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Higher Order Complex Differential Equations with Analytic Coefficients in the Unit Disc

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Abstract The growth of solutions of the following differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0$$

is studied, where $A_j(z)$ is analytic in the unit disc $\mathbb{D}=\{z:|z|<1\}$ for $j=0,1,\ldots,k-1$. Some precise estimates of $[p,q]$-order of solutions of the equation are obtained by using a notion of new $[p,q]$-type on coefficients.

Keywords complex differential equation; $[p,q]$-order of growth; $[p,q]$-type; analytic function; unit disc

MR(2010) Subject Classification 34M10; 30D35

1. Introduction and main results

Since 1980s, it has been developed that the theory of complex linear differential equations in the unit disc [1]. In general, many authors consider the following linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.1}$$

where $A_j(z)$ is analytic in the unit disc $\mathbb{D}=\{z:|z|<1\}$, $j=0,1,\ldots,k-1$, $k\geq 2$. One of the main tools is Nevanlinna theory of meromorphic functions which can be found in [2–4]. It is well known that all solutions of (1.1) are analytic functions in $\mathbb{D}$ if the coefficients of (1.1) are analytic functions in $\mathbb{D}$. Heittokangas investigated the complex differential equations in the unit disc, and obtained many valuable results in [5]. From that time, it has been very interesting to study the growth of analytic solutions of linear differential equations in the unit disc. Later on, more and more results concerning the growth of solutions of (1.1) were done by different researchers, for example, see [6–10] and reference therein. Recently, Zemirni and Belaïdi investigated the growth of solutions of second order complex differential equations with analytic coefficients in $\mathbb{D}$ by using a new notion called new type of analytic function, more details can be found in [11]. Here we are going to study the growth of solutions of (1.1) by using two kinds of manners. On the one

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hand, the growth of solutions of (1.1) is studied by using Zemirni-Belaïdi's idea, these results are shown in Section 2 which are improvement of previous results from Zemirni and Belaïdi. On the other hand, the growth of solutions of (1.1) is studied by using the notion of the lower order of growth of analytic function.

In order to state our results, we firstly recall some notations. For a meromorphic function $f(z)$ in $\mathbb{D}$, the order and the lower order are defined respectively by

$$\sigma(f)=\limsup_{r\to 1^-}\frac{\log^+ T(r,f)}{\log\frac{1}{1-r}},\quad \mu(f)=\liminf_{r\to 1^-}\frac{\log^+ T(r,f)}{\log\frac{1}{1-r}},$$

where $T(r,f)=m(rf)+N(r,f)$ is the Nevanlinna characteristic function of $f(z)$, where the proximity function $m(r,f)$ and the counting function $N(r,f)$ are defined as follows

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\mathrm{d}\theta,\quad N(r,f)=\int_0^r\frac{n(t,f)-n(0,f)}{t}\mathrm{d}t+n(0,f)\log r,$$

where $n(r,f)$ denotes the number of poles of $f(z)$ in the region $\{z:|z|\le r\}$ counting its multiplicities, and $\log^+ x=\max\{0,\log x\}$, more details can be found in [2]. It is easy to see that $\mu(f)\le\sigma(f)$.

Let $f(z)$ be an analytic function in $\mathbb{D}$, its order and lower order can also be defined by

$$\sigma_M(f)=\limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f)}{\log\frac{1}{1-r}},\quad \mu_M(f)=\liminf_{r\to 1^-}\frac{\log^+\log^+ M(r,f)}{\log\frac{1}{1-r}},$$

respectively, where $M(r,f)=\max\{|f(z)|:|z|\le r\}$.

Proposition 1.1 *Suppose that $f(z)$ is analytic in $\mathbb{D}$. It follows from Tsuji's result [12, Theorem V. 13, p. 205] that*

$$\sigma(f)\le\sigma_M(f)\le\sigma(f)+1,$$

and there exists a function $f(z)=e^{\frac{1}{(1-z)^n}}$ such that $\sigma_M(f)=n$ and $\sigma(f)=n-1$, where $n>1$.

