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Differentiability of Interval Valued Function and Its Application in Interval Valued Programming

Yu-e BAO*, **Na LI**, **Linfen ZHANG**
College of Mathematics and Physics, Inner Mongolia University for Nationalities, Inner Mongolia 028043*, P. R. China*

Abstract In this paper, the differentiability of interval valued function is discussed by using the idea of total differential of real valued function, the concept of D-differentiability of interval valued function is established and some basic properties are given. By discussing the optimality condition of unconstrained interval programming, the necessary conditions for obtaining the optimal solution of a class of constrained interval valued programming with real valued function constraints are given. Meanwhile, the sufficient conditions for obtaining the optimal solution are given for the convex interval value programming problem with real value function constraints.

Keywords interval-valued function; D-differentiability; interval-valued programming; KKT optimal condition

MR(2010) Subject Classification 58C25; 90C30; 90C46

1. Introduction

Since the operation theory of interval number was firstly proposed in 1966 by Moore [1], the interval number theory and its application have been improved greatly with a lot of researchers' effort. Specially, various concepts of differentiability about interval valued function on the foundation of optimal theory and differential equation theory [2–9] of interval valued function are imported.

There are two methods to define the differentiability of interval valued function. The first one is H-derivative which is given by H-difference from the nonempty subset in real number space to the interval valued function in interval number space, and H-partial derivative which is given from the subset in n-dimensional Euclid space to the interval valued function in interval number space [8–10]. The second one is gH-derivative which is given by gH-difference from the nonempty subset in real number space to the interval valued function in interval number space, and gH-partial derivative which is given from the subset in n-dimensional Euclid space to the interval valued function in interval number space [8, 9]. These concepts of differentiability only concern about the changing rate of interval valued function in axis direction rather than other special

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* Corresponding author
E-mail address: byebed@y163.com (Yu-e BAO); 1024951096@qq.com (Na LI); 1475094089@qq.com (Linfen ZHANG)

direction. We built the concepts of $H(gH)$-directional differential and derivative in [11, 12], and proved that $H(gH)$-derivative and $H(gH)$-partial derivative are directional derivatives in axis direction. At the same time, we introduced the concepts of D-directional derivative and D-partial derivative in order to avoid the troubles brought by the difference between interval numbers. We discussed the relationship between H-directional differential and D-directional differential, and proved that H-directional differential leads to D-directional differential, whose reverse is not always true by using example.

The optimal theory has wide applications as an important branch of mathematics. Because of the uncertainty of some data and information, we use the interval number to express the variation range of data or information in math programming [13–16]. KKT optimal condition, which plays an important role in the field of optimal theory, has been researched for more than one century. Wu studied the KKT optimal condition of convex interval valued programming whose constrained condition is real valued function under the condition of H-differential [7–9]. Sing, Dar and Goyal [17] proposed the KKT condition of optimal problem whose target function and constrained function are both interval valued function. Chalco-Cano, Lodwick and Rufian-Lizana [18] and Singh, Dar and Kim [19] studied the KKT optimal condition of convex interval-valued optimal problems whose constrained function is real valued function. Zhang, Liu, Li and Feng [20] studied the KKT optimal condition of non-convex interval-valued optimal problem.

We are inspired by [21,22] and introduce the concept of D-differentiability to avoid the difficulty brought by the difference of interval number. D-differentiability allows us to discuss the KKT optimal condition of interval-valued optimal problem and enriches the theory of interval-valued function. In Section 3, we build the concepts of D-differentiability and its gradient and propose the operation properties and characterization of D-differentiable interval-valued function. In Section 4, we discuss the optimal condition of unconstrained interval-valued programming whose target function is D-differentiable. In Section 5, we discuss the KKT condition of constrained interval-valued programming whose target function is D-differentiable. We also discuss the optimal condition of interval-valued programming whose constrained condition is real valued function.

