rho_1+|g|)\cdot|q^k|.\end{aligned}$$

By the Gronwall's inequality, we obtain

$$|q^k(t)| \leq \rho_0 \cdot e^{-[\nu(N+1)^2+r]t} + \frac{4C_0\rho_2\rho_1 + |g|}{\nu(N+1)^2+r}(1 - e^{-[\nu(N+1)^2+r]t}).$$

So there exists $t_1'(B) = \frac{1}{\nu(N+1)^2+r}\ln\frac{[\nu(N+1)^2+r]\rho_0}{4C_0\rho_2\rho_1+|g|}$, such that

$$|q^k(t)| \leq \frac{2(4C_0\rho_2\rho_1 + |g|)}{\nu(N+1)^2+r}.$$

When $N_1' \in \mathbf{N}$ is large enough, satisfying $\frac{2(4C_0\rho_2\rho_1+|g|)}{\nu(N+1)^2+r} \leq \rho_0$, we get

$$|q^k(t)| \leq \rho_0, \quad t \geq t_1'(B). \tag{3.6}$$

Considering the inner product of (3.4) with $-\Delta q^k$, we have

$$\begin{aligned}\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|\nabla q^k|^2 + \nu|\Delta q^k|^2 + r|\nabla q^k|^2 &\leq (|B(u^{k-1}, u^{k-1})| + |g|)\cdot|\Delta q^k| \\ &\leq (4C_0\rho_2\rho_1 + |g|)\cdot|\Delta q^k|.\end{aligned}$$

From Cauchy inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla q^k|^2 + [\nu(N+1)^2 + 2r]|\nabla q^k|^2 \leq \frac{(4C_0\rho_2\rho_1 + |g|)^2}{\nu}.$$

By the Gronwall's inequality, we obtain

$$|\nabla q^k|^2 \leq \rho_1^2 \cdot e^{-[\nu(N+1)^2+2r]t} + \frac{(4C_0\rho_2\rho_1 + |g|)^2}{\nu[\nu(N+1)^2+2r]}(1 - e^{-[\nu(N+1)^2+2r]t}).$$

So there exists $t_2'(B) = \frac{1}{\nu(N+1)^2+2r}\ln\frac{\nu[\nu(N+1)^2+2r]\rho_1^2}{(4C_0\rho_2\rho_1+|g|)^2}$, such that

$$|\nabla q^k(t)|^2 \leq \frac{2(4C_0\rho_2\rho_1 + |g|)^2}{\nu[\nu(N+1)^2+2r]}.$$

When $N_2' \in \mathbf{N}$ is large enough, satisfying $\frac{2(4C_0\rho_2\rho_1+|g|)^2}{\nu[\nu(N+1)^2+2r]} \leq \rho_1^2$, we get

$$|\nabla q^k(t)|^2 \leq \rho_1^2, t \geq t_2'(B). \tag{3.7}$$

So (3.5) holds for k.

By principle of induction, $N_0 \in \mathbf{N}$ is the minimum natural number satisfying (3.6) and (3.7), and $t_0(B)$=max$\{t_1(B), t_2(B), t_1'(B), t_2(B)'\}$, where $t_i(B), t_i'(B), N_i$ and N_i' $(i = 1, 2)$ are independent of k. Thus (3.5) holds for all $k \in N$. □

Theorem 3.2 *Suppose that $u(x,t)$ is the solution of system (2.1) corresponding to the initial value $u_0 \in B$, and q^k $(k = 0, 1, 2, \ldots)$ are given by (3.3) and (3.4), then there exist a positive integer N^* and a positive constant $t^*(B) > 0$, such that*

$$|q^k - q|_V \to 0, \quad k \to \infty, \ t \geq t^*(B), \ N \geq N^*.$$

Proof Let $\omega^k = q^k - q$. Then from (3.2) and (3.4), we have

$$\omega_t^k - \nu\Delta\omega^k + r\omega^k + Q_N B(u^{k-1}, u^{k-1}) - Q_N B(u, u) = 0, \quad k = 1, 2, \ldots, \tag{3.8}$$

where $B(u^{k-1}, u^{k-1}) - B(u, u)$=$B(\omega^{k-1}, u^{k-1}) + B(u, \omega^{k-1})$.