For the following definitions, we define $\exp_0 r=r$, $\exp_1 r=e^r$ and $\exp_{n+1} r=\exp(\exp_n r)$, $\log_0 r=r$, $\log_1 r=\log r$, $\log_{n+1} r=\log(\log_n r)$; moreover, we denote by $\log_{-n} r=\exp_n r$ and $\exp_{-n} r=\log_n r$, where $n\in N$. Let $p\ge q\ge 1$, and $f(z)$ be meromorphic in $\mathbb{D}$. We define respectively the $[p,q]$-order and the lower $[p,q]$-order of $f(z)$ by

$$\sigma_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log_q\frac{1}{1-r}},\quad \mu_{[p,q]}(f)=\liminf_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log_q\frac{1}{1-r}}.$$

If $f(z)$ is an analytic function in $\mathbb{D}$, similarly, the $[p,q]$-order and the lower $[p,q]$-order of $f(z)$ are defined respectively by

$$\sigma_{[p,q],M}(f)=\limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_q\frac{1}{1-r}},\quad \mu_{[p,q],M}(f)=\liminf_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_q\frac{1}{1-r}}.$$

Proposition 1.2 *Let $f(z)$ be analytic in $\mathbb{D}$. It follows from [9, 13, 14, 16] that the following statements hold:*

(i) If $p=q$, then $\sigma_{[p,q]}(f)\le\sigma_{[p,q],M}(f)\le\sigma_{[p,q]}(f)+1$;

(ii) If $p=q\ge 2$ and $\sigma_{[p,q]}(f)<1$, then $\sigma_{[p,q]}(f)\le\sigma_{[p,q],M}(f)\le 1$;

(iii) If $p = q \geq 2$ and $\sigma_{[p,q]}(f) \geq 1$ or $p > q \geq 1$, then $\sigma_{[p,q]}(f) = \sigma_{[p,q],M}(f)$.

By [11, 14], we have the similar statements as follows:

Proposition 1.3 *Let $f(z)$ be analytic in $\mathbb{D}$. The following statements hold:*

(i) If $p = q = 1$, then $\mu_{[p,q]}(f) \leq \mu_{[p,q,M]}(f) \leq \mu_{[p,q]}(f) + 1$;

(ii) If $p = q \geq 2$ and $\mu_{[p,q]}(f) < 1$ then $\mu_{[p,q]}(f) \leq \mu_{[p,q],M}(f) \leq 1$;

(iii) If $p = q \geq 2$ and $\mu_{[p,q]}(f) \geq 1$, or $p > q \geq 1$ then $\mu_{[p,q]}(f) = \mu_{[p,q],M}(f)$.

Let $p \geq q \geq 1$, and $f(z)$ be a meromorphic function in $\mathbb{D}$ with $0 < \sigma_{[p,q]}(f) = \sigma < +\infty$ and $0 < \mu_{[p,q]}(f) = \mu < +\infty$. Then the $[p,q]$-type of $f(z)$ and the lower $[p,q]$-type of $f(z)$ are given respectively by

$$\tau_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log^+_{p-1} T(r,f)}{(\log_{q-1} \frac{1}{1-r})^{\sigma}}, \quad \underline{\tau}_{[p,q]}(f) = \liminf_{r\to 1^-} \frac{\log^+_{p-1} T(r,f)}{(\log_{q-1} \frac{1}{1-r})^{\mu}}.$$

Similarly, if $f(z)$ is an analytic function in $\mathbb{D}$, we can also get the definition of the $[p,q]$-type and the lower $[p,q]$-type of $f(z)$ as follows,

$$\tau_{[p,q],M}(f) = \limsup_{r\to 1^-} \frac{\log^+_{p} M(r,f)}{(\log_{q-1} \frac{1}{1-r})^{\sigma}}, \quad \underline{\tau}_{[p,q],M}(f) = \liminf_{r\to 1^-} \frac{\log^+_{p} M(r,f)}{(\log_{q-1} \frac{1}{1-r})^{\mu}}.$$

Proposition 1.4 *Let $f(z)$ be analytic in $\mathbb{D}$. Then the following statements hold:*

(i) If $p = q \geq 2$ and $\sigma_{[p,q]}(f) \geq 1$ or $p > q \geq 1$, then $\tau_{[p,q]}(f) = \tau_{[p,q],M}(f)$ from [14, Proposition 1.3];

(ii) If $p = q \geq 2$ and $\mu_{[p,q]}(f) \geq 1$, or $p > q \geq 1$, then $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{[p,q],M}(f)$ by Propsition 1.3.

