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On $F(p,s)$-Teichmüller Space

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Abstract In this paper, we determine the differential of the Bers projection at the origin in the $F(p,s)$-Teichmüller space.

Keywords quasiconformal mapping; $F(p,s)$-Teichmüller space; Bers projection

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1. Introduction

This is a continuous work of our previous paper [1], where we introduced the $F(p,s)$-Teichmüller space and investigated its Schwarzian derivative model and pre-logarithmic derivative model. In particular, we proved that the Bers projection is holomorphic. In this paper, we shall determine the differential of Bers projection at the origin in the $F(p,s)$-Teichmüller space. We start with some notations and definitions.

Let $\Delta=\{z:|z|<1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\Delta^*=\{z:|z|>1\}$ be the outside of the unit disk. For any $a\in\Delta$, set $\varphi_a(z)=\frac{z-a}{1-\bar{a}z}$, $z\in\Delta$. For $p\geq 2$, and $s\geq 0$, the space $F(p,s)$ consists of all holomorphic functions f on the unit disk Δ with the following finite norm

$$\|f\|^p_{F_{p,s}}=\sup_{a\in\Delta}\iint\limits_{\Delta}|f'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)^s\mathrm{d}x\mathrm{d}y<\infty. \tag{1.1}$$

This space $F(p,s)$, as a special case of $F(p,q,s)$ which is introduced by Zhao [2], is a Banach space.

An orientation preserving homeomorphism f from domain Ω onto $f(\Omega)$ is quasiconformal if f has locally L^2 integrable distributional derivative on Ω and satisfies the following equation

$$f_{\bar{z}}=\mu(z)f_z,$$

for some measurable functions μ with $\|\mu\|_\infty<1$. Here we use the notations

$$f_{\bar{z}}=\frac{1}{2}(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}),\quad f_z=\frac{1}{2}(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}),$$

and the function μ is called the complex dilatation of f (see [3] for more details).

The universal Teichmüller space T is defined as $T=M(\Delta^*)/\sim$, where $M(\Delta^*)$ denotes the unit ball of the Banach space $L^\infty(\Delta^*)$ of bounded measurable functions on Δ^*. Let f_μ be the

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unique quasiconformal mapping whose complex dilatation is μ in Δ^* and zero in Δ, normalized by

$$f_\mu(0) = f'_\mu(0) - 1 = f''_\mu(0) = 0.$$

We say that two Beltrami coefficients μ_1 and μ_2 in $M(\Delta^*)$ are Teichmüller equivalent and denoted by $\mu_1 \sim \mu_2$ if $f_{\mu_1}(\Delta) = f_{\mu_2}(\Delta)$ (see [3–5] for more details).

Let $s > 0$. A positive measure λ defined in a simply connected domain Ω is called s-Carleson measure if

$$\|\lambda\|_c = \sup\{\frac{\lambda(\Omega \cap D(z,r))}{r^s} : z \in \partial\Omega, 0 < r < \infty\} < \infty, \tag{1.2}$$

where $D(z,r)$ denotes a disk with center z and radius r. 1-Carleson measure is the classical Carleson measure. We denote by $CM_s(\Omega)$ the set of all s-Carleson measures on Ω.

Here and in what follows, we assume that $2 \le p$ and $0 < s < 2$. Denote by $M_{p,s}(\Delta^*)$ the Banach space of all essentially bounded measurable functions μ each of which induces an s-Carleson measure

$$\lambda_\mu(z) := \frac{\|\mu(z)|^p}{(|z|-1)^{2-s}} \mathrm{d}x\mathrm{d}y \in CM_s(\Delta^*).$$

The norm of $\mu \in M_{p,s}(\Delta^*)$ is defined as

$$\| \mu \|_s = \| \mu \|_\infty + \|\lambda_\mu\|_{C,s}^{1/p}, \tag{1.3}$$

where $\|\lambda_\mu\|_{C,s}$ is the s-Carleson norm of λ_μ on Δ^*. The $F(p,s)$-Teichmüller space $T_{F(p,s)}$ is defined as $T_{F(p,s)} = M^1_{p,s}(\Delta^*)/\sim$, where $M^1_{p,s}(\Delta^*) = M_{p,s}(\Delta^*) \cap M(\Delta^*)$. It should be pointed out that $F(2,1)$-Teichmüller space is the BMO-Teichmüller space [6–8], the limit case $F(2,0)$-Teichmüller space is the Weil-Petersson Teichmüller space [9, 10] and $F(p,0)$-Teichmüller space is the p-integrable Teichmüller space [11–13].