2. Preliminary

We denote by $[\mathbb{R}]$ the set of all closed and bounded intervals in real line $\mathbb{R}$, i.e.,

$$[\mathbb{R}] = \{a = [\underline{a}, \overline{a}] | \underline{a}, \overline{a} \in R, \underline{a} \leq \overline{a}\}.$$

Let M be a nonempty subset in n-dimensional Euclid space $\mathbb{R}^n$. A mapping $F : M \to [\mathbb{R}]$ is called the interval valued function. In this case $F(x) = [\underline{F}(x), \overline{F}(x)]$, where $\underline{F}$ and $\overline{F}$ are real valued functions defined on M satisfying $\underline{F}(x) \leq \overline{F}(x)$ for any $x \in M$.

For $a = [\underline{a}, \overline{a}], b = [\underline{b}, \overline{b}] \in [\mathbb{R}]$ and $k \in \mathbb{R}$ $(k \geq 0)$, we define the operations of addition, multiplication, scalar multiplication and partial ordering of interval numbers [9, 21]:

(1) $a + b = [\underline{a+b}, \overline{a+b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$;

(2) $ab = [\underline{ab}, \overline{ab}] = [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}]$;

(3) $ka = [\underline{ka}, \overline{ka}]$;

(4) $a \leq b \Leftrightarrow \underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$;

(5) $a < b \Leftrightarrow \underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

For $a = [\underline{a}, \overline{a}]$, $b = [\underline{b}, \overline{b}] \in [\mathbb{R}]$, define $D_H(a,b) = \max\left\{|\underline{a} - \underline{b}|, \left|\overline{a} - \overline{b}\right|\right\}$, then $([\mathbb{R}], D_H)$ is a complete metric space. And for $a, b, c \in [\mathbb{R}]$ and $k \in \mathbb{R}$, the following properties hold:

$$D_H(a+c, b+c) = D_H(a,b),\ D_H(ka, kb) = |k|\, D_H(a,b).$$

For $a_i \in [\mathbb{R}]$ $(i = 1, 2, \ldots n)$, we define $a = (a_1, a_2, \ldots, a_n)$ as n-dimensional interval valued vector in $[\mathbb{R}]$. Let $[\mathbb{R}]^n$ denote the set of all n-dimensional interval valued vectors in $[\mathbb{R}]$. For

$$a = (a_1, a_2, \ldots, a_n),\ b = (b_1, b_2, \ldots, b_n) \in [\mathbb{R}]^n;\ x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n,$$

we define the following operations:

(1) $a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$;

(2) $kx = (kx_1, kx_2, \ldots, kx_n)(k \geq 0)$;

(3) $\langle x, a\rangle = \sum_{i=1}^n x_i a_i$, $\underline{\langle x, a\rangle} = \langle x, \underline{a}\rangle = \sum_{i=1}^n x_i \underline{a}_i$, $\overline{\langle x, a\rangle} = \langle x, \overline{a}\rangle = \sum_{i=1}^n x_i \overline{a}_i$, where $x_i \geq 0$, $i = 1, 2, \ldots, n$; $\underline{a} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$, $\overline{a} = (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n) \in \mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$.

Definition 2.1 ([8,10]) *Let $a, b \in [\mathbb{R}]$. If there exists $c \in [\mathbb{R}]$ such that $a = b + c$, then c is called the Hukuhara difference (H-difference) between a and b, denoted as $c = a -_H b$.*

Proposition 2.2 ([12]) *Let $a, b \in [\mathbb{R}]$. If H-difference $a -_H b$ exists, then*

$$D_H(a, b + c) = D_H(a -_H b, c).$$

In this paper, for $y \in \mathbb{R}^n$, the unit vector of y is denoted as y_ε.

