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Liouville-Type Theorem for Stable Solutions of the Kirchhoff Equation with Negative Exponent

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Abstract In this paper, we consider the Liouville-type theorem for stable solutions of the following Kirchhoff equation

$$M\Big(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\Delta u=g(x)u^{-q},\quad x\in\mathbb{R}^N,$$

where $M(t)=a+bt^{\theta},a>0,b,\theta\geq 0,\theta=0$ if and only if $b=0$. $N\geq 2,q>0$ and the nonnegative function $g(x)\in L^1_{\rm loc}(\mathbb{R}^N)$. Under suitable conditions on $g(x),\theta$ and q, we investigate the nonexistence of positive stable solution for this problem.

Keywords Kirchhoff equation; negative exponent; stable solution; nonexistence

MR(2010) Subject Classification 35J60; 35A01; 35B53; 35B35

1. Introduction

In this paper, we are concerned with Liouville-type theorem for stable solution of the Kirchhoff equation

$$M\Big(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\Delta u=g(x)u^{-q},\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $M(t)=a+bt^{\theta},a>0,b,\theta\geq 0,\theta=0$ if and only if $b=0$. $N\geq 2,q>0$, $g(x)\in L^1_{\rm loc}(\mathbb{R}^N)$ is nonnegative, the exact assumption on $g(x)$ will be given latter. Such problem is often referred to as being nonlocal because of the presence of the integral over the entire domain $\mathbb{R}^N$. When $\theta=1$, problem (1.1) is analogous to the stationary problem of a model introduced by Kirchhoff [1]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho u_{tt}-\Big(\frac{p_0}{h}+\frac{E}{2L}\int_0^L u_x^2\mathrm{d}x\Big)u_{xx}=0,\quad t>0,\quad x\in(0,L),\tag{1.2}$$

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where ρ, p_0, h, E, L are all positive constants. This equation extends the classical D'Alembert wave equation. For the bounded domain Ω, the problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\mathrm{d}x)\Delta u = f(x,u), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega \end{cases} \tag{1.3}$$

is related to the stationary analogue of (1.2). Such nonlocal elliptic problem like (1.3) has received a lot of attention and some important and interesting results have been established, see [2–5] and the references therein.

Recently, much attention has been paid for Kirchhoff elliptic equation on $\mathbb{R}^N$. Li et al. [6] studied the following problem

$$\Big(a+\lambda\int_{\mathbb{R}^N}(|\nabla u|^2+bu^2)\mathrm{d}x\Big)(-\Delta u+bu)=f(u),\quad x\in\mathbb{R}^N, \tag{1.4}$$

where $N\ge 3$ and a,b are positive constants, $\lambda\ge 0$ is a parameter. They proved the existence of a positive solution to problem (1.4) with small $\lambda\in[0,\lambda_0)$. Fan and Liu [7] studied the existence of multiple solutions for the Kirchhoff equation

$$\Big(a+\mu\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\mathrm{d}x\Big)(-\Delta u+V(x)u)=f(x,u)+g(x)|u|^{q-2}u,\quad x\in\mathbb{R}^N, \tag{1.5}$$

where $\mu\ge 0$ is small, $N\ge 3, 1<q<2, a>0$, and the potential function $V(x)\in C(\mathbb{R}^N)$ satisfying $\inf_{x\in\mathbb{R}^N}V(x)>0$ and $\mathrm{meas}(\{x\in\mathbb{R}^N|V(x)\le M\})<\infty$ for each $M>0$. This assumption guarantees that the embedding $W^{1,2}(\mathbb{R}^N)\hookrightarrow L^s(\mathbb{R}^N)$ is compact for each $2\le s<2^*$. For problem (1.5), the function $f(x,u)$ verifies $|f(x,t)|\le C(1+|t|^{p-1})$, $\lim_{t\to 0}t^{-1}f(t)=0$, $\lim_{t\to\infty}t^{-1}f(t)=\infty$, and $\|g\|_{q'}(q'=2^*/(2^*-q)$ is small.