The Bers projection Φ is defined by seeding $\mu \in M(\Delta^*)$ to the Schwarzian derivative S_{f_μ} of f_μ on the unit disc Δ. Recall that for a conformal mapping f on Δ, its Schwarzian derivative S_f is defined as

$$S_f = (N_f)' - \frac{1}{2}(N_f)^2, \quad N_f = (\log f')'.$$

The Bers projection maps the universal Teichmüller space into the complex Banach space $\mathcal{B}(\Delta)$ which consists of all holomorphic functions on Δ with norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{z\in\Delta} |\varphi(z)|(1-|z|^2)^2 < \infty.$$

It is well known that the map $\Phi : M(\Delta^*) \to \mathcal{B}(\Delta)$ is a holomorphic split submersion [4, 5].

In [1], we proved that the Bers projection maps $F(p,s)$-Teichmüller space into the complex Banach space $N(p,s)$ which consists of all holomorphic functions f on Δ with the following finite norm

$$\|f\|^p_{N_{p,s}} = \sup_{a\in\Delta} \iint_\Delta |f(z)|^p (1-|z|^2)^{s+2p-2} \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} \mathrm{d}x\mathrm{d}y, \tag{1.4}$$

and $\Phi : M^1_{p,s}(\Delta^*) \to N(p,s)$ is holomorphic. We note that a holomorphic function $f \in N(p,s)$ if and only if [14]

$$|f(z)|^p (1-|z|^2)^{s+2p-2} \mathrm{d}x\mathrm{d}y \in CM_s(\Delta).$$

In this paper, we shall prove the following result.

Theorem 1.1 *Let $p \geq 2$ and $0 < s < 2$. The differential $D_0\Phi$ of Bers projection $\Phi : M^1_{p,s}(\Delta^*) \to N(p,s)$ at the origin is given by the following correspondence*

$$\nu \mapsto \frac{-6}{\pi} \iint_{\Delta^*} \frac{\nu(w)}{(w-z)^4} \mathrm{d}u\mathrm{d}v, \tag{1.5}$$

which is a bounded mapping from $M_{p,s}(\Delta^*)$ to $N(p,s)$.

2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. In what follows, $C(\cdot)$ will denote constant that depends only on the elements put in the bracket.

Proof of Theorem 1.1 Let $p \geq 2$ and $0 < s < 2$. We first show that the differential $D_0\Phi$ of Bers projection $\Phi : M^1_{p,s}(\Delta^*) \to N(p,s)$ at the origin is given by the correspondence (1.5).

Let $\mu \in M_{p,s}(\Delta^*)$. We can choose $\epsilon > 0$ such that $t\mu \in M^1_{p,s}(\Delta^*)$ for all $t < 2\epsilon$. For simplicity of notation, we use $\beta(t)$ to denote the Schwarzian derivative $S_{f_{t\mu}}$. It is well known [5, 15] that $\beta(t)$ is an analytic function of t for every $z \in \Delta$ and

$$\frac{\partial \beta(t)(z)}{\partial t}|_{t=0} = -\frac{6}{\pi} \iint_{\Delta^*} \frac{\mu(w)}{(w-z)^4} \mathrm{d}u\mathrm{d}v.$$

