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On Characterization of Monoids by Properties of Generators

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Abstract Kilp and Knauer in (Comm. Algebra, 1992, 20(7), 1841–1856) gave characterizations of monoids when all generators in category of right S-acts (S is a monoid) satisfy properties such as freeness, projectivity, strong flatness, Condition (P), principal weak flatness, principal weak injectivity, weak injectivity, injectivity, divisibility, strong faithfulness and torsion freeness. Sedaghtjoo in (Semigroup Forum, 2013, 87: 653–662) characterized monoids by some other properties of generators including weak flatness, Condition (E) and regularity. To our knowledge, the problem has not been studied for properties mentioned above of (finitely generated, cyclic, monocyclic, Rees factor) right acts. In this article we answer the question corresponding to these properties and also fg-weak injectivity.

Keywords generator; finitely generated; Subannihilator congruence; injective

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1. Introduction

Kilp and Knauer in [1], investigated monoids over which all generators in category of right S-acts satisfy an special flatness property. Continuing this study Sedaghatjoo in [2], investigated monoids over which all generators are weakly flat, satisfy Condition (E) or regular. Here we investigate the corresponding problem for (finitely generated, cyclic, monocyclic, Rees factor) right acts.

For a monoid S, a nonempty set A is called a right S-act, usually denoted by A_S, if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \longrightarrow A$, $(a, s) \longmapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A_S$ and all $s, t \in S$. Throughout this article, S will always stand for a monoid and A_S is a right S-act. For basic definitions and terminology relating semigroups and acts over monoids, we refer the reader to [3] and [4].

Let $\mathbf{C}$ be a category. An object $G \in \mathbf{C}$ is called a generator in $\mathbf{C}$ if the functor $\mathrm{Mor}_{\mathbf{C}}(G, -)$ is faithful, i.e., for any $X, Y \in \mathbf{C}$ and any $f, g \in \mathrm{Mor}_{\mathbf{C}}(X, Y)$ with $f \neq g$ there exists $\alpha \in \mathrm{Mor}_{\mathbf{C}}(G, X)$ such that $f\alpha \neq g\alpha$.

We recall from [4, II, 3.16] that G_S is a generator if and only if there exists an epimorphism $\pi : G_S \longrightarrow S_S$. Hence S_S is a generator in **Act**-S. Also note that if A_S is a generator then for

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every right S-act B_S such that $\mathrm{Hom}(B, S) \neq \emptyset$, $A_S \sqcup B_S$ is a generator. Thus for every right ideal I of S, $A_S \sqcup I$ is a generator, whenever A_S is a generator.

The following lemma will be useful in our main results later.

Lemma 1.1 *Let A_S be a right S-act such that $\mathrm{Hom}(A_S, S_S) \neq \emptyset$. Then A_S is a retract of $S \times (S \times A_S)$.*

Proof Let A_S be a right S-act such that $\mathrm{Hom}(A_S, S_S) \neq \emptyset$ and let $f \in \mathrm{Hom}(A_S, S_S)$. Then by universal property, there exists a homomorphism $h : A_S \longrightarrow S \times A_S$ making the following diagram commutative.

$$\begin{tikzcd}
 & A_S \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "\mathrm{id}_A"] & \\
S_S & S \times A_S \arrow[l, "p_1"] \arrow[r, "p_2"'] & A_S
\end{tikzcd}$$

Diagram 1 (P_1)

Therefore, A_S is a retract of $S \times A_S$. Since $\mathrm{Hom}(S \times A_S, S_S) \neq \emptyset$, $S \times A_S$ is a retract of $S \times (S \times A_S)$ and so A_S is a retract of $S \times (S \times A_S)$ as required. □

Theorem 1.2 ([2, Theorem 1.2]) *Let S be a monoid and α be an act property which is preserved under retraction and transferred from coproducts to their components. The following assertions are equivalent:*

(1) All generators satisfy property α;

(2) $S \times A_S$ satisfies property α for every right S-act A_S;

(3) A right S-act A_S satisfies property α if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$.

Corollary 1.3 ([2, Corollary 1.3]) *Let S be a monoid and α be an act property which is transferred from coproducts to their components. If all generators satisfy property α, then for each nonempty set I, S^I satisfies property α.*

In the following we show the condition that property α is transferred from coproduct to their components in Theorem 1.2 and Corollary 1.3 is redundant and can be omitted. Moreover, we add a new equivalent statement to statements in Theorem 1.2. In fact the following theorem will form the main object of our concern in sequel.

Theorem 1.4 *Let S be a monoid and α be an act property which is preserved under retraction. Then the following statements are equivalent:*

(1) All generators satisfy property α;

(2) $S \times A_S$ satisfies property α for every right S-act A_S;

(3) $S \times A_S$ satisfies property α for every generator A_S;

(4) A right S-act A_S satisfies property α if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$.

Proof (1)⇒(2). Since $S \times A_S$ is a generator, it is obvious.

(2)⇒ (3). It is obvious.

(3)⇒ (4). Let A_S be a right S-act such that $\mathrm{Hom}(A_S, S_S) \neq \emptyset$. Since $S \times A_S$ is a generator, $S \times (S \times A_S)$ satisfies property α by assumption. Thus by assumption and Lemma 1.1, A_S is a retract of $S \times (S \times A_S)$, and so the result follows.

(4)⇒ (1). Follows from Theorem 1.2. □

Lemma 1.5 *Let S be a monoid and α be an act property. If all generators satisfy property α, then for every nonempty set I, S^I satisfies property α.*

Proof Since S^I is a generator for every nonempty set I, the result follows. □

For comparison with what follows, we recall the next corollary from [2].