Li and Su [8], Nie and Wu [9], also considered problem (1.5), where the potential $V(x)$ is radially symmetric function. The other class of potential $V(x)\in C(\mathbb{R}^N)$ satisfying

$$0<V_0=\inf_{x\in\mathbb{R}^N}V(x)\le\lim_{|x|\to\infty}V(x)=\sup_{x\in\mathbb{R}^N}V(x):=V_\infty<\infty \tag{1.6}$$

has also been studied, see [10–12] and the references therein.

We note that, in the above works, one always assumes that the potential function $V(x)\ge 0$ and $V(x)\not\equiv 0$ in $\mathbb{R}^N$.

On the other hand, the nonexistence and stability of solutions to nonlinear partial differential equations also have been studied in recent years. We refer the readers to [13–21] and the references therein. It is worth pointing out that for the singular elliptic equation (1.1), Ma and Wei in [20] obtained:

Theorem 1.1 *Let $q>0$, $g(x)=1$ and $M(t)=1$ in (1.1). Moreover, if*

$$2\le N<2+\frac{4}{1+q}(q+\sqrt{q^2+q}), \tag{1.7}$$

then there are no stable positive solutions to (1.1) *in* $\mathbb{R}^N$.

Remark 1.2 Obviously, if $2<N<10$, then (1.7) implies that

$$q>p_0:=-1-\frac{4(N-4+2\sqrt{N-1})}{(N-2)(N-10)}. \tag{1.8}$$

Motivated by [18, 22, 23], we will study the nonexistence of positive stable solution of (1.1). We now introduce the main results in this paper.

As in [24], let $X = \mathcal{D}^{1,2}(\mathbb{R}^N)$ be the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm of $\|u\|_X = (\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x)^{1/2}$. Then there exists a positive constant S such that

$$\Big(\int_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x\Big)^{1/2^*} \leq S\Big(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)^{1/2}, \text{ or } \|u\|_{2^*} \leq S\|u\|_X,\ \forall u \in X \tag{1.9}$$

which is called the Sobolev's inequality in [25], where 2^* is the Sobolev critical exponent.

Now we consider the energy functional of (1.1) $\mathcal{I} : X \to \mathbb{R}$ defined by

$$\mathcal{I}(u) = \frac{a}{2}\|u\|_X^2 + \frac{b}{2(\theta+1)}\|u\|_X^{2(\theta+1)} + \frac{1}{1-q}\int_{\mathbb{R}^N} g(x)u^{1-q}\mathrm{d}x. \tag{1.10}$$

It is well known that if $u \in X$ is a weak solution of (1.1), then for any $\zeta \in X$, the function $E(t) = \mathcal{I}(u + t\zeta)$ satisfies $E'(0) = 0$, that is,

$$E'(0) = \mathcal{I}'(u)\zeta = \Big(a + b\|u\|_X^{2\theta}\Big)\int_{\mathbb{R}^N} \nabla u \cdot \nabla\zeta \mathrm{d}x + \int_{\mathbb{R}^N} g(x)u^{-q}\zeta\mathrm{d}x = 0,\quad \forall\, \zeta \in X. \tag{1.11}$$

As in [13], we say that the positive solution u of (1.1) is stable if $E''(0) \geq 0$. A routine calculation shows that

$$\begin{aligned}\frac{E'(t) - E'(0)}{t} &= \frac{\mathcal{I}'(u+t\zeta)\zeta - \mathcal{I}'(u)\zeta}{t} = \frac{b(\|u+t\zeta\|_X^{2\theta} - \|u\|_X^{2\theta})}{t}\int_{\mathbb{R}^N} \nabla u\cdot\nabla\zeta\mathrm{d}x + \\ &(a + b\|u+t\zeta\|_X^{2\theta})\int_{\mathbb{R}^N} |\nabla\zeta|^2\mathrm{d}x + \frac{1}{t}\int_{\mathbb{R}^N} g(x)((u+t\zeta)^{-q} - u^{-q})\zeta\mathrm{d}x\end{aligned} \tag{1.12}$$