Note that the Bers projection $\Phi : M^1_{p,s}(\Delta^*) \to N(p,s)$ is holomorphic [1]. It is sufficient to show that the limit

$$\lim_{t \to 0} \frac{\beta(t) - \beta(0)}{t} = \frac{\partial \beta(t)}{\partial t}|_{t=0} \tag{2.1}$$

exsits in $N(p,s)$. In fact, for every $a \in \Delta$, the Cauchy formula and Fubini theorem yield

$$\begin{aligned}
&\iint_{\Delta} |\frac{\beta(t)(z) - \beta(0)(z)}{t} - \frac{\partial \beta(t)(z)}{\partial t}|_{t=0}|^p (1-|z|^2)^{s+2p-2} |\varphi_a'(z)|^s \mathrm{d}x\mathrm{d}y \\
&= \frac{|t|^p}{2^p \pi^p} \iint_{\Delta} \Big| \int_{|s|=2\varepsilon} \frac{\beta(s)(z)}{(s-t)s^2} \mathrm{d}s \Big|^p (1-|z|^2)^{s+2p-2} |\varphi_a'(z)|^s \mathrm{d}x\mathrm{d}y \\
&\leq \frac{|t|^p}{2^p \pi^p \varepsilon^{3p}} \iint_{\Delta} \Big(\int_{|s|=2\varepsilon} |\beta(s)(z)||\mathrm{d}s| \Big)^p (1-|z|^2)^{s+2p-2} |\varphi_a'(z)|^s \mathrm{d}x\mathrm{d}y \\
&\leq C(\epsilon)|t|^p \iint_{\Delta} \int_{|s|=2\varepsilon} |\beta(s)(z)|^p |\mathrm{d}s| (1-|z|^2)^{s+2p-2} |\varphi_a'(z)|^s \mathrm{d}x\mathrm{d}y \\
&= C(\epsilon)|t|^p \int_{|s|=2\varepsilon} \iint_{\Delta} |\beta(s)(z)|^p (1-|z|^2)^{s+2p-2} |\varphi_a'(z)|^s \mathrm{d}x\mathrm{d}y |\mathrm{d}s| \\
&\leq C(\epsilon)|t|^p.
\end{aligned} \tag{2.2}$$

The assertion follows.

Now, we prove that the linear operator $D_0\Phi : M_{p,s}(\Delta^*) \to N(p,s)$ is bounded. For any $\zeta \in \partial\Delta$ and $0 < r < \infty$, let $B(\zeta, r) = \Delta \cap D(\zeta, r)$ and $B^*(\zeta, r) = \Delta^* \cap D(\zeta, r)$, where $D(\zeta, r)$ is the disk with center ζ and radius r. Set

$$\lambda_\mu^{1/p}(w) := \frac{|\mu(w)|}{(|w|^2-1)^{p_1}}, \quad p_1 = \frac{2-s}{p}.$$

Then

$$\begin{aligned}
&\iint\limits_{B(\zeta,r)} \Big|\iint\limits_{\Delta^*} \frac{\mu(w)}{(w-z)^4}\mathrm{d}u\mathrm{d}v\Big|^p (1-|z|^2)^{s+2p-2}\mathrm{d}x\mathrm{d}y \\
&\le 2^p \iint\limits_{B(\zeta,r)} \Big(\iint\limits_{B^*(\zeta,2r)} \frac{\lambda_\mu^{1/p}(w)(1-|z|^2)^{2-p_1}(|w|-1)^{p_1}}{|z-w|^4}\mathrm{d}u\mathrm{d}v\Big)^p \mathrm{d}x\mathrm{d}y + \\
&\quad 2^p \iint\limits_{B(\zeta,r)} (1-|z|^2)^{s+2p-2}\Big(\iint\limits_{\Delta^*\setminus B^*(\zeta,2r)} \frac{\lambda_\mu^{1/p}(w)(|w|-1)^{p_1}}{|z-w|^4}\mathrm{d}u\mathrm{d}v\Big)^p \mathrm{d}x\mathrm{d}y \\
&= 2^p(I_1+I_2).
\end{aligned} \tag{2.3}$$

For any $z \in \Delta$, let $D_z = \{w : |w-z| > 1-|z|\}$. Consider kernel function

$$K(z,w) = \frac{(1-|z|^2)^{2-p_1}(|w|-1)^{p_1}}{|z-w|^4}.$$

Then,

$$\iint\limits_{\Delta^*} K(z,w)\mathrm{d}u\mathrm{d}v \le \iint\limits_{D_z} \frac{(1-|z|^2)^{2-p_1}}{|z-w|^{4-p_1}}\mathrm{d}u\mathrm{d}v \le (1-|z|^2)^{2-p_1} 2\pi \int_{1-|z|}^{\infty} \frac{1}{r^{3-p_1}}\mathrm{d}r = C_0.$$