For our results, the following notation is also needed. Let $p > 2$, and $f(z)$ be a meromorphic function in $\mathbb{D}$ with $0 < \sigma_{[p,q]}(f) = \sigma < +\infty$ and $0 < \tau_{[p,q]}(f) = \tau < +\infty$. Then, $\tau^*_{[p,q]}(f)$ is defined by

$$\tau^*_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log^+_{p-2} T(r,f)}{\exp(\tau(\log_{q-1} \frac{1}{1-r})^{\sigma})}.$$

If $f(z)$ satisfies $0 < \mu_{[p,q]}(f) = \mu < +\infty$ and $0 < \underline{\tau}_{[p,q]}(f) = \underline{\tau} < +\infty$, then $\underline{\tau}^*_{[p,q]}(f)$ is defined by

$$\underline{\tau}^*_{[p,q]}(f) = \liminf_{r\to 1^-} \frac{\log^+_{p-2} T(r,f)}{\exp(\underline{\tau}(\log_{q-1} \frac{1}{1-r})^{\mu})}.$$

Let $f(z)$ be analytic in $\mathbb{D}$. In similar way, we define $\tau^*_{[p,q],M}(f)$ and $\underline{\tau}^*_{[p,q],M}(f)$ respectively by

$$\tau^*_{[p,q],M}(f) = \limsup_{r\to 1^-} \frac{\log^+_{p-1} M(r,f)}{\exp(\tau_M(\log_{q-1} \frac{1}{1-r})^{\sigma_M})}, \quad \underline{\tau}^*_{[p,q],M}(f) = \liminf_{r\to 1^-} \frac{\log^+_{p-1} M(r,f)}{\exp(\underline{\tau}_M(\log_{q-1} \frac{1}{1-r})^{\mu_M})},$$

where $0 < \sigma_{[p,q],M}(f) = \sigma_M < +\infty$, $0 < \tau_{[p,q],M}(f) = \tau_M < +\infty$, $0 < \mu_{[p,q],M}(f) = \mu_M < +\infty$ and $0 < \underline{\tau}_{[p,q],M}(f) = \underline{\tau}_M < +\infty$.

Proposition 1.5 *Let $f(z)$ be analytic in $\mathbb{D}$. By Propsitions 1.2–1.4, we have the following statements:*

(i) If $p = q \geq 2$ and $\sigma_{[p,q]}(f) \geq 1$ or $p > q \geq 1$, then $\tau^*_{[p,q]}(f) = \tau^*_{[p,q],M}(f)$;

(ii) If $p = q \geq 2$ and $\mu_{[p,q]}(f) \geq 1$, or $p > q \geq 1$, then $\underline{\tau}^*_{[p,q]}(f) = \underline{\tau}^*_{[p,q],M}(f)$.

The Proposition 1.5 shows that $\tau^*_{[p,q]}(A_j)$ can be replaced by $\tau^*_{[p,q],M}(A_j)$ in Theorem 1.6 and Theorem 1.8 when $\sigma_{[p,q]}(f) \geq 1$ or $p > q \geq 1$. And $\underline{\tau}^*_{[p,q]}(A_j)$ can also be replaced by $\underline{\tau}^*_{[p,q],M}(A_j)$ in Theorem 1.7 and Theorem 1.9 when $\mu_{[p,q]}(f) \geq 1$ or $p > q \geq 1$.