and

$$\begin{aligned}E''(0) = \lim_{t\to 0}\frac{E'(t) - E'(0)}{t} &= 2b\theta\|u\|_X^{2(\theta-1)}\Big(\int_{\mathbb{R}^N}\nabla u\cdot\nabla\zeta\mathrm{d}x\Big)^2 + \\ &(a + b\|u\|_X^{2\theta})\int_{\mathbb{R}^N}|\nabla\zeta|^2\mathrm{d}x - q\int_{\mathbb{R}^N} g(x)u^{-q-1}\zeta^2\mathrm{d}x.\end{aligned} \tag{1.13}$$

Set $Q_u(\zeta) = E''(0)$. We now define stability as follows.

Definition 1.3 *A positive weak solution $u \in X$ of (1.1) is stable if $Q_u(\zeta) \geq 0$ for any $\zeta \in X$.*

Remark 1.4 The quadratic form Q_u is called the second variation of the energy functional $\mathcal{I}$. Then, the stability condition translates into the fact that the second variation of the energy functional is non-negative. Thus, all the minima of the functional $\mathcal{I}$ are stable solutions of (1.1), see [13].

Remark 1.5 If $u \in X$ is a stable positive weak solution of (1.1), applying Hölder inequality and (1.13), we deduce that

$$q\int_{\mathbb{R}^N} g(x)u^{-q-1}\zeta^2\mathrm{d}x \leq A\int_{\mathbb{R}^N}|\nabla\zeta|^2\mathrm{d}x,\quad \forall\,\zeta \in X \tag{1.14}$$

with

$$A = a + b(1+2\theta)\|u\|_X^{2\theta}. \tag{1.15}$$

Throughout this paper, we give the following assumption on $g(x)$.

(H_1) $g(x) \in L^1_{\rm loc}(\mathbb{R}^N)$ is nonnegative in $\mathbb{R}^N$. Moreover, there exist $k > -2$ and $R_0, c_0 > 0$ such that $g(x) \geq c_0|x|^k$, $\forall\, |x| \geq R_0$.

Our main result can be included in the following theorem:

Theorem 1.6 *Let (H_1) and $M(t) = a + bt^\theta, a > 0, b, \theta \geq 0$ hold, $\theta = 0$ if and only if $b = 0$. Assume that one of the following conditions is satisfied:*

(H_2) $\theta \geq 0$ and $N = 2$, $q > 0$;

(H_3) $0 \leq \theta \leq \frac{3}{2}$ and $2 < N < 2 + \frac{4(2+k)}{1+2\theta}$, $q > \alpha_0$;

(H_4) $\theta > \frac{3}{2}$ and $2 < N < 2 + \frac{4(2+k)}{1+2\theta}$, $q > \beta_0$;

(H_5) $\theta > \frac{3}{2}$ and $N = 2 + \frac{4(2+k)}{1+2\theta}$, $q > \frac{4}{2\theta-3}$;

(H_6) $\theta > \frac{3}{2}$ and $2 + \frac{4(2+k)}{1+2\theta} < N < 2 + \frac{(1+\sqrt{1+2\theta})(2+k)}{2\theta}$, $\beta_1 < q < \beta_2$,

where

$$\alpha_0 = -1 - \frac{2(2+k)[N-4-k+\sqrt{(N+k)^2-(N-2)^2(1+2\theta)}]}{(N-2)[(N-2)(1+2\theta)-4(2+k)]}. \tag{1.16}$$

$$\beta_0 = -1 - \frac{2(2+k)[N-4-k+\sqrt{(N+k)^2-(N-2)^2(1+2\theta)}]}{(N-2)[(N-2)(1+2\theta)-4(2+k)]}. \tag{1.17}$$

$$\beta_{1,2} = -1 - \frac{2(2+k)[N-4-k\pm\sqrt{(N+k)^2-(N-2)^2(1+2\theta)}]}{(N-2)[(N-2)(1+2\theta)-4(2+k)]}. \tag{1.18}$$

Then (1.1) has no positive weak stable solution.