Corollary 1.6 ([2, Corollary 1.4]) *Let S be a monoid and α be an act property which is transferred from coproducts to their components. If all generators satisfy property α, then all right ideals of S satisfy property α.*

In light of Theorem 1.4 and regarding the fact that for every right ideal I of S, $\mathrm{Hom}(I, S) \neq \emptyset$, Corollary 1.6 is extended to the following:

Corollary 1.7 *Let S be a monoid and α be an act property which is preserved under retraction or transferred from coproducts to their components. If all generators satisfy property α, then all right ideals of S satisfy property α.*

We recall from [2] that a right congruence ρ on S_S is a right subannihilator congruence if $\rho \leq \ker \lambda_s$ for some $s \in S$.

Theorem 1.8 *Let S be a monoid and α be an act property which is preserved under retraction. If the following three statements*

(1) All generators satisfy property α;

(2) S_S satisfies property α;

(3) $D(S) = (S \times S)_S$ satisfies property α;

are equivalent, then the following statements are also equivalent:

(1) All generators satisfy property α;

(2) All finitely generated generators satisfy property α;

(3) All cyclic generators satisfy property α;

(4) All monocyclic generators satisfy property α;

(5) $S \times A_S$ satisfies property α for every right S-act A_S;

(6) $S \times A_S$ satisfies property α for every finitely generated right S-act A_S;

(7) $S \times A_S$ satisfies property α for every cyclic right S-act A_S;

(8) $S \times A_S$ satisfies property α for every monocyclic right S-act A_S;

(9) $S \times A_S$ satisfies property α for every Rees factor right S-act A_S;

(10) $S \times A_S$ satisfies property α for every generator A_S;

(11) $S \times A_S$ satisfies property α for every finitely generated generator A_S;

(12) *$S \times A_S$ satisfies property α for every cyclic generator A_S;*

(13) *$S \times A_S$ satisfies property α for every monocyclic generator A_S;*

(14) *A right S-act A_S satisfies property α if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;*

(15) *A finitely generated right S-act A_S satisfies property α if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;*

(16) *A cyclic right S-act A_S satisfies property α if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;*

(17) *A monocyclic right S-act A_S satisfies property α if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;*

(18) *For every right subannihilator congruence ρ, S/ρ satisfies property α;*

(19) *$(S_S)^I$ satisfies property α for every nonempty set I;*

(20) *$(S_S)^k$ satisfies property α for every $k \in \mathbb{N}$.*

Proof Implications (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (10) $\Leftrightarrow$ (14) are clear from Theorem 1.4.

Implications $(5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8), (10) \Rightarrow (11) \Rightarrow (12) \Rightarrow (13), (7) \Rightarrow (9), (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4), (14) \Rightarrow (15) \Rightarrow (16) \Rightarrow (17), (19) \Rightarrow (20), (1) \Rightarrow (19)$ and $(20) \Rightarrow (1)$ are obvious.

Also it is easy to prove implications (9) $\Rightarrow$ (1), (8) $\Rightarrow$ (1), (4) $\Rightarrow$ (1), (17) $\Rightarrow$ (1), (18) $\Rightarrow$ (1). Now we show implication (14) $\Rightarrow$ (18).

Let ρ be a right subannihilator congruence. Thus there exists $s \in S$ such that $\rho \leq \ker \lambda_s$. Define $f : S/\rho \longrightarrow S_S$ by $f([t]_\rho) = st$. Clearly, f is an S-homomorphism and so $\mathrm{Hom}(S/\rho, S_S) \neq \emptyset$. Thus S/ρ satisfies property α by assumption. □

It is obvious that all properties under discussion here are preserved under retraction.

2. Monoids over which all generators are torsion free or weakly flat

In this section we begin our investigation with the weakest of flatness property. An act A_S is called torsion free, if for any $x, y \in A_S$ and any right cancellable element $c \in S$ the equality $xc = yc$ implies $x = y$ (see [4]). We recall from [1] the following theorem.

Theorem 2.1 ([1, Theorem 3.1]) *The following conditions on a monoid S are equivalent.*

(1) *All generators are torsion free;*

(2) *All right S-acts are torsion free;*

(3) *Every right cancellable element of S is right invertible.*

Lemma 2.2 *Any generator contains a generator cyclic subact.*

Proof Let A_S be a generator. Then there exists an epimorphism $\pi : A_S \longrightarrow S_S$. Since π is an epimorphism, there exists $z \in A_S$ such that $\pi(z) = 1$. Let $A^\star = zS$, then $\pi|_{A^\star} : A^\star \longrightarrow S_S$ is an epimorphism, and so $A^\star$ is a generator cyclic subact of A_S. □

In the following theorem we give some more equivalent conditions to the conditions in the above theorem.

Theorem 2.3 *For any monoid S the following statements are equivalent:*

(1) *All generators are torsion free;*

(2) *All finitely generated generators are torsion free;*

(3) *All generators generated by at most three elements are torsion free;*

(4) $S \times A_S$ *is torsion free for every generator* A_S*;*

(5) $S \times A_S$ *is torsion free for every finitely generated generator* A_S*;*

(6) $S \times A_S$ *is torsion free for every generator* A_S *generated by at most three elements;*

(7) $S \times A_S$ *is torsion free for every right* S*-act* A_S*;*

(8) $S \times A_S$ *is torsion free for every finitely generated right* S*-act* A_S*;*

(9) $S \times A_S$ *is torsion free for every right* S*-act* A_S *generated by at most two elements;*

(10) A right S *act* A_S *is torsion free if* $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(11) A finitely generated right S *act* A_S *is torsion free if* $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(12) A right S *act* A_S *generated by at most two elements is torsion free if* $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(13) All right S*-acts are torsion free;*

(14) Every right cancellable element of S *is right invertible.*

Proof Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) $\Leftrightarrow$ (13) $\Leftrightarrow$ (14) are clear from Theorems 1.4 and 2.1.