A similar argument gives that for any $w \in \Delta^*$,

$$\iint\limits_{\Delta} K(z,w)\mathrm{d}x\mathrm{d}y \le C_1. \tag{2.4}$$

It follows from the Schur's theorem [16, Theorem 3.6] that the operator

$$Kf(z) = \iint\limits_{\Delta^*} K(z,w)f(w)\mathrm{d}u\mathrm{d}v$$

is bounded from $L^p(\Delta^*)$ to $L^p(\Delta)$. Let $g(w) = \lambda_\mu^{1/p}(w)\chi_{B^*(\zeta,2r)}$, where $\chi_{B^*(\zeta,2r)}$ is the characteristic functions of $B^*(\zeta,2r)$. Thus,

$$\begin{aligned}
\iint\limits_{\Delta} \Big|\iint\limits_{\Delta^*} K(z,w)g(w)\mathrm{d}u\mathrm{d}v\Big|^p \mathrm{d}x\mathrm{d}y &\le C_2 \iint\limits_{\Delta^*} |g(w)|^p \mathrm{d}u\mathrm{d}v \\
&= C_2 \iint\limits_{B^*(\zeta,2r)} \frac{|\mu(w)|^p}{(|w|-1)^{2-s}}\mathrm{d}u\mathrm{d}v \le C_2\|\lambda_\mu\|_c r^s.
\end{aligned} \tag{2.5}$$

We proceed to estimate I_2. Since $\lambda_\mu(w)\mathrm{d}u\mathrm{d}v$ is an s-Carleson measure in Δ^*, the Hölder inequality yields

$$\Big(\iint\limits_{B^*(\zeta,r)} \lambda_\mu^{1/p}(w)\mathrm{d}u\mathrm{d}v\Big)^p \le (\pi r^2)^{p-1} \iint\limits_{B^*(\zeta,r)} \lambda_\mu(w)\mathrm{d}u\mathrm{d}v \le \pi^{p-1}\|\lambda_\mu\|_c r^{2p-2+s}.$$

Thus, we get

$$\iint\limits_{B(\zeta,r)} (1-|z|^2)^{s+2p-2}\Big(\sum_{i=1}^{\infty} \iint\limits_{B^*(\zeta,2^{i+1}r)\setminus B^*(\zeta,2^i r)} \frac{\lambda_\mu^{1/p}(w)(|w|-1)^{p_1}}{|z-w|^4}\Big)^p \mathrm{d}x\mathrm{d}y$$

$$\leq C_3\|\lambda_\mu\|_c \iint\limits_{B(\zeta,r)} (1-|z|^2)^{s+2p-2}\Big(\sum_{i=1}^{\infty}\frac{(2^{i+1}r)^{2-p_1}}{(2^i r)^{4-p_1}}\Big)^p \mathrm{d}x\mathrm{d}y$$

$$\leq C_4\|\lambda_\mu\|_c r^s. \tag{2.6}$$

Combining (2.3), (2.5) with (2.6) gives $\|D_0\Phi\|_{N(p,s)} \leq \|\lambda_\mu\|_s$. The proof is completed. $\Box$

3. Some remarks

Let $p \geq 2$ and $0 < s < 2$. We denote by $\widetilde{T}_{F(p,s)}$ the set of functions $\log f'$, where f is conformal on Δ and admits a quasiconformal extension to the whole plane $\mathbb{C}$ such that its complex dilatation μ satisfies

$$\frac{|\mu(z)|^p}{(|z|-1)^{2-s}}\mathrm{d}x\mathrm{d}y \in CM_s(\Delta^*). \tag{3.1}$$

It is known [1] that $\widetilde{T}_{F(p,s)}$ is a disconnected subset of the space $F(p,s)$. Furthermore, $\widetilde{T}_b = \{\log f' \in \widetilde{T}_{F(p,s)} : f(\Delta) \text{ is bounded }\}$ and $\widetilde{T}_\theta = \{\log f' \in \widetilde{T}_{F(p,s)} : f(e^{i\theta}) = \infty\}$, $\theta \in [0,2\pi)$, are the connected components of $\widetilde{T}_{F(p,s)}$.