Remark 1.7 (i) If $\theta = 0$, we obtain

$$\alpha_0 = -1 - \frac{2(2+k)[N-4-k+\sqrt{(2N-2+k)(2+k)}]}{(N-2)(N-10-4k)}. \tag{1.19}$$

Then α_0 is equal to the exponent $p(N, \alpha)$ in [26].

(ii) If $\theta = 0$, $k = 0$, we obtain

$$\alpha_0 = -1 - \frac{4(N-4+2\sqrt{N-1})}{(N-2)(N-10)}. \tag{1.20}$$

Then α_0 is equal to the exponent $q_c(2, N)$ in [27] and $\alpha_0 = p_0$, where p_0 is the critical exponent (1.8) and coincides with that in [20].

2. Proof of Theorem 1.6

To prove the nonexistence of positive weak stable solution of (1.1), we use the test function method, which has been used in [18, 22, 23]. Since Kirchhoff equation (1.1) is nonlocal, some modification in choosing test functions is necessary.

We first establish the following lemma.

Lemma 2.1 *Let $u \in C^{1,\omega}_{\rm loc}(\mathbb{R}^N)$ $(0 < \omega < 1)$ be a positive weak stable solution of (1.1) with $q > 0$. Then for every $\gamma \in (\gamma(q), -1)$, where*

$$\gamma(t) = -\frac{1+2\theta+2t+2\sqrt{t(t+1+2\theta)}}{1+2\theta}, \quad t > 0 \tag{2.1}$$

and for any constant $m \geq \frac{q-\gamma}{q+1}$, there exists a positive constant C depending on $q, \gamma, m, a, b, \theta$ such that

$$\int_{\mathbb{R}^N} (g(x)u^{\gamma-q} + |\nabla u|^2 u^{\gamma-1})\varphi^{2m} \mathrm{d}x \leq CA^{\frac{q-\gamma}{q+1}} \int_{\mathbb{R}^N} g(x)^{\frac{\gamma+1}{q+1}} |\nabla \varphi|^{\frac{2(q-\gamma)}{q+1}} \mathrm{d}x, \tag{2.2}$$

where $\varphi(x) \in C_0^1(\mathbb{R}^N)$ is any nonnegative function with $0 \leq \varphi(x) \leq 1$ and A is given by (1.15).

Proof Let $u \in C^{1,\omega}_{\mathrm{loc}}(\mathbb{R}^N)$ $(0 < \omega < 1)$ be a positive weak stable solution of (1.1) and $\gamma < -1$. Choosing $\zeta = u^\gamma \varphi^2$ as a test function in (1.11), we obtain

$$|\gamma| A_1 \int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1} \varphi^2 \mathrm{d}x \leq 2A_1 \int_{\mathbb{R}^N} |\nabla u||\nabla \varphi| u^\gamma \varphi \mathrm{d}x + \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x, \tag{2.3}$$

where $A_1 = a + b\|u\|_X^{2\theta}$. Applying Young's inequality with any $\varepsilon \in (0,1)$, we get

$$2\int_{\mathbb{R}^N} |\nabla u||\nabla \varphi| u^\gamma \varphi \mathrm{d}x \leq \varepsilon \int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1} \varphi^2 \mathrm{d}x + C_1 \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x. \tag{2.4}$$

Here and in what follows, we denote by C_j a positive constant depending on ε and q, γ, θ. Combining (2.3) with (2.4) enables us to deduce

$$(|\gamma| - \varepsilon) A_1 \int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1} \varphi^2 \mathrm{d}x \leq \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x + C_1 A_1 \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x. \tag{2.5}$$