Some estimations for the solutions of the second order complex differential equation were obtained by using the notion of new type above, more details can be found in [11]. In the paper, we investigate the growth of solutions of equation (1.1) by the new type $\tau^*_{[p,q]}(f)$ and $\underline{\tau}^*_{[p,q]}(f)$, and $p \geq q \geq 1$. Moreover, we denote the linear measure and logarithmic measure of a set $E \subset [0, 1)$ by $\mathrm{m}(E) = \int_E \mathrm{d}t$ and $\mathrm{m_l}(E) = \int_E \frac{1}{1-t}\mathrm{d}t$.

Here, we get the following results by $[p,q]$-order, $[p,q]$-type and a new $[p,q]$-type, in which the coefficient $A_0(z)$ is a dominant coefficient by the manner of the new $[p,q]$-type.

Theorem 1.6 *Let $p > 2$ and $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\mathbb{D}$ satisfying the following conditions:*

(i) $0 < \max\{\sigma_{[p,q]}(A_j) : j \neq 0\} \leq \sigma_{[p,q]}(A_0) < +\infty$;

(ii) $0 < \max\{\tau_{[p,q]}(A_j) : \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0), j \neq 0\} \leq \tau_{[p,q]}(A_0) < +\infty$;

(iii) $0 < \max\{\tau^*_{[p,q]}(A_j) : \tau_{[p,q]}(A_j) = \tau_{[p,q]}(A_0), \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0), j \neq 0\} < \tau^*_{[p,q]}(A_0) < +\infty$.

Then, every nontrivial solution $f(z)$ of (1.1), satisfies $\sigma_{[p,q]}(f) = +\infty$ and $\sigma_{[p,q]}(A_0) \leq \sigma_{[p+1,q]}(f) \leq \max\{\sigma_{[p,q],M}(A_j) : j = 0, 1, \ldots, k-1\}$. Furthermore, if $p > q$, then $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

In the Theorem 1.7, the $[p,q]$-order of growth of solutions of (1.1) can be estimated by the lower $[p,q]$-order of growth of coefficients $A_0(z)$, which is dominant in terms of the lower new type $\underline{\tau}^*_{[p,q]}$.

Theorem 1.7 *Let $p > 2$ and $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\mathbb{D}$ satisfying the following conditions:*

(i) $0 < \max\{\mu_{[p,q]}(A_j) : j \neq 0\} \leq \mu_{[p,q]}(A_0) < +\infty$;

(ii) $0 < \max\{\underline{\tau}_{[p,q]}(A_j) : \mu_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\} \leq \underline{\tau}_{[p,q]}(A_0) < +\infty$;

(iii) $0 < \max\{\underline{\tau}^*_{[p,q]}(A_j) : \underline{\tau}_{[p,q]}(A_j) = \underline{\tau}_{[p,q]}(A_0), \mu_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \neq 0\} < \underline{\tau}^*_{[p,q]}(A_0) < +\infty$, and there exists a set having infinite logarithmic measure such that the conclusion of Lemma 2.4 holds for all the coefficients A_j $(j \neq 0)$.

Then, every nontrivial solution $f(z)$ of (1.1), satisfies $\sigma_{[p,q]}(f) = +\infty$ and $\mu_{[p,q]}(A_0) \leq \sigma_{[p+1,q]}(f)$.

In the two theorems above, the growth of the solutions of (1.1) has been estimated by $A_0(z)$. From that, we are going to consider the following results, in which the coefficient $A_s(z)$ is a dominant coefficient, where $s \neq 0$.

Theorem 1.8 *Let $p > 2$ and $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\mathbb{D}$ satisfying the following conditions:*

(i) $0 < \max\{\sigma_{[p,q]}(A_j) : j \neq s\} \leq \sigma_{[p,q]}(A_s) < +\infty$;

(ii) $0 < \max\{\tau_{[p,q]}(A_j) : \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_s), j \neq s\} \leq \tau_{[p,q]}(A_0) < +\infty$;

(iii) $0 < \max\{\tau^*_{[p,q]}(A_j) : \tau_{[p,q]}(A_j) = \tau_{[p,q]}(A_s), \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_s), j \neq s\} < \tau^*_{[p,q]}(A_s) < +\infty$.