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) and (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(9) $\Rightarrow$ (7). Let A_S be a right S-act, $(s, x)c = (t, y)c$, for $s, t \in S, x, y \in A_S$ and right cancellable element $c \in S$ and let $A^\star = xS \cup yS$. Then $S \times A^\star_S$ is torsion free by assumption. Thus $(s, x)c = (t, y)c$ implies that $(s, x) = (t, y)$ and so $S \times A_S$ is torsion free.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $\mathrm{Hom}(A_S, S_S) \neq \emptyset$ and suppose $xc = yc$, for $x, y \in A_S$ and right cancellable element $c \in S$. Since $\mathrm{Hom}(A_S, S_S) \neq \emptyset$ there exists homomorphism $f : A_S \longrightarrow S_S$. If $A^\star = xS \cup yS$ and $f^\star = f|_{A^\star}$, then $xc = yc$ implies $x = y$ by assumption and so A_S is torsion free.

(3) $\Rightarrow$ (1). Let A_S be a generator and $xc = yc$, for $x, y \in A_S$ and right cancellable element $c \in S$. Then there exists $z \in A_S$ such that $\pi(z) = 1$ and $A^\star = xS \cup yS \cup zS$ is a generator, by Lemma 2.2. Thus $A^\star$ is torsion free by assumption. Hence $xc = yc$ implies $x = y$, as required.

(6) $\Rightarrow$ (1). Let A_S be a generator and $xc = yc$, for $x, y \in A_S$ and right cancellable element $c \in S$. If $A^\star = xS \cup yS \cup zS$ is as in the proof of (3) $\Rightarrow$ (1), then $(1, x)c = (1, y)c$ in $S \times A^\star$. Clearly, $A^\star$ is a generator and so $S \times A^\star$ is torsion free by assumption, thus $(1, x)c = (1, y)c$ implies that $(1, x) = (1, y)$. Hence $x = y$ as required. □

An act A_S is called *flat* if the functor $A_S \otimes {}_S-$ preserves all monomorphisms of left S-acts. If the functor $A_S \otimes {}_S-$ preserves embeddings of (principal) left ideal into S, then A_S is called (principally) weakly flat [4]. By [2], a right S-act A_S is called almost weakly flat if A_S is principally weakly flat and satisfies Condition

(W') If $as = a't$, and $Ss \cap St \neq \emptyset$, for $a, a' \in A_S, s, t \in S$, then there exist $a'' \in A_S$, $u \in Ss \cap St$ such that $as = a't = a''u$.

It is proved in [2, Theorem 3.4] that all generators are weakly flat if and only if all right S-acts are almost weakly flat.

Lemma 2.4 ([4, III, 11.4]) *An act* A_S *is weakly flat if and only if it is principally weakly flat and satisfies Condition*

(W) If $as = a't$ for $a, a' \in A_S, s, t \in S$, then there exist $a'' \in A_S$, $u \in Ss \cap St$, such that $as = a't = a''u$.

Theorem 2.5 *For any monoid S the following statements are equivalent:*

(1) All generators are weakly flat;

(2) All finitely generated generators are weakly flat;

(3) All generators, generated by at most three elements are weakly flat;

(4) $S \times A_S$ is weakly flat for every generator A_S;

(5) $S \times A_S$ is weakly flat for every finitely generated generator A_S;

(6) $S \times A_S$ is weakly flat for every generator A_S generated by at most three elements;

(7) $S \times A_S$ is weakly flat for every right S-act A_S;

(8) $S \times A_S$ is weakly flat for every finitely generated right S-act A_S;

(9) $S \times A_S$ is weakly flat for every right S-act A_S generated by at most two elements;

(10) A right S-act A_S is weakly flat if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;

(11) A finitely generated right S-act A_S is weakly flat if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;

(12) A right S-act A_S generated by at most two elements is weakly flat if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;

(13) All right S-acts are almost weakly flat;

(14) S is regular and for each $s, t \in S$ with $Ss \cap St \neq \phi$ there exists $w \in Ss \cap St$ such that $1(\ker\lambda_s \vee \ker\lambda_t)w$.

Proof Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (10) $\Leftrightarrow$ (13) $\Leftrightarrow$ (14) are obvious from [2, Theorems 3.4, 3.8] and Theorem 1.4.

Implications (7) $\Rightarrow$ (8) $\Rightarrow$ (9), (10) $\Rightarrow$ (11) $\Rightarrow$ (12), (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(9) $\Rightarrow$ (7). Let A_S be a right S-act. First we show that $S \times A_S$ satisfies Condition (W). Suppose that $(l_1, a)s = (l_2, a')t$, for $a, a' \in A_S$, $l_1, l_2, s, t \in S$ and let $A^\star = aS \cup a'S$. Then $S \times A^\star_S$ satisfies Condition (W) and so there exists $(l, a'') \in S \times A^\star_S \subseteq S \times A_S$ and $u \in Ss \cap St$, such that $(l_1, a)s = (l_2, a')t = (l, a'')u$. Now let $(w_1, b)s = (w_2, b')s$, for $(w_1, b), (w_2, b') \in S \times A_S, s \in S$ and let $B = bS \cup b'S$. Since $S \times B_S$ is principally weakly flat the equality $(w_1, b)s = (w_2, b')s$ in $S \times B_S$ implies $(w_1, b) \otimes s = (w_2, b') \otimes s$ in $(S \times B_S) \otimes Ss \subseteq (S \times A_S) \otimes Ss$, and so $S \times A_S$ is principally weakly flat. Thus $S \times A_S$ is weakly flat by Lemma 2.4.

(12) $\Rightarrow$ (10). Let A_S be a right S-act such that $\mathrm{Hom}(A_S, S_S) \neq \emptyset$ and suppose $as = a't$, for $a, a' \in A_S, s, t \in S$. Since $\mathrm{Hom}(A_S, S_S) \neq \emptyset$, there exists a homomorphism $f : A_S \longrightarrow S_S$. If $A^\star = aS \cup a'S$ and $f^\star = f|_{A^\star}$, then $A^\star$ satisfies Condition (W) by assumption and so $as = a't$ in $A^\star$ implies that there exists $a'' \in A^\star \subseteq A_S$ and $u \in Ss \cap St$ such that $as = a't = a''u$, this implies that A_S satisfies Condition (W). Now let $as = a's$ for $a, a' \in A_S$ and $s \in S$. If $B = aS \cup a'S$ and $g = f\mid_{B_S}$, Then B is principally weakly flat by assumption and so $as = a's$ in B_S implies $a \otimes s = a' \otimes s$ in $B \otimes Ss \subseteq A_S \otimes Ss$. Thus A_S is principally weakly flat and so A_S is weakly flat by Lemma 2.4.