Considering the inner product of (3.8) with $-\Delta\omega^k$, and using the Sobolev embedding $H_p^1(Q)$ into $L^4(Q)$, we have

$$\begin{aligned}
&\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\omega^k|^2+\nu|\Delta\omega^k|^2+r|\nabla\omega^k|^2\le|B(u^{k-1},u^{k-1})-B(u,u)|\cdot|\Delta\omega^k|\\
&\quad\le(|B(\omega^{k-1},u^{k-1})|+|B(u,\omega^{k-1})|)\cdot|\Delta\omega^k|\\
&\quad\le(|\omega^{k-1}|_{L^4}|\nabla u^{k-1}|_{L^4}+|u^k|_{L^\infty}|\nabla\omega^{k-1}|)\cdot|\Delta\omega^k|\\
&\quad\le C_1\rho_2|\nabla\omega^{k-1}||\Delta\omega^k|.
\end{aligned}$$

From Cauchy inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\omega^k|^2+[\nu(N+1)^2+2r]|\nabla\omega^k|^2\le\frac{C_1^2\rho_2^2}{\nu}\cdot|\nabla\omega^{k-1}|^2. \tag{3.9}$$

Since $\omega^0=q^0-q$, by (3.2) and (3.3), we have

$$\omega_t^0-\nu\Delta\omega^0+r\omega^0+Q_NB(p,p)-Q_NB(u,u)=0. \tag{3.10}$$

Then taking the inner product of (3.10) with $-\Delta\omega^0$, we get

$$\begin{aligned}
&\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\omega^0|^2+\nu|\Delta\omega^0|^2+r|\nabla\omega^0|^2\le|B(u,u)-B(p,p)|\cdot|\Delta\omega^0|\\
&\quad\le(|B(u,q)|+|B(q,p)|)\cdot|\Delta\omega^0|\le(|u|_{L^\infty}|\nabla q|+|q|_{L^\infty}|\nabla p|)\cdot|\Delta\omega^0|\\
&\quad\le C_1\rho_2\rho_1|\Delta\omega^0|.
\end{aligned}$$

From Cauchy inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\omega^0|^2+[\nu(N+1)^2+2r]|\nabla\omega^0|^2\le\frac{C_1^2\rho_2^2\rho_1^2}{\nu}.$$

By Gronwall's inequality, we get

$$|\nabla\omega^0|^2\le|\nabla\omega^0(0)|^2\cdot e^{-[\nu(N+1)^2+2r]t}+\frac{C_1^2\rho_2^2\rho_1^2}{\nu[\nu(N+1)^2+2r]}(1-e^{-[\nu(N+1)^2+2r]t}).$$

So there exists $t_0^*(B)=\frac{1}{\nu(N+1)^2+2r}\ln\frac{\nu[\nu(N+1)^2+2r]\cdot|\nabla\omega^0(0)|^2}{C_1^2\rho_2^2\rho_1^2}$, such that

$$|\nabla\omega^0(t)|^2\le\frac{2C_1^2\rho_2^2\rho_1^2}{\nu[\nu(N+1)^2+2r]},\quad t>t_0^*(B).$$

Let $k=1$ in (3.9). We have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\omega^1|^2+[\nu(N+1)^2+2r]|\nabla\omega^1|^2\le\frac{C_1^2\rho_2^2}{\nu}\cdot|\nabla\omega^0|^2.$$

By Gronwall's inequality, we get

$$|\nabla\omega^1|^2\le|\nabla\omega^1(0)|^2\cdot e^{-[\nu(N+1)^2+2r]t}+\frac{C_1^2\rho_2^2\cdot|\nabla\omega^0|^2}{\nu[\nu(N+1)^2+2r]}(1-e^{-[\nu(N+1)^2+2r]t}).$$

So there exists $t_1^*(B)=\frac{1}{\nu(N+1)^2+2r}\ln\frac{\nu[\nu(N+1)^2+2r]\cdot|\nabla\omega^1(0)|^2}{C_1^2\rho_2^2\cdot|\nabla\omega^0|^2}$, such that

$$|\nabla\omega^1(t)|^2\le\frac{2C_1^2\rho_2^2\cdot|\nabla\omega^0|^2}{\nu[\nu(N+1)^2+2r]},\quad t>t_1^*(B).$$

By iteration method for (3.9), we find that there exists $t_k^*(B)>0$, such that

$$|\nabla\omega^k|^2\le(\frac{2C_1^2\rho_2^2}{\nu[\nu(N+1)^2+2r]})^k\cdot|\nabla\omega^0|^2,\quad t\ge t_k^*(B),\ k=1,2,\ldots,$$

and $|\nabla\omega^k| \to 0$ as $k \to \infty$. Thus $N^* \in \mathbf{N}$ is the minimum natural number satisfying

$$\frac{2C_1^2\rho_2^2}{\nu[\nu(N+1)^2+2r]} < 1. \tag{3.11}$$

The proof is completed. □

Remark 3.3 The proofs of Theorems 3.1 and 3.2 show that the asymptotic solution sequence approximates infinitely the true solution.

Remark 3.4 The dimension of the asymptotic attractor is the dimension of the asymptotic solution sequence, namely

$$N_{B^k} = \min\{N \in \mathbf{N} : \frac{2(4C_0\rho_2\rho_1+|g|)}{\nu(N+1)^2+r} \le \rho_0; \frac{2(4C_0\rho_2\rho_1+|g|)^2}{\nu[\nu(N+1)^2+2r]} \le \rho_1^2; \frac{2C_1^2\rho_2^2}{\nu[\nu(N+1)^2+2r]} < 1\}.$$

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