Then, every nontrivial solution $f(z)$ of (1.1), in which $f^{(n)}$ just has finite many zeros for all $n < s$ $(n = 0, \ldots, s-1)$, satisfies $\sigma_{[p,q]}(f) = +\infty$ and $\sigma_{[p,q]}(A_s) \leq \sigma_{[p+1,q]}(f) \leq \max\{\sigma_{[p,q],M}(A_j) : j = 0, 1, \ldots, k-1\}$. If $p > q$, then $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$.

The last result shows the estimate of $[p,q]$-order of solutions of (1.1) using the lower $[p,q]$-order of $A_s(z)$, as same as the Theorem 1.9, the coefficient $A_s(z)$ is dominant coefficient, where $s \neq 0$.

Theorem 1.9 *Let $p > 2$ and $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\mathbb{D}$ satisfying the following conditions:*

(i) $0 < \max\{\mu_{[p,q]}(A_j) : j \neq s\} \leq \mu_{[p,q]}(A_s) < +\infty$;

(ii) $0 < \max\{\underline{\tau}_{[p,q]}(A_j) : \mu_{[p,q]}(A_j) = \mu_{[p,q]}(A_s), j \neq s\} \leq \underline{\tau}_{[p,q]}(A_s) < +\infty$;

(iii) $0 < \max\{\underline{\tau}^*_{[p,q]}(A_j) : \underline{\tau}_{[p,q]}(A_j) = \underline{\tau}_{[p,q]}(A_s), \mu_{[p,q]}(A_j) = \mu_{[p,q]}(A_s), j \neq s\} < \underline{\tau}^*_{[p,q]}(A_s) < +\infty$, *and there exists a set having infinite logarithmic measure such that the conclusion of Lemma 2.4 holds for all the coefficients $A_j (j \neq s)$.*

Then, every nontrivial solution $f(z)$ of (1.1), in which $f^{(n)}$ just has finite many zeros for all $n < s$ $(n = 0, \ldots, s-1)$, satisfies $\sigma_{[p,q]}(f) = +\infty$ and $\mu_{[p,q]}(A_s) \leq \sigma_{[p+1,q]}(f)$.

2. Auxiliary results

In the section, some preliminaries results are given for proving Theorems 1.6–1.9. The first result comes from [2, p. 55], and can also be found in [5, p. 8].

Lemma 2.1 *Let $f(z)$ be a nonconstant meromorphic function in $\mathbb{D}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f), \quad r \notin E,$$

where $k \in N$, $S(r,f) = O\left(\log^+ T(r,f) + \log \frac{1}{1-r}\right)$ and $\int_E \frac{1}{1-r} \mathrm{d}t < \infty$.

For the new type $\tau^*_{[p,q]}(f)$, an estimation is obtained in [11] which plays an important role in proving our results.

Lemma 2.2 ([11]) *Let $p \geq 2$ and $f(z)$ be a nonconstant meromorphic function in $\mathbb{D}$ such that $0 < \sigma_{[p,q]}(f) = \sigma < +\infty, 0 < \tau_{[p,q]}(f) = \tau < +\infty, 0 < \tau^*_{[p,q]}(f) = \tau^* < +\infty$. Then for any given $\beta < \tau^*$, there exists a subset $E \in [0,1)$ that has infinite logarithmic measure such that for all $r \in E$, we have*

$$\log_{p-2} T(r,f) > \beta \exp\left(\tau \left(\log_{q-1} \frac{1}{1-r}\right)^{\sigma}\right).$$

The following result shows the relationship between the growth of solution of (1.1) and the growth of coefficients of (1.1).