(3) $\Rightarrow$ (1). Let A_S be a generator and $a \otimes s = a' \otimes s$ in $A \otimes S$, for $a, a' \in A_S, s \in S$. Then $as = a's$ in A_S and by Lemma 2.2, there exists $a'' \in A_S$ such that $\pi(a'') = 1$ and

$A^\star = aS \cup a'S \cup a''S$ is a generator. Thus $as = a's$ in A_S implies that $as = a's$ in $A^\star$ and so $a \otimes s = a' \otimes s$ in $A^\star \otimes S$. Since $A^\star$ is a generator so $A^\star$ is weakly flat by assumption. Thus $a \otimes s = a' \otimes s$ in $A^\star_S \otimes {}_S Ss \subseteq A_S \otimes {}_S Ss$. Hence A_S is principally weakly flat. It is easy to see that A_S satisfies Condition (W) and so A_S is weakly flat.

(6) $\Rightarrow$ (4). Let A_S be a generator and $(l_1, a) \otimes s = (l_2, a') \otimes s$ in $(S \times A)_S \otimes {}_S S$, for $l_1, l_2, s \in S$ and $a, a' \in A_S$. Let $A^\star = aS \cup a'S \cup a''S$ be as in the proof of (3) $\Rightarrow$ (1). Thus $S \times A^\star$ is weakly flat by assumption. Hence, $(l_1, a) \otimes s = (l_2, a') \otimes s$ in $(S \times A^\star)_S \otimes {}_S Ss \subseteq (S \times A)_S \otimes {}_S Ss$ and so $S \times A$ is principally weakly flat. Now we show that $S \times A$ satisfies Condition (W). Let $(l_1, a)s = (l_2, a')t$ in $S \times A$, for $l_1, l_2, s, t \in S$ and $a, a' \in A_S$. Similar to that of (3) $\Rightarrow$ (1) if $A^\star = aS \cup a'S \cup a''S$, clearly, $A^\star$ is a generator and so $S \times A^\star$ satisfies Condition (W) by assumption. Thus there exists $(l, a'') \in S \times A^\star \subseteq S \times A$ and $u \in Ss \cap St$ such that $(l, a'')u = (l_1, a)s = (l_2, a')t$. Therefore, $S \times A_S$ is weakly flat by Lemma 2.4. $\square$

3. Monoids over which all generators satisfy Condition (E)

In this section we use Theorem 1.4 to give a characterization of monoids for which all generators satisfy Condition (E).

An S-act A_S satisfies Condition (E), if for all $a \in A_S, s, s' \in S$, $as = as' \Rightarrow (\exists a' \in A_S)(\exists u \in S)(a = a'u$ and $us = us')$.

We recall that a monoid S is left (right) collapsible if for every $s, t \in S$ there exists $u \in S$ such that $us = ut$ ($su = tu$). Let S be a monoid and $x, y \in S$. Then $l(x, y) := \{z \in S \mid zx = zy\}$. Evidently $l(x, y) = \emptyset$ or $l(x, y)$ is a left ideal. If S is a left collapsible monoid, then for every $x, y \in S$, $l(x, y) \neq \emptyset$, and so $l(x, y)$ is a left ideal.

In [2, Theorem 2.2], some equivalents conditions were obtained for all generators to satisfy Condition (E). Here we find some more equivalent conditions, as follows:

Theorem 3.1 *For any monoid S the following statements are equivalent:*

(1) All generators satisfy Condition (E);

(2) All finitely generated generators satisfy Condition (E);

(3) All generators, generated by at most two elements satisfy Condition (E);

(4) $S \times A_S$ satisfies Condition (E) for every generator A_S;

(5) $S \times A_S$ satisfies Condition (E) for every finitely generated generator A_S;

(6) $S \times A_S$ satisfies Condition (E) for every generator A_S generated by at most two elements;

(7) $S \times A_S$ satisfies Condition (E) for every right S-act A_S;

(8) $S \times A_S$ satisfies Condition (E) for every finitely generated right S-act A_S;

(9) $S \times A_S$ satisfies Condition (E) for every cyclic right S-act A_S;

(10) $S \times A_S$ satisfies Condition (E) for every monocyclic right S-act A_S;

(11) A right S-act A_S satisfies Condition (E) if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;

(12) A finitely generated right S-act A_S satisfies Condition (E) if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;

(13) A cyclic right S-act A_S satisfies Condition (E) if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;

(14) A monocyclic right S-act A_S satisfies Condition (E) if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;

(15) $(\forall x, y \in S)(l(x,y) = \emptyset \vee (\exists e \in E(S),\ \rho(x,y) = \ker\lambda_e))$;

(16) *For every right subannihilator congruence* ρ, S/ρ *satisfies Condition (E);*

(17) $(\forall x, y \in S)(l(x,y) = \emptyset \vee S/\rho(x,y)$ *satisfies Condition (E));*

(18) $(\forall x, y \in S)(l(x,y) = \emptyset \vee (\exists u \in S, ux = uy\ \wedge\ 1\ \rho(x,y)\ u))$;

(19) $(\forall x, y, t \in S)(l(tx,ty) = \emptyset \vee S/\rho(tx,ty)$ *satisfies Condition (E));*

(20) $(\forall x, y, t \in S)(l(tx,ty) = \emptyset \vee (\exists u \in S,\ t\ \rho(tx,ty)\ u\ \wedge\ ux = uy))$;

(21) $(\forall x, y \in S)(l(x,y) = \emptyset \vee (\exists e \in E(S),\ ex = ey\ \wedge 1\ \rho(x,y)\ e))$.