On the other hand, using the stability assumption with $\zeta = u^{\frac{\gamma+1}{2}} \varphi$ in (1.14) yields

$$\begin{aligned} q \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x \leq & \frac{(1+\gamma)^2}{4} A \int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1} \varphi^2 \mathrm{d}x + A \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x + \\ & (1+|\gamma|) A \int_{\mathbb{R}^N} |\nabla u||\nabla \varphi| u^\gamma \varphi \mathrm{d}x. \end{aligned} \tag{2.6}$$

By Young's inequality, it follows that

$$(1+|\gamma|) \int_{\mathbb{R}^N} |\nabla u||\nabla \varphi| u^\gamma \varphi \mathrm{d}x \leq \varepsilon \int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1} \varphi^2 \mathrm{d}x + C_2 \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x, \tag{2.7}$$

where ε coincides with that in (2.4). Plugging (2.7) into (2.6), we can deduce

$$q \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x \leq \frac{[(1+\gamma)^2 + 4\varepsilon]}{4} A \int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1} \varphi^2 \mathrm{d}x + C_3 A \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x. \tag{2.8}$$

Furthermore, from (2.5) and (2.8) we have

$$\begin{aligned} q \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x & \leq \frac{[(1+\gamma)^2 + 4\varepsilon] A}{4(|\gamma| - \varepsilon) A_1} \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x + C_4 A \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x \\ & \leq \frac{[(1+\gamma)^2 + 4\varepsilon](1+2\theta)}{4(|\gamma| - \varepsilon)} \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x + C_4 A \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x, \end{aligned} \tag{2.9}$$

that is

$$q_\varepsilon \int_{\mathbb{R}^N} g(x) u^{\gamma-q} \varphi^2 \mathrm{d}x \leq C_4 A \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x, \tag{2.10}$$

with

$$q_\varepsilon = q - \frac{[(1+\gamma)^2 + 4\varepsilon](1+2\theta)}{4(|\gamma| - \varepsilon)}, \quad \lim_{\varepsilon \to 0^+} q_\varepsilon = q_0 = q - \frac{(1+\gamma)^2 (1+2\theta)}{4|\gamma|}. \tag{2.11}$$

Thanks to $\gamma \in (\gamma(q), -1)$, $q_0 > 0$ holds, where $\gamma(t)$ is defined by (2.1). Thus, we can select sufficiently small $\varepsilon > 0$ such that $q_\varepsilon > 0$.

Applying (2.10) and (2.5), we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1}\varphi^2 \mathrm{d}x \le C_5 \int_{\mathbb{R}^N} |\nabla \varphi|^2 u^{\gamma+1} \mathrm{d}x. \tag{2.12}$$

Now we claim that (2.2) is true. In fact, we can choose some constant m large enough satisfying

$$\frac{(m-1)(\gamma-q)}{\gamma+1} \ge m, \text{ or } m \ge \frac{q-\gamma}{q+1}. \tag{2.13}$$

By virtue of $0 \le \varphi(x) \le 1$ in $\mathbb{R}^N$, one can achieve

$$[\varphi(x)]^{\frac{2(m-1)(\gamma-q)}{(\gamma+1)}} \le [\varphi(x)]^{2m}, \quad \forall\, x \in \mathbb{R}^N. \tag{2.14}$$

Replacing φ in (2.10) with φ^m and utilizing the Hölder inequality, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x) u^{\gamma-q}\varphi^{2m} \mathrm{d}x \le C_6 A \int_{\mathbb{R}^N} |\nabla\varphi|^2 \varphi^{2(m-1)} u^{\gamma+1} \mathrm{d}x \\
&\le C_6 A\Big(\int_{\mathbb{R}^N} g(x) u^{\gamma-q}\varphi^{\frac{2(m-1)(\gamma-q)}{\gamma+1}} \mathrm{d}x\Big)^{\frac{\gamma+1}{\gamma-q}} \Big(\int_{\mathbb{R}^N} g(x)^{\frac{\gamma+1}{q+1}} |\nabla\varphi|^{\frac{2(q-\gamma)}{q+1}} \mathrm{d}x\Big)^{\frac{q+1}{q-\gamma}} \\
&\le C_6 A\Big(\int_{\mathbb{R}^N} g(x) u^{\gamma-q}\varphi^{2m} \mathrm{d}x\Big)^{\frac{\gamma+1}{\gamma-q}} \Big(\int_{\mathbb{R}^N} g(x)^{\frac{\gamma+1}{q+1}} |\nabla\varphi|^{\frac{2(q-\gamma)}{q+1}} \mathrm{d}x\Big)^{\frac{q+1}{q-\gamma}}.
\end{aligned} \tag{2.15}$$