Lemma 2.3 ([13, Lemma 2.5]) *If $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ are analytic functions of $[p,q]$-order* *in $\mathbb{D}$, then every nontrivial solutions f of (1.1) satisfies*

$$\sigma_{[p+1,q]}(f) \le \sigma_{[p+1,q],M}(f) \le \max\{\sigma_{[p,q],M}(A_j) : j = 0, 1, \ldots, k-1\}.$$

Lemma 2.4 ([11]) *Let $p \ge 2$ and $f(z)$ be a nonconstant meromorphic function in $\mathbb{D}$ such that $0 < \mu_{[p,q]}(f) = \mu < +\infty, 0 < \underline{\tau}_{[p,q]}(f) = \underline{\tau} < +\infty$ and $0 < \underline{\tau}^*_{[p,q]}(f) = \underline{\tau}^* < +\infty$. Then for any given $\varepsilon > 0$, there exists a subset $E \in [0,1)$ that has infinite logarithmic measure such that for all $r \in E$, we have*

$$T(r,f) \le \exp_{p-2}\left((\underline{\tau}^* + \varepsilon)\exp\left(\underline{\tau}\left(\log_{q-1}\frac{1}{1-r}\right)^{\mu}\right)\right).$$

The following result is lemma on the logarithmic derivative, which can be found in [15].

Lemma 2.5 *Let k and j be integers satisfying $k > j \ge 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If $f(z)$ is meromorphic in $\mathbb{D}$ such that $f^{(j)}$ does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-r}, T\left(s(r), f\right)\right\}\right)^{k-j}, \quad r \notin E,$$

where $s(r) = 1 - d(1-r)$, and the set $E \subset [0,1)$ has finite logarithmic measure.

3. Proofs of Theorems 1.6 and 1.7

Now, we apply Lemmas 2.1–2.4 to prove Theorems 1.6 and 1.7. Firstly, Theorem 1.6 is proved as follows.

Proof of Theorem 1.6 Let σ, τ, τ^* be positive finite numbers satisfying $\sigma_{[p,q]}(A_0) = \sigma$, $\tau_{[p,q]}(A_0) = \tau$, $\tau^*_{[p,q]}(A_0) = \tau^*$. From the conditions of Theorem 1.6, there exist two real constants α and β satisfying $0 < \max\{\tau^*_{[p,q]}(A_j) : \tau_{[p,q]}(A_j) = \tau_{[p,q]}(A_0), \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0)\} < \alpha < \beta < \tau^*$. Then for all $r \to 1^-$, we have

$$m(r, A_j) \le \exp_{p-2}\left(\alpha\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right). \tag{3.1}$$

By Lemma 2.2, there exists a set $E_1 \in [0,1)$ that has infinite logarithmic measure such that for all $r \in E_1$, we have

$$m(r, A_0) \ge \exp_{p-2}\left(\beta\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right). \tag{3.2}$$

By (1.1), we get

$$-A_0(z) = \frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f},$$

then

$$m(r, A_0) \le \sum_{i=1}^{k} m\left(r, \frac{f^{(i)}}{f}\right) + \sum_{j=1}^{k-1} m(r, A_j) + \log k.$$

It follows from the inequality above and by Lemma 2.1, there exists a set $E_2 \subset [0,1)$ that has finite logarithmic measure, for all $r \in [0,1)\backslash E_2$, we can get

$$m(r,A_0) \le O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right) + \sum_{j=1}^{k-1} m(r,A_j) + \log k. \tag{3.3}$$

Combining (3.1)–(3.3), for all $r \in E_1\backslash E_2$ and $r \to 1^-$, we have

$$\exp_{p-2}\left(\beta\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right) < O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right) + t(k-1)\exp_{p-2}\left(\alpha\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right).$$

Since $\beta > \alpha$, for $p > 2$ and $r \to 1^-$ we have

$$\exp_{p-2}\left(\beta\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right) > (k-1)\exp_{p-2}\left(\alpha\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right),$$

which implies that

$$\exp_{p-2}\left(\beta\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right) < O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right).$$