Proof Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) $\Leftrightarrow$ (11) $\Leftrightarrow$ (15) $\Leftrightarrow$ (16) $\Leftrightarrow$ (17) are clear from Theorem 1.4 and [2, Theorem 2.2].

Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10), (11) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14), (17) $\Rightarrow$ (19) are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and $as = at$, for $a \in A_S$ and $s, t \in S$. Hence by Lemma 2.2, there exists $a' \in A_S$ such that $\pi(a') = 1$ and $A^\star = aS \cup a'S$ is a generator. Thus $A^\star$ satisfies Condition (E) by assumption. Hence $as = at$ in $A^\star$ implies the existence of $a'' \in A^\star \subseteq A$ and $u \in S$ such that $a = a''u$ and $us = ut$. Therefore, A_S satisfies Condition (E), as required.

(6) $\Rightarrow$ (4). Let A_S be a generator and $(l, a)s = (l, a)t$, for $a \in A_S$ and $l, s, t \in S$. If $A^\star = aS \cup a'S$ is as in the proof of (3) $\Rightarrow$ (1), then $A^\star$ is a generator and so $S \times A^\star$ satisfies Condition (E) by assumption. Thus there exist $(l', a'') \in S \times A^\star \subseteq S \times A$ and $u \in S$ such that $(l, a) = (l', a'')u$ and $us = ut$. Therefore, $S \times A$ satisfies Condition (E).

(17) $\Rightarrow$ (18). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then $S/\rho(x, y)$ satisfies Condition (E) and so $[1]_\rho x = [1]_\rho y$ implies that there exist $\alpha, u_1 \in S$, such that $[1]_\rho = [\alpha]_\rho u_1$ and $u_1 x = u_1 y$. If $\alpha u_1 = u$, then $[1]_\rho = [u]_\rho$ and $ux = uy$. Hence $1\rho(x, y)u$ and $ux = uy$.

(18) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$. Then by assumption there exists $u_1 \in S$ such that $u_1 tx = u_1 ty$ and $1\rho(tx, ty)u_1$. Thus $t\rho(tx, ty)u_1 t$ and $u_1 tx = u_1 ty$. If $u_1 t = u$, then $t\rho(tx, ty)u$ and $ux = uy$ as required.

(19) $\Rightarrow$ (20). Let $x, y, t \in S$ such that $l(tx, ty) \neq \emptyset$, and suppose $\rho(tx, ty) = \rho$. Since $[t]_\rho x = [t]_\rho y$ and S/ρ satisfies Condition (E), there exist $\alpha, u_1 \in S$, such that $[t]_\rho = [\alpha]_\rho u_1$ and $u_1 x = u_1 y$. If $\alpha u_1 = u$, then $t\rho(tx, ty)u$ and $ux = uy$.

(14) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x, y) \leq \ker\lambda_z$. Define the mapping $f : S/\rho(x, y) \longrightarrow S_S$ by $f([t]_{\rho(x,y)}) = zt$, for $t \in S$. Clearly, f is well defined and is an S-homomorphism. Therefore, $\mathrm{Hom}(S/\rho(x, y), S_S) \neq \emptyset$ and so $S/\rho(x, y)$ satisfies Condition (E) by assumption, as required.

(20) $\Rightarrow$ (21). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. If $t = 1$, then there exists $u \in S$ such that $ux = uy$ and $1\rho(x, y)u$. If $\rho = \rho(x, y)$, then $(x, y) \in \ker\lambda_u$ implies that $\rho \subseteq \ker\lambda_u$. Since $1\rho u$ we have $(1, u) \in \ker\lambda_u$, that is, $u = u^2$ and so u is an idempotent. Let $u = e$. Since $(x, y) \in \rho \subseteq \ker\lambda_e$ implies that $ex = ey$ and $1\rho(x, y)e$, we are done.

(21) $\Rightarrow$ (15). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$ and let $\rho = \rho(x, y)$. By assumption there exists $e \in E(S)$ such that $ex = ey$ and $1\rho e$. Then $ex = ey$ implies that $\rho \subseteq \ker\lambda_e$. Let $l_1, l_2 \in S$, such that $(l_1, l_2) \in \ker\lambda_e$. Then $el_1 = el_2$, and since $1\rho e$ we have $l_1\rho el_1$, $l_2\rho el_2$ and so $l_1\rho l_2$.

Thus, $\ker \lambda_e \subseteq \rho$, and so $\ker \lambda_e = \rho$, as required.

(10) $\Rightarrow$ (17). Let $x, y \in S$ such that $l(x,y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$ and so $\rho(x,y) \leq \ker \lambda_z$. Suppose $\rho(x,y) = \rho$ and let $l_1, l_2 \in S$ such that $l_1 \rho l_2$, then $zl_1 = zl_2$. Thus $(z, [1]_\rho)l_1 = (z, [1]_\rho)l_2$ in $S \times S/\rho$. The last equality implies by assumption that there exist $(w, [a]_\rho) \in S \times S/\rho$ and $v \in S$ such that $(z, [1]_\rho) = (w, [a]_\rho)v, vl_1 = vl_2$. If $av = u$, then we have $1 \rho u$, and $ul_1 = ul_2$. Thus $S/\rho(x,y)$ satisfies Condition (E) by [4, III, 14.8]. $\square$

Lemma 3.2 ([2, Corollary 2.6]) *Let S be a monoid over which all generators satisfy Condition (E). Then for each pair (x,y) in $S \times S$, $l(x,y) = \emptyset$ or $l(x,y) = S$ or $xS \cup yS = S$.*

The following lemma will be used in our next result.