Definition 2.3 ([12]) *Let $F : M \to [\mathbb{R}]$ be an interval valued function. For $x \in M$ and $y \in \mathbb{R}^n$, if there exits $\delta > 0$ such that $x + hy_e \in M$ $(x - hy_e \in M)$ for any $h \in (0, \delta)$, and there exists $a_+ \in [\mathbb{R}]$ $(a_- \in [\mathbb{R}])$ such that*

$$\lim_{h\to 0^+} \frac{D_H(F(x + hy_e), F(x) + ha_+)}{h} = 0,\ \lim_{h\to 0^+} \frac{D_H(F(x), F(x - hy_e) + ha_-)}{h} = 0,$$

then F is right (left) D-differentiable in direction y at x, a_+ $(a_-$ $)$ is the right (left) D-derivative of in directiony at x, and denoted as $F_+^D(x, y) = a_+ (F_-^D(x, y) = a_-)$.

If $F_+^D(x, y) = F_-^D(x, y)$, then we say that F is D-differentiable in direction y at x, and denoted as $F^D(x, y) = F_+^D(x, y) = F_-^D(x, y)$. And $F^D(x, y)$ is called the D-directional derivative of F in direction y at x.

Definition 2.4 ([2,8]) *Let $F : M \to ([\mathbb{R}], D_H)$ be an interval valued function and*

$$x^0 = (x_1^0, \ldots, x_n^0) \in M.$$

If F is H-differentiable at x^0, then the gradient $\nabla F(x^0)$ of F at x^0 is defined by

$$\nabla F(x^0) = ((\partial F/\partial x_1)(x_0), (\partial F/\partial x_2)(x_0), \ldots, (\partial F/\partial x_n)(x_0)).$$

In this paper, we denote the gradient given in Definition 2.4 by $\nabla F^H(x^0)$.

Definition 2.5 ([16]) *Let M be a nonempty convex subset of $\mathbb{R}^n$ and F be an interval valued function on M. We say that F is convex on M if*

$$F(\lambda x+(1-\lambda)y)\le \lambda F(x)+(1-\lambda)F(y)$$

for any $x,y\in M$ and $\lambda\in[0,1]$.

It is easy to obtain that F is convex interval valued function on M if and only if $\underline{F}$ and $\overline{F}$ are both convex real valued function on M in [16, Proposition 2.2].

Lemma 2.6 ([23])(Gordan Theorem) *Let A be an $m\times n$ matrix. The inequality $A\overline{x}<0$ has a solution if and only if there is nonzero and non-negative $y\in R$ such that $A^Ty=0$.*

3. Differentiability of interval valued function

For $x^0=(x_1^0,x_2^0,\ldots,x_n^0)$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, let

$$(x-x^0)^+=((x_1-x_1^0)^+,(x_2-x_2^0)^+,\ldots,(x_n-x_n^0)^+),$$

$$(x-x^0)^-=((x_1-x_1^0)^-,(x_2-x_2^0)^-,\ldots,(x_n-x_n^0)^-),$$

where

$$(x_i-x_i^0)^+=\begin{cases} x_i-x_i^0, & x_i\ge x_i^0\\ 0, & x_i<x_i^0\end{cases}\ (i=1,2,\ldots,n),$$

$$(x_i-x_i^0)^-=\begin{cases} x_i^0-x_i, & x_i\le x_i^0\\ 0, & x_i>x_i^0\end{cases}\ (i=1,2,\ldots,n).$$

Then

$$x-x^0=(x-x^0)^+-(x-x^0)^-$$

and for $\lambda\ge 0$, we have

$$\lambda(x-x^0)^+=(\lambda(x-x^0))^+,\quad \lambda(x-x^0)^-=(\lambda(x-x^0))^-.$$

Definition 3.1 *Let $F:M\to([\mathbb{R}],D_H)$ be an interval valued function,*

$$x^0=(x_1^0,x_2^0,\ldots,x_n^0)\in\operatorname{int}M.$$

If there exists $a=(a_1,a_2,\ldots,a_n)\in[\mathbb{R}]^n$ such that

$$\lim_{x\to x^0}\frac{D_H(F(x)+\langle(x-x^0)^-,a\rangle,F(x^0)+\langle(x-x^0)^+,a\rangle)}{d(x,x^0)}=0$$

for any $x=(x_1,x_2,\ldots,x_n)\in M$, then F is D-differentiable at x^0, and $(a_1,a_2,\ldots,a_n)$ is the D-gradient of F at x^0 which is denoted as $\nabla F^D(x^0)=(a_1,a_2,\ldots,a_n)$.