Therefore

$$\sigma_{[p,q]}(f) = +\infty$$

and

$$\sigma_{[p+1,q]}(f) \ge \sigma = \sigma_{[p,q]}(A_0).$$

On the other hand, by Lemma 2.3, we have

$$\sigma_{[p+1,q]}(f) \le \max\{\sigma_{[p,q],M}(A_j) : j = 0,1,\ldots,k-1\}.$$

This implies that

$$\sigma_{[p,q]}(A_0) \le \sigma_{[p+1,q]}(f) \le \max\{\sigma_{[p,q],M}(A_j) :\ j = 0,1,\ldots,k-1\}.$$

If $p > q$, then

$$\max\{\sigma_{[p,q],M}(A_j) :\ j = 0,1,\ldots,k-1\} = \sigma_{[p,q]}(A_0).$$

Therefore, we conclude that $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. $\square$

Proof of Theorem 1.7 Let μ, $\underline{\tau}$, $\underline{\tau}^*$ be positive finite numbers satisfying $\mu_{[p,q]}(A_0) = \mu$, $\underline{\tau}_{[p,q]}(A_0) = \underline{\tau}$, $\underline{\tau}^*_{[p,q]}(A_0) = \underline{\tau}^*$. By the conditions of Theorem 1.7, there exist two real constants α and β such that

$$0 < \max\{\underline{\tau}^*_{[p,q]}(A_j) : \underline{\tau}_{[p,q]}(A_j) = \underline{\tau}_{[p,q]}(A_0), \mu_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j \ne 0\} < \alpha < \beta < \underline{\tau}^*.$$

Then for any given ε satisfying $0 < \varepsilon < \min\{\frac{\underline{\tau}^*-\beta}{2}, \frac{\beta-\alpha}{2}\}$, there exists $r_0 \in [0,1)$, such that for all $r \in (r_0,1)$,

$$m(r,A_0) \ge \exp_{p-2}\left((\underline{\tau}^* - \varepsilon)\exp\left(\underline{\tau}\left(\log_{q-1}\frac{1}{1-r}\right)^{\mu}\right)\right). \tag{3.4}$$

By Lemma 2.4 and the conditions of Theorem 1.7, there exists a set $E_3 \subset [0,1)$ that has infinite logarithmic measure such that for all $r \in E_3$ and $j \neq 0$,

$$m(r, A_j) \leq \exp_{p-2}\left((\alpha+\varepsilon)\exp\left(\underline{\tau}\left(\log_{q-1}\frac{1}{1-r}\right)^{\mu}\right)\right). \tag{3.5}$$

Combining (3.3)–(3.5), for all $r \in E_3 \backslash E_2$ and using the similar way in the proof of Theorem 1.6, we can easily get the conclusion. $\square$

4. Proofs of Theorems 1.8 and 1.9

Here, we prove Theorems 1.8 and 1.9 by using Lemmas 2.2–2.5 as follows.

Proof of Theorem 1.8 By Lemma 2.5, there exists a set E_4 that has finite logarithmic measure, for all $k > i$ $(i, k \in N)$ and $r \in [0,1)\backslash E_4$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(i)}(z)}\right| \leq \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\max\left\{\log\frac{1}{1-r}, T(s(r), f)\right\}\right)^{k-i}.$$

If r tends to 1^-, then $\log\frac{1}{1-r} > 1$ and $T(s(r), f) = T(r, f)$. Thus we get

$$\begin{aligned}
m\left(r, \frac{f^{(k)}}{f^{(i)}}\right) &\leq \log^+\left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\left(\log\frac{1}{1-r} + T(s(r), f)\right)\right)^{k-i}\\
&\leq (k-i)\log^+\left(\left(\frac{1}{1-r}\right)^{2+\varepsilon}\left(\log\frac{1}{1-r} + T(r, f)\right)\right)\\
&\leq (k-i)(2+\varepsilon)\log^+\frac{1}{1-r} + (k-i)\log^+\log\frac{1}{1-r} + (k-i)\log^+ T(r,f)\\
&\leq O\left(\log^+ T(r,f) + \log^+\frac{1}{1-r}\right), \quad r \to 1.
\end{aligned}$$

Then, for $r \in [0,1)\backslash E_4$ and $r \to 1$, we obtain

$$m\left(r, \frac{f^{(k)}}{f^{(i)}}\right) \leq O\left(\log^+ T(r,f) + \log^+\frac{1}{1-r}\right). \tag{4.1}$$