Lemma 3.3 *Suppose for every $x, y \in S$, $(l(x,y) = \emptyset \ \vee \ l(x,y) = S \ \vee \ xS \cup yS = S)$. If $x' \in S$ is the right inverse of x, then it is also the left inverse of x.*

Proof Let $x' \in S$ be the right inverse of x. Thus $xx' = 1$ and so $x'x \in E(S)$. If $E(S) = \{1\}$, then $xx' = 1 = x'x$. Now suppose that $|E(S)| \geq 2$. Then there exists $e \in E(S) \setminus \{1\}$, and so $ex.x' = ex.x'e$, which implies that $ex \in l(x', x'e)$. Therefore, $l(x', x'e) \neq \emptyset$, and so $l(x', x'e) = S$ or $x'S \cup x'eS = S$. If $l(x', x'e) = S$, then $x' = x'e$. Thus $xx' = xx'e$ and so $e = 1$, a contradiction. Therefore, $x'S \cup x'eS = S$ which implies that $1 \in x'eS$ or $1 \in x'S$. If $1 \in x'eS$, then there exists $l \in S$ such that $1 = x'el$ and so $x'x = 1$. If $1 \in x'S$, then there exists $t \in S$ such that $1 = x't$. Hence $x = xx't = 1t = t$, and so $xx' = 1 = x'x$. $\square$

Lemma 3.4 ([1, Corollary 1.5]) *If S is commutative or if the identity 1 of S is externally adjointed, then all cyclic xS generators are isomorphic to S_S.*

Corollary 3.5 *Let S be a monoid over which all generators satisfy Condition (E). Then all cyclic generators are isomorphic to S_S.*

Proof Let aS be a cyclic generator. Then there exists an epimorphism $\pi : aS \longrightarrow S_S$. Let $\pi(a) = t$. Then there exists $t'' \in S$ such that $tt'' = 1$. Define the mapping $\varphi : aS \longrightarrow S_S$ by $\varphi(as) = s$, $s \in S$. If $as = at'$ for $s, t' \in S$, then by Lemmas 3.2 and 3.3, $t''t = 1$ and so $s = t'$. Hence φ is well defined and so $aS \cong S$. $\square$

4. Monoids over which all generators are strongly faithful

Kilp and Knauer in [1] showed that over a monoid S all generators are strongly faithful if and only if S is left cancellative. Now in the following corollary we add some more equivalent conditions for all generators to be strongly faithful.

Lemma 4.1 *Let S be a monoid. Then the following statements are equivalent:*

(1) All generators are strongly faithful;

(2) $D(S) = (S \times S)_S$ is strongly faithful;

(3) S_S is strongly faithful;

(4) S_S *is left cancellative;*

(5) $(\forall x, y \in S)$ $(l(x, y) = \emptyset \ \vee \ l(x, y) = S)$.

Proof (1) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator and $as = at$, for $a \in A_S, s, t \in S$. Then there exists an epimorphism $\pi : A_S \longrightarrow S_S$ and also $\pi(a)s = \pi(a)t$. Since S_S is strongly faithful, we have $s = t$, that is, A_S is strongly faithful.

(1) $\Rightarrow$ (2). Since all generators are strongly faithful, $S \times A_S$ is strongly faithful for every right S-act A_S and so $D(S) = (S \times S)_S$ is strongly faithful.

(2) $\Rightarrow$ (3). Let $as = at$, for $a, s, t \in S$. Then $(a, a)s = (a, a)t$ and that $D(S)$ is strongly faithful, we have $s = t$, that is S_S, is strongly faithful.

(3) $\Rightarrow$ (2). It is obvious.

(4) $\Rightarrow$ (5). Let $x, y \in S$ such that $l(x, y) \neq \emptyset$. Then there exists $z \in S$ such that $zx = zy$. Since S is left cancellative, $x = y$ and so $l(x, y) = S$.

(5) $\Rightarrow$ (4). Let $x, y, z \in S$ such that $zx = zy$. Thus $z \in l(x, y)$, that is, $l(x, y) \neq \emptyset$ and so $l(x, y) = S$ by assumption. Thus $x = y$ and so S is left cancellative as required.

(1) $\Leftrightarrow$ (4). It follows from [1, Proposition 1.3]. $\square$

Corollary 4.2 *Let S be a monoid. Then all the statements in Lemma 4.1 and Theorem 1.8 are equivalent when the property α is strongly faithful.*

5. Monoids over which all generators are regular

Author in [2] gave three equivalent conditions under which all generators are regular. Now in Theorem 5.1, we give more equivalent conditions to these conditions. We recall from [4] that an element $a \in A_S$ is called act-regular if there exists a homomorphism $f : aS \longrightarrow S$ such that $af(a) = a$, and A_S is called a regular act if every $a \in A_S$ is an act-regular element. By [4, III, 19.3] it is equivalent to saying that every cyclic subact of A is projective.

Theorem 5.1 *For any monoid S the following statements are equivalent:*

(1) All generators are regular;

(2) All finitely generated generators are regular;

(3) All generators generated by at most two elements are regular;

(4) $S \times A_S$ *is regular for every right S-act A_S;*

(5) $S \times A_S$ *is regular for every generator A_S;*

(6) $S \times A_S$ *is regular for every finitely generated generator A_S;*

(7) $S \times A_S$ *is regular for every generator A_S generated by at most two elements;*

(8) $S \times A_S$ *is regular for every finitely generated right S-act A_S;*

(9) $S \times A_S$ *is regular for every cyclic right S-act A_S;*

(10) A right S-act A_S is regular if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(11) A finitely generated right S-act A_S is regular if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(12) A cyclic right S-act A_S is regular if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(13) For every right subannihilator congruence ρ, S/ρ is regular.

(14) For every right subannihilator congruence ρ and for every $s \in S$ there exists an idempotent $e \in S$ such that $\rho s = \ker \lambda_e$.

Proof Implications (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (10) $\Leftrightarrow$ (14) are clear from Theorem 1.4 and [4, III, 19.5].