Consequently, we obtain

$$\int_{\mathbb{R}^N} g(x) u^{\gamma-q}\varphi^{2m} \mathrm{d}x \le C_7 A^{\frac{q-\gamma}{q+1}} \int_{\mathbb{R}^N} g(x)^{\frac{\gamma+1}{q+1}} |\nabla\varphi|^{\frac{2(q-\gamma)}{q+1}} \mathrm{d}x. \tag{2.16}$$

Analogously, with φ replaced by φ^m in (2.12), it follows from (2.12), (2.15) and (2.16) that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1}\varphi^{2m} \mathrm{d}x &\le C_8 \int_{\mathbb{R}^N} |\nabla\varphi|^2 \varphi^{2(m-1)} u^{\gamma+1} \mathrm{d}x \\
&\le C_9 A^{\frac{q-\gamma}{q+1}} \int_{\mathbb{R}^N} g(x)^{\frac{\gamma+1}{q+1}} |\nabla\varphi|^{\frac{2(q-\gamma)}{q+1}} \mathrm{d}x.
\end{aligned} \tag{2.17}$$

Combining (2.16) with (2.17) enables us to deduce (2.2). The proof is completed. □

Proof of Theorem 1.6 Define $\varphi_0(s) \in C_0^1[0, +\infty)$ with

$$\varphi_0(s) = \begin{cases} 1, & 0 \le s \le 1, \\ 0, & s > 2. \end{cases} \tag{2.18}$$

Let $\varphi(x) = \varphi_0(\frac{|x|}{R})$ for $R \ge R_0$, where R_0 is given in (H$_1$). Obviously, $\varphi(x) \in C_0^1(\mathbb{R}^N)$ with $0 \le \varphi(x) \le 1$. A direct calculation shows that there exists $C > 0$ such that $|\nabla\varphi(x)| \le CR^{-1}$, $x \in \overline{B}_{2R} \setminus \overline{B}_R$ and $|\nabla\varphi(x)| = 0$, $x \in \overline{B}_R \cup \overline{B}_{2R}^c$, where $B_r = \{x \in \mathbb{R}^N : |x| < r\}$.

Suppose on the contrary that (1.1) admits a positive weak stable solution, then utilizing the assumption (H_1) and the estimate (2.2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x) u^{\gamma-q}\varphi^{2m} \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^2 u^{\gamma-1}\varphi^{2m} \mathrm{d}x \\
&\le CA^{\frac{q-\gamma}{q+1}} R^{\frac{-2(q-\gamma)}{q+1}} \int_{R<|x|\le 2R} |x|^{\frac{k(\gamma+1)}{q+1}} \mathrm{d}x
\end{aligned}$$

$$\leq CA^{\frac{q-\gamma}{q+1}} R^{N-\frac{2(q-\gamma)-k(\gamma+1)}{q+1}}, \tag{2.19}$$

where C denote various positive constants.