Let σ, τ, τ^* be positive finite numbers such that $\sigma_{[p,q]}(A_s) = \sigma$, $\tau_{[p,q]}(A_s) = \tau$, $\tau^*_{[p,q]}(A_s) = \tau^*$. Set

$$\max\{\tau^*_{[p,q]}(A_j) : \tau_{[p,q]}(A_j) = \tau_{[p,q]}(A_s), \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_s), j \neq s\} = \beta_1,$$

and let β_2 be a real constant such that $\beta_1 < \beta_2 < \tau^*$. Then, for any given ε satisfying $0 < \varepsilon < \min\{\frac{\beta_2-\beta_1}{2}, \frac{\tau^*-\beta_2}{2}\}$, there exists $r_0 \in [0,1)$ such that for all $r \in (r_0, 1)$,

$$m(r, A_j) \leq \exp_{p-2}\left((\beta_1+\varepsilon)\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right). \tag{4.2}$$

By Lemma 2.2, there exists a set $E_5 \subset [0,1)$ that has infinite logarithmic measure such that for all $r \in E_5$, we have

$$m(r, A_s) \geq \exp_{p-2}\left((\tau^*-\varepsilon)\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right). \tag{4.3}$$

On the other hand, by the First Main Theorem in Nevanlinna theory and the hypotheses of Theorem 1.8, for $n < s$, we get

$$T\left(r, \frac{f^{(n)}}{f^{(s)}}\right) = T\left(r, \frac{f^{(s)}}{f^{(n)}}\right) + O(1) = m\left(r, \frac{f^{(s)}}{f^{(n)}}\right) + N\left(r, \frac{f^{(s)}}{f^{(n)}}\right) + O(1).$$

Since $f^{(n)}$ just has finite many zeros for all $n < s$ ($n = 0, \dots, s-1$), by the properties of analytic functions and the definition of the counting function, we obtain

$$N(r, \frac{f^{(s)}}{f^{(n)}}) = O(1).$$

Thus

$$m\left(r, \frac{f^{(n)}}{f^{(s)}}\right) \le T\left(r, \frac{f^{(n)}}{f^{(s)}}\right) \le m\left(r, \frac{f^{(s)}}{f^{(n)}}\right) + O(1). \tag{4.4}$$

By (1.1), we get

$$-A_s = \frac{f^{(k)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_0\frac{f}{f^{(s)}},$$

and then

$$m(r, A_s) \le \sum_{j\ne s} m(r, A_j) + \sum_{s<i\le k} m\left(r, \frac{f^{(i)}}{f^{(s)}}\right) + \sum_{0\le n<s} m\left(r, \frac{f^{(n)}}{f^{(s)}}\right) + \log k. \tag{4.5}$$

Combining (4.1)–(4.5), for all $r \in E_5 \backslash E_4$ and $r \to 1^-$, we have

$$\begin{aligned} &\exp_{p-2}\left((\tau^* - \varepsilon)\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right) \\ &\le O\left(\log^+ T(r,f) + \log^+\frac{1}{1-r}\right) + \exp_{p-2}\left((\beta_1 + \varepsilon)\exp\left(\tau\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right)\right). \end{aligned}$$

By using the similar way as in the proof of Theorem 1.6, we deduce that

$$\sigma_{[p,q]}(f) = +\infty$$

and

$$\sigma_{[p+1,q]}(f) \ge \sigma = \sigma_{[p,q]}(A_s).$$

By Lemma 2.3, and using the same reasoning as in the proof of Theorem 1.6, we get $\sigma_{[p+1,q]}(f) \le \max\{\sigma_{[p,q],M}(A_j) : j = 0, 1, \dots, k-1\}$, and $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$ for $p > q$. □

Proof of Theorem 1.9 By Lemmas 2.4 and 2.5 and using similar method as in the proof of Theorem 1.8, the conclusion can be obtained, here we omit the details. □

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