Implications $(1) \Rightarrow (2) \Rightarrow (3), (5) \Rightarrow (6) \Rightarrow (7), (4) \Rightarrow (8) \Rightarrow (9)$ and $(10) \Rightarrow (11) \Rightarrow (12)$ are obvious.

(3) $\Rightarrow$ (1). Let A_S be a generator. We show that aS is a projective right S-act for $a \in A_S$. By Lemma 2.2, there exists $a' \in A_S$ such that $\pi(a') = 1$ and $A^\star = aS \cup a'S$ is a generator generated by two elements. Thus $A^\star$ is regular by assumption and so aS is projective.

(7) $\Rightarrow$ (5). Let A_S be a generator and $(s,a)S$ be a cyclic subact of $S \times A$. If $A^\star = aS \cup a'S$ similar to that of (3) $\Rightarrow$ (1), $A^\star$ is a generator generated by two elements. Thus $S \times A^\star$ is regular by assumption and so $(s,a)S$ is projective. Hence $S \times A$ is regular.

(12) $\Rightarrow$ (13). Let ρ be a right subannihilator congruence. Then $\mathrm{Hom}(S/\rho, S_S) \neq \emptyset$ and so S/ρ is regular by assumption.

(13) $\Rightarrow$ (1). Let A_S be a generator and $a \in A_S$. Since $\ker \lambda_a$ is a right subannihilator congruence, $aS \cong S/\ker \lambda_a$ is regular and so it is projective. Thus A_S is regular as required.

(9) $\Rightarrow$ (10). Let A_S be a right S-act such that $\mathrm{Hom}(A_S, S_S) \neq \emptyset$. Let $a \in A_S$ and suppose $f : A_S \longrightarrow S_S$ is an S-homomorphism. Consider $(f(a), a) \in S \times aS$. Since $S \times aS$ is regular by assumption, $(f(a), a)S$ is projective by [4, III, 19.3]. Hence by [4, III, 17.9] there exists an idempotent $e \in E(S)$ such that $\ker \lambda_{(f(a),a)} = \ker \lambda_e$. So we have $\ker \lambda_e = \ker \lambda_{(f(a),a)} = \ker \lambda_{f(a)} \cap \ker \lambda_a \subseteq \ker \lambda_a$. It can easily be seen that $\ker \lambda_a \subseteq \ker \lambda_e$ and so $S/\ker \lambda_a \cong aS$ is projective by [4, III, 17.9]. Hence A_S is regular. □

6. Monoids over which all generators are divisible, principally weakly injective, fg-weakly injective, weakly injective, injective or completely reducible

We recall from [4] that an act A_S is called divisible if $Ac = A$ for any left cancellable element $c \in S$. Kilp and Knauer in [1] showed that over a monoid S all generators are divisible if and only if every left cancellable element is left invertible. In Corollary 6.2 we will give some more equivalent conditions when all generators are divisible. As an immediate consequence of [4, III, 2.2], [5, Proposition 6.1] and [1, Theorem 4.1] we have the following lemma.

Lemma 6.1 *Let S be a monoid. Then the following statements are equivalent:*

(1) All generators are divisible;

(2) $D(S) = (S \times S)_S$ is divisible;

(3) S_S is divisible;

(4) All right S-acts are divisible;

(5) Every left cancellable element is left invertible.

Corollary 6.2 *Let S be a monoid and the property α be divisible. Then all the statements in Lemma 6.1 and Theorem 1.8 are equivalent.*

Now we continue our investigation over monoids when all generators are (principally, fg-weakly or weakly) injective.

Theorem 6.3 *For any monoid S the following statements are equivalent:*

(1) All generators are principally weakly injective;

(2) $S \times A_S$ is principally weakly injective for every generator A_S;

(3) $S \times A_S$ is principally weakly injective for every right S-act A_S;

(4) $S \times A_S$ is principally weakly injective for every finitely generated right S-act A_S;

(5) $S \times A_S$ is principally weakly injective for every cyclic right S-act A_S;

(6) A right S-act A_S is principally weakly injective if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(7) A finitely generated right S-act A_S is principally weakly injective if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(8) A cyclic right S-act A_S is principally weakly injective if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(9) For every right subannihilator congruence ρ, S/ρ is principally weakly injective;

(10) All right S-acts are principally weakly injective;

(11) S is regular.

Proof Implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (6) are clear from Theorem 1.4 and (1) $\Leftrightarrow$ (10) $\Leftrightarrow$ (11) from [1, Corollary 4.3].

Implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) $\Rightarrow$ (8), (10) $\Rightarrow$ (6) are obvious.

(8) $\Rightarrow$ (9). Let ρ be a right subannihilator congruence. Thus $\mathrm{Hom}(S/\rho, S) \neq \emptyset$, and so S/ρ is principally weakly injective.

(9) $\Rightarrow$ (10). For any $s \in S$, $\ker \lambda_s$ is a right subannihilator congruence. So by assumption $S/\ker \lambda_s$ is principally weakly injective for every $s \in S$. Since $S/\ker \lambda_s \cong sS$, thus all principal right ideals are principally weakly injective, hence all right S-acts are principally weakly injective by [4, IV, 1.6].

(5) $\Rightarrow$ (10). Let A_S be a right S-act. We show that A_S is principally weakly injective. Let $\ker \lambda_s \leq \ker \lambda_a$, for $a \in A_S, s \in S$. Consider $(s, a) \in S \times aS$. Since $\ker \lambda_{(s,a)} = \ker \lambda_s \cap \ker \lambda_a = \ker \lambda_s$. Thus $\ker \lambda_s \leq \ker \lambda_{(s,a)}$. By assumption $S \times aS$ is principally weakly injective. Thus by [4, III, 3.2], there exists $(w, al) \in S \times aS$ such that $(s, a) = (w, al)s$. Hence $a = (al)s$ and again by [4, III, 3.2], A_S is principally weakly injective as required. $\square$

Theorem 6.4 *For any monoid S the following statements are equivalent:*

(1) All generators are fg-weakly injective;

(2) $S \times A_S$ is fg-weakly injective for every right S-act A_S;

(3) $S \times A_S$ is fg-weakly injective for every generator A_S;

(4) $S \times A_S$ is fg-weakly injective for every finitely generated right S-act A_S;

(5) A right S-act A_S is fg-weakly injective if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(6) A finitely generated right S-act A_S is fg-weakly injective if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$*;*

(7) All right S-acts are fg-weakly injective.