Set

$$\rho = N - \frac{2(q-\gamma)-k(\gamma+1)}{q+1}. \tag{2.20}$$

Obviously, if $\rho < 0$, passing to the limits as $R \to +\infty$ in (2.19), we deduce a contradiction. Next, we are devoted to choosing some appropriate γ such that $\rho < 0$. To do this, we define the function

$$h(t) = \frac{2[t-\gamma(t)]-k[\gamma(t)+1]}{t+1}, \quad t > 0, \tag{2.21}$$

where $\gamma(t)$ is given by (2.1). A direct calculation leads to

$$\lim_{t\to 0^+} \gamma(t) = -1, \quad \gamma'(t) < 0, \ t > 0, \quad \lim_{t\to+\infty} \gamma(t) = -\infty, \tag{2.22}$$

and

$$\lim_{t\to 0^+} h(t) = 2 \leq N, \quad \lim_{t\to+\infty} h(t) = 2 + \frac{4(2+k)}{1+2\theta}, \tag{2.23}$$

$$h'(t) = \frac{(2+k)\big(2\sqrt{t(t+1+2\theta)}+1+2\theta+t(1-2\theta)\big)}{(1+2\theta)\sqrt{t(t+1+2\theta)}(t+1)^2}, \quad t > 0. \tag{2.24}$$

A routine calculation shows that if $0 \leq \theta \leq \frac{3}{2}$, then $h(t)$ is strictly increasing on $(0,+\infty)$; if $\theta > \frac{3}{2}$, then $h(t)$ is strictly increasing on $(0, \frac{1+2\theta+2\sqrt{1+2\theta}}{2\theta-3})$ and strictly decreasing on $(\frac{1+2\theta+2\sqrt{1+2\theta}}{2\theta-3}, +\infty)$. Moreover, $h(\frac{1+2\theta+2\sqrt{1+2\theta}}{2\theta-3}) = 2 + \frac{(1+\sqrt{1+2\theta})(2+k)}{2\theta}$, $h(\frac{4}{2\theta-3}) = 2 + \frac{4(2+k)}{1+2\theta}$.

Therefore, if $N = 2$ and $\theta \geq 0$, then $N < h(t)$, $\forall\, t > 0$. So if we fix $\gamma \in (\gamma(t), -1)$ suitably near $\gamma(t)$, we obtain

$$N < \frac{2(t-\gamma)-k(\gamma+1)}{t+1}. \tag{2.25}$$

Letting $R \to +\infty$ in (2.19), we get a contradiction.

If $2 < N < 2 + \frac{4(2+k)}{1+2\theta}$ and $0 \leq \theta \leq \frac{3}{2}$, by the properties of the function $h(t)$, there exists a unique $\alpha_0 > 0$ such that $N < h(t)$, $t > \alpha_0$. From this, taking $R \to +\infty$ in (2.19), we deduce a contradiction. Clearly, α_0 may be deduced from the equation $N = h(q)$, which is given in (1.16).

If $2 < N < 2 + \frac{4(2+k)}{1+2\theta}$ and $\theta > \frac{3}{2}$, by the properties of the function $h(t)$, there exists a unique $\beta_0 > 0$ such that $N < h(t)$, $t > \beta_0$. From this, letting $R \to +\infty$ in (2.19), we get a contradiction. Clearly, β_0 may be deduced from the equation $N = h(q)$, which is given in (1.17).

If $N = 2 + \frac{4(2+k)}{1+2\theta}$ and $\theta > \frac{3}{2}$, note that $h(t) > h(\frac{4}{2\theta-3}) = 2 + \frac{4(2+k)}{1+2\theta}$, $t > \frac{4}{2\theta-3}$, we have $N < h(t)$, $t > \frac{4}{2\theta-3}$. From this, letting $R \to +\infty$ in (2.19), we get a contradiction.

Assume now $2+\frac{4(2+k)}{1+2\theta} < N < 2+\frac{(1+\sqrt{1+2\theta})(2+k)}{2\theta}$ and $\theta > \frac{3}{2}$, by the properties of the function $h(t)$, there exist $\beta_{1,2} > \frac{4}{2\theta-3}$ such that $N < h(t)$ for $\beta_1 < t < \beta_2$. From this, letting $R \to +\infty$ in (2.19), we get a contradiction. Clearly, $\beta_{1,2}$ may be deduced from the equation $N = h(q)$, which is given in (1.18). The proof of Theorem 1.6 is completed. $\square$

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