Example 3.2 The interval valued function $F:M\to([\mathbb{R}],D_H)$ is defined by

$$F(x)=[\|x\|^2-1,\|x\|^2+1],x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n.$$

Since $f(x) = \|x\|^2$ is a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$ and $f_i'(x) = 2x_i$ $(i = 1, 2, \dots, n)$, for $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{R}^n$, we have

$$f(x) = f(x^0) + \sum_{i=1}^{n} 2x_i^0(x_i - x_i^0) + o(d(x, x^0))$$

i.e.,

$$\lim_{x \to x_0} \frac{\|x\|^2 - \|x^0\|^2 - 2\sum_{i=1}^{n}(x_i - x_i^0)x_i^0}{d(x, x^0)} = 0.$$

On the other hand,

$$\sum_{i=1}^{n}(x_i - x_i^0)^{-} 2x_i^0 - \sum_{i=1}^{n}(x_i - x_i^0)^{+} 2x_i^0 = -2\sum_{i=1}^{n}(x_i - x_i^0)x_i^0.$$

Let $\tilde{x}_i^0 = [x_i^0, x_i^0] \in [\mathbb{R}]$ $(i = 1, 2, \dots, n)$. Then $2\tilde{x}^0 = (2\tilde{x}_1^0, 2\tilde{x}_2^0, \dots, \tilde{x}_n^0) \in [\mathbb{R}]^n$ and

$$\begin{aligned}
&\left|\underline{F(x) + \langle (x - x^0)^{-}, 2\tilde{x}^0\rangle} - \underline{F(x^0) + \langle (x - x^0)^{+}, 2\tilde{x}^0\rangle}\right| \\
&= \left|\underline{F(x)} - \underline{F(x^0)} + \underline{\langle (x - x^0)^{-}, 2\tilde{x}^0\rangle} - \underline{\langle (x - x^0)^{+}, 2\tilde{x}^0\rangle}\right| \\
&= \left|\underline{F}(x) - \underline{F}(x^0) + \langle (x - x^0)^{-}, \underline{2\tilde{x}^0}\rangle - \langle (x - x^0)^{+}, \underline{2\tilde{x}^0}\rangle\right| \\
&= \left|\underline{F}(x) - \underline{F}(x^0) + \langle (x - x^0)^{-}, 2x^0\rangle - \langle (x - x^0)^{+}, 2x^0\rangle\right| \\
&= \left|\|x\|^2 - \|x^0\|^2 - 2\sum_{i=1}^{n}(x_i - x_i^0)x_i^0\right|.
\end{aligned}$$

Similarly, we can obtain that

$$\left|\overline{F(x) + \langle (x - x^0)^{-}, 2\tilde{x}^0\rangle} - \overline{F(x^0) + \langle (x - x^0)^{+}, 2\tilde{x}^0\rangle}\right| = \left|\|x\|^2 - \|x^0\|^2 - 2\sum_{i=1}^{n}(x_i - x_i^0)x_i^0\right|.$$