Let $\mu \in M^1_{p,s}(\Delta^*)$ and $z_0 \in \Delta^*$. We use $f^{z_0}_\mu$ to denote the quasiconformal mapping on $\mathbb{C}$ whose complex dilatation is μ in Δ^* and zero in Δ, normalized by

$$f_\mu(0) = f'_\mu(0) - 1 = 0, \quad f_\mu(z_0) = \infty.$$

The pre-Bers projection mapping L_{z_0} on $M^1_{p,s}(\Delta^*)$ is defined by setting $L_{z_0}(\mu) = \log(f^{z_0}_\mu)'$. Then $\cup_{z_0\in\Delta^*} L_{z_0}(M^1_{p,s}(\Delta^*)) = \widetilde{T}_{F(p,s)} \cap F(p,s)^0$, where $F(p,s)^0$ consists of all functions $\varphi \in F(p,s)$ with $\varphi(0) = 0$.

In [1], we proved that the pre-Bers projection $L_{z_0} : M^1_{p,s}(\Delta^*) \to F(p,s)^0$ is holomorphic. Analogous to Theorem 1.1, we obtain

Theorem 3.1 *Let $p \geq 2$ and $0 < s < 2$. The differential D_0L_∞ of pre-Bers projection $L_\infty : M^1_{p,s}(\Delta^*) \to F(p,s)^0$ at the origin is given by the following correspondence*

$$\nu \mapsto \frac{-2}{\pi}\iint\limits_{\Delta^*}\frac{\nu(w)}{(z-w)^3}\mathrm{d}u\mathrm{d}v, \tag{3.2}$$

which is a bounded mapping from $M_{p,s}(\Delta^*)$ to $F(p,s)$.

Proof For abbreviation, we use $f_t(z)$ to stand for $f_{t\mu}(z)$ for $t\mu \in M^1_{p,s}(\Delta^*)$. Then $f_0(z) = z$. Ahlfors and Bers [15] proved that for any fixed z, the function $f_t(z)$ is an analytic function of t and

$$\frac{\partial f_t(z)}{\partial t}\Big|_{t=0} = F_\mu(z) = -\frac{1}{\pi}\iint\limits_{\Delta^*}\Big(\frac{1}{w-z} - \frac{z}{w-1} + \frac{z-1}{w}\Big)\mu(w)\mathrm{d}u\mathrm{d}v.$$

We denote t-derivative by a dot and z-derivative by a prime. Noting that $f'_t(z)|_{t=0} = 1$ and $f''_t(z)|_{t=0} = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{f''_t(z)}{f'_t(z)}\Big)\Big|_{t=0} = \frac{\dot{f}''_t(z)f'_t(z) - \dot{f}'_t(z)f''_t(z)}{f'_t(z)^2}\Big|_{t=0}$$

$$= F''_\mu(z) = \frac{-2}{\pi} \iint_{\Delta^*} \frac{\nu(w)}{(z-w)^3} \mathrm{d}u\mathrm{d}v.$$

By using the similar argument as in Theorem 1.1, we can prove that the differential D_0L_∞ of pre-Bers projection $L_\infty : M^1_{p,s}(\Delta^*) \to F(p,s)^0$ at the origin is $\frac{\mathrm{d}}{\mathrm{d}t}\frac{f''_t}{f'_t}|_{t=0}$ and it is a bounded operator from $M_{p,s}(\Delta^*)$ to $F(p,s)$. We omit the details. □

It is well known that there are holomorphic local right inverses (i.e., section) to the Bers projections of universal Teichmüller space [4, 5] and BMO-Teichmüller space [7]. Thus, it is natural to ask the following

Problem 3.2 *Whether there are holomorphic local sections to the Bers projection and pre-Bers projection of $F(p,s)$-Teichmüller space?*

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