(8) *S is regular and all finitely generated right ideals of S are principal.*

Proof Implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) are clear from Theorem 1.4.

Implications (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (5) and (7) $\Rightarrow$ (1) are obvious.

(4) $\Rightarrow$ (7). Let A_S be a finitely generated right S-act. At first we show that A_S is fg-weakly injective. Let K_S be a finitely generated right ideal of S and $f : K_S \longrightarrow A_S$ be an S-homomorphism. Define $f^* : K_S \longrightarrow S \times A_S$ by $f^*(k) = (k, f(k))$. Then clearly, f^* is an S-homomorphism. Since $S \times A_S$ is fg-weakly injective by [4, III, 4.2], there exists $(w, z) \in S \times A_S$ such that $f^*(k) = (w, z)k$, for every $k \in K_S$. Hence $f^*(k) = (k, f(k)) = (w, z)k = (wk, zk)$, for every $k \in K_S$. Thus $f(k) = zk$, for every $k \in K_S$, and again by [4, III, 4.2], A_S is fg-weakly injective. Consequently, all finitely generated right ideals are fg-weakly injective. Hence all right S-acts are fg-weakly injective by [4, IV, 2.17].

(6) $\Rightarrow$ (7). Let K_S be a finitely generated right ideal of S. Then $\mathrm{Hom}(K_S, S_S) \neq \emptyset$ and so K_S is fg-weakly injective by assumption. Thus all finitely generated right ideals are fg-weakly injective and so by [4, IV, 2.17] all right S-acts are fg-weakly injective.

(7) $\Leftrightarrow$ (8). By [4, IV, 2.17]. $\square$

We recall from [6] that, an element $e \in S$ is called special idempotent if $e^2 = e$ and for any congruence $\equiv$ on S_S there exists $c \in eS$ such that $ce \equiv e$ and $a \equiv b$ implies $ca \equiv cb$ for any $a, b \in S$. From Theorem 1.4, [1, Corollaries 4.4, 4.5], we have the following theorems.

Theorem 6.5 *For any monoid S the following statements are equivalent:*

(1) *All generators are weakly injective;*

(2) *$S \times A_S$ is weakly injective for every right S-act A_S;*

(3) *$S \times A_S$ is weakly injective for every generator A_S;*

(4) *A right S-act A_S is weakly injective if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;*

(5) *All right S-acts are weakly injective;*

(6) *S is regular and all right ideals of S are principal.*

Theorem 6.6 *For any monoid S the following statements are equivalent:*

(1) *All generators are injective;*

(2) *$S \times A_S$ is injective for every right S-act A_S;*

(3) *$S \times A_S$ is injective for every generator A_S;*

(4) *A right S-act A_S is injective if $\mathrm{Hom}(A_S, S_S) \neq \emptyset$;*

(5) *All right S-acts are injective;*

(6) *S contains a left zero and all right ideals of S are generated by special idempotents.*

Theorem 6.7 ([1, Theorem 2.7]) *All generators are completely reducible if and only if S is a group.*

Lemma 6.8 *Let S be a monoid. Then the following statements are equivalent:*

(1) *All generators are completely reducible;*

(2) *$D(S) = (S \times S)_S$ is completely reducible;*

(3) S_S *is completely reducible;*

(4) All right S*-acts are completely reducible;*

(5) S *is group.*

Proof Implications (4) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (5) are clear from [4, I, 5.34] and Theorem 6.7.

(1) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). By [4, I, 5.33] and Theorem 6.7 the result follows.

(1) $\Rightarrow$ (2). Since $S \times A$ is a generator for every right S-act A_S, by assumption it is completely reducible. Hence $D(S) = (S \times S)_S$ is completely reducible and the result follows.

(2) $\Rightarrow$ (1). Since $D(S) = (S \times S)_S$ is completely reducible, by definition $D(S) = (S \times S)_S = \dot{\bigcup}_{i \in I} B_i$, where B_i's are disjoint simple subacts of $D(S)$. Let $s \in S$. Since $(1, s) \in D(S)$, there exists $i_0 \in I$ such that $(1, s) \in B_{i_0}$. Since $(1, s)S \leq B_{i_0}$ and B_{i_0} is simple, thus $(1, s)S = B_{i_0}$. On the other hand $(s, s^2) \in D(S) = \dot{\bigcup}_{i \in I} B_i$, and so there exists $j_0 \in I$ such that $(s, s^2) \in B_{j_0}$. Since $(s, s^2)S \leq B_{j_0}$ and B_{j_0} is simple thus $(s, s^2)S = B_{j_0}$. Therefore, $(s, s^2) \in B_{i_0} \cap B_{j_0}$. Since B_i's $(i \in I)$ are disjoint thus $i_0 = j_0$ and so $(1, s)S = (s, s^2)S$. Therefore, $(1, s) \in (1, s)S = (s, s^2)S$ implies that there exists $x \in S$ such that $(1, s) = (s, s^2)x = (sx, s^2x)$. Hence $sx = 1$. Thus $sS = S$ and so S is a group. Hence by Theorem 6.7, the result follows. $\Box$

Now in the following corollary we add some more equivalent conditions when all generators are completely reducible.

Corollary 6.9 *Let* S *be a monoid and* α *be the property completely reducible. Then all the statements in Lemma 6.8 and Theorem 1.8 are equivalent.*

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