Thus we have

$$\begin{aligned}
&\lim_{x \to x^0} \frac{D_H(F(x) + \langle (x - x^0)^{-}, 2\tilde{x}^0\rangle, F(x^0) + \langle (x - x^0)^{+}, 2\tilde{x}^0\rangle)}{d(x, x^0)} \\
&= \lim_{x \to x^0} \frac{\max\left\{\left|\|x\|^2 - \|x^0\|^2 - 2\sum_{i=1}^{n}(x_i - x_i^0)x_i^0\right|, \left|\|x\|^2 - \|x^0\|^2 - 2\sum_{i=1}^{n}(x_i - x_i^0)x_i^0\right|\right\}}{d(x, x^0)} \\
&= \lim_{x \to x^0} \frac{\left|\|x\|^2 - \|x^0\|^2 - 2\sum_{i=1}^{n}(x_i - x_i^0)x_i^0\right|}{d(x, x^0)} \\
&= 0.
\end{aligned}$$

According to Definition 3.1, F is D-differentiable at x^0, and $\nabla F^D(x^0) = (2\tilde{x}_1^0, 2\tilde{x}_2^0, \dots, 2\tilde{x}_n^0)$.

Theorem 3.3 *Let $F: M \to ([\mathbb{R}], D_H)$ be an interval valued function, $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$. If F is D-differentiable at x_0, and for any $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n \setminus \{0\}$, the H-difference*

$$\langle y_e^{+}, \nabla F^D(x^0)\rangle -_H \langle y_e^{-}, \nabla F^D(x^0)\rangle$$

exists, then $F^D(x^0, y)$ exists and $F^D(x^0, y) = \langle y_e^{+}, \nabla F^D(x^0)\rangle -_H \langle y_e^{-}, \nabla F^D(x^0)\rangle$.

Proof For $y \in \mathbb{R}^n \setminus \{0\}$, we have $y_e = (y_{1e}, y_{2e}, \dots, y_{ne}) \in \mathbb{R}^n \setminus \{0\}$. Thus there exists $\delta > 0$ such that $x^0 + hy_e,\ x^0 - hy_e \in M$, for any $h \in (0, \delta)$ and $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \operatorname{int} M$.

Denote $x = (x_1, x_2, \dots, x_n) = x^0 + hy_e$, then we have $x \to x^0 (h \to 0^+)$,

$$hy_{ie} = x_i - x_i^0 (y_{ie} = \frac{y_i}{\|y\|}, i = 1, 2, \dots, n),\ d(x, x^0) = \sqrt{\sum_{i=1}^{n} (hy_{ie})^2} = h\|y_e\| = h.$$

Let F be D-differentiable at x^0 , $\nabla F^D(x^0) = (a_1, a_2, \dots, a_n)$ and H-difference

$$\langle y_e^+, \nabla F^D(x^0)\rangle -_H \langle y_e^-, \nabla F^D(x^0)\rangle$$

exist. Then

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{D_H(F(x^0 + hy_e), F(x^0) + h(\langle y_e^+, \nabla F^D(x^0)\rangle -_H \langle y_e^-, \nabla F^D(x^0)\rangle))}{h} \\
&= \lim_{h\to 0^+} \frac{D_H(F(x^0 + hy_e) + h\langle y_e^-, \nabla F^D(x^0)\rangle, F(x^0) + h\langle y_e^+, \nabla F^D(x^0)\rangle)}{h} \\
&= \lim_{x\to x_0} \frac{D_H(F(x) + \langle (x - x^0)^-, \nabla F^D(x^0)\rangle, F(x^0) + \langle (x - x^0)^+, \nabla F^D(x^0)\rangle)}{d(x^0, x)} \\
&= 0.
\end{aligned}$$

So $F_+^D(x^0, y)$ exists, and

$$F_+^D(x^0, y) = \langle y_e^+, \nabla F^D(x^0)\rangle -_H \langle y_e^-, \nabla F^D(x^0)\rangle.$$

Likewise we denote $x = (x_1, x_2, \dots, x_n) = x^0 - hy_e$, then $x \to x^0$ $(h \to 0^+)$,

$$hy_{ie} = x_i^0 - x_i\ (i = 1, 2, \dots, n),\ d(x, x^0) = \sqrt{\sum_{i=1}^{n} (hy_{ie})^2} = h\|y_e\| = h.$$

We can also obtain the existence of $F_-^D(x^0, y)$ and

$$F_-^D(x^0, y) = \langle y_e^+, \nabla F^D(x^0)\rangle -_H \langle y_e^-, \nabla F^D(x^0)\rangle.$$

Therefore, $F^D(x^0, y)$ exists and

$$F^D(x^0, y) = \langle y_e^+, \nabla F^D(x^0)\rangle -_H \langle y_e^-, \nabla F^D(x^0)\rangle. \ \square$$

Corollary 3.4 *Let $F : M \to ([\mathbb{R}], D_H)$ be an interval valued function, $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$. If F is D-differentiable at x_0, and $\nabla F^D(x^0) = (a_1, a_2, \dots, a_n)$, then $F^D(x^0, e_i) = a_i$ $(i = 1, 2, \dots, n)$.*

Proof Let $y = e_i$ $(i = 1, 2, \dots, n)$ in Theorem 3.3. Then

$$\langle y_e^+, \nabla F^D(x^0)\rangle -_H \langle y_e^-, \nabla F^D(x^0)\rangle = \langle e_i, \nabla F^D(x^0)\rangle -_H \langle 0, \nabla F^D(x^0)\rangle = a_i \in [\mathbb{R}].$$

Thus $F^D(x^0, e_i) = a_i$ $(i = 1, 2, \dots, n)$. $\square$

By using Corollary 3.4 and Theorem 3.3 and Example 3.2, we can obtain the following corollary.

Corollary 3.5 *Let $F : M \to ([\mathbb{R}], D_H)$ be an interval valued function, $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$. Then*

(1) *If F is H-differentiable at x^0, then F is D-differentiable at x^0, and $\nabla F^D(x^0) = \nabla F^H(x^0)$.*

(2) *If F is D-differentiable at x^0, then F is not necessarily H-differentiable at x^0.*

Theorem 3.6 *Let $F, G : M \to ([\mathbb{R}], D_H)$ be two interval valued functions. If F and G are D-differentiable at $x^0 \in \operatorname{int} M$, then λF $(\lambda > 0)$ and $F+G$ are D-differentiable at x^0, and*

$$\nabla(\lambda F)^D(x^0) = \lambda \nabla F^D(x^0),\ \nabla(F+G)^D(x^0) = \nabla F^D(x^0) + \nabla G^D(x^0).$$

Proof (1) Let F be D-differentiable at x^0. Then for any $x = (x_1, x_2, \ldots, x_n) \in M$, there exists $a = (a_1, a_2, \ldots, a_n) \in [R]^n$ such that

$$\lim_{x \to x^0} \frac{D_H(F(x) + \langle (x-x^0)^-, a \rangle, F(x^0) + \langle (x-x^0)^+, a \rangle)}{d(x, x^0)} = 0.$$

And for any $\lambda > 0$,

$$\begin{aligned}
&\lim_{x \to x^0} \frac{D_H(\lambda F(x) + \langle (x-x^0)^-, \lambda a \rangle, \lambda F(x^0) + \langle (x-x^0)^+, \lambda a \rangle)}{d(x, x^0)} \\
&= \lim_{x \to x_0} \frac{\lambda D_H(F(x) + \langle (x-x^0)^-, a \rangle, F(x^0) + \langle (x-x^0)^+, a \rangle)}{d(x^0, x)} = 0.
\end{aligned}$$

Therefore,

$$\nabla(\lambda F)^D(x^0) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n) = \lambda(a_1, a_2, \ldots, a_n) = \lambda \nabla F^D(x^0).$$

(2) Denote $\nabla F^D(x^0) = (a_1, a_2, \ldots, a_n)$, $\nabla G^D(x^0) = (b_1, b_2, \ldots, b_n)$, then

$$\lim_{x \to x_0} \frac{D_H(F(x) + \langle (x-x^0)^-, a \rangle, F(x^0) + \langle (x-x^0)^+, a \rangle)}{d(x^0, x)} = 0,$$

$$\lim_{x \to x_0} \frac{D_H(G(x) + \langle (x-x^0)^-, b \rangle, G(x^0) + \langle (x-x^0)^+, b \rangle)}{d(x^0, x)} = 0.$$

According to

$$\begin{aligned}
0 &\le \lim_{x \to x_0} \frac{D_H((F+G)(x) + \langle (x-x^0)^-, a+b \rangle, (F+G)(x^0) + \langle (x-x^0)^+, a+b \rangle)}{d(x^0, x)} \\
&\le \lim_{x \to x_0} \frac{D_H(F(x) + \langle (x-x^0)^-, a \rangle, F(x^0) + \langle (x-x^0)^+, a \rangle)}{d(x^0, x)} + \\
&\quad \lim_{x \to x_0} \frac{D_H(G(x) + \langle (x-x^0)^-, b \rangle, G(x^0) + \langle (x-x^0)^+, b \rangle)}{d(x^0, x)} = 0,
\end{aligned}$$

we can obtain that

$$\lim_{x \to x_0} \frac{D_H((F+G)(x) + \langle (x-x^0)^-, a+b \rangle, (F+G)(x^0) + \langle (x-x^0)^+, a+b \rangle)}{d(x^0, x)} = 0.$$

Thus $F+G$ is also D-differentiable at x^0, and

$$\begin{aligned}
\nabla(F+G)^D(x^0) &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \\
&= (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \\
&= \nabla F^D(x^0) + \nabla G^D(x^0). \quad \square
\end{aligned}$$

Theorem 3.7 *Let $F : M \to ([\mathbb{R}], D_H)$ be an interval valued function, $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$, $a = (a_1, a_2, \dots, a_n) \in [\mathbb{R}]^n$. Then*

$$\nabla F^D(x^0) = (a_1, a_2, \dots, a_n) \Leftrightarrow \begin{cases} \nabla \underline{F}(x^0) = \underline{\nabla F^D(x^0)} = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n) \\ \nabla \overline{F}(x^0) = \overline{\nabla F^D(x^0)} = (\overline{a}_1, \overline{a}_2, \dots, \overline{a}_n) \end{cases}.$$

Proof Let $\nabla F^D(x^0) = (a_1, a_2, \dots, a_n)$. Then

$$\lim_{x \to x^0} \frac{D_H(F(x) + \langle a, (x - x^0)^- \rangle, F(x^0) + \langle a, (x - x^0)^+ \rangle)}{d(x, x^0)} = 0$$

$$\Leftrightarrow \lim_{x \to x^0} \max \left\{ \begin{array}{l} \frac{\left|\underline{F}(x) + \sum_{i=1}^n (x_i - x_i^0)^- \underline{a}_i - \underline{F}(x^0) - \sum_{i=1}^n (x_i - x_i^0)^+ \underline{a}_i\right|}{d(x, x^0)}, \\ \frac{\left|\overline{F}(x) + \sum_{i=1}^n (x_i - x_i^0)^- \overline{a}_i - \overline{F}(x^0) - \sum_{i=1}^n (x_i - x_i^0)^+ \overline{a}_i\right|}{d(x, x^0)} \end{array} \right\} = 0$$

$$\Leftrightarrow \begin{cases} \lim_{x \to x^0} \frac{\left|\underline{F}(x) - \underline{F}(x^0) - \sum_{i=1}^n (x_i - x_i^0) \underline{a}_i\right|}{d(x, x^0)} = 0 \\ \lim_{x \to x^0} \frac{\left|\overline{F}(x) - \overline{F}(x^0) - \sum_{i=1}^n (x_i - x_i^0) \overline{a}_i\right|}{d(x, x^0)} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \underline{F}(x) - \underline{F}(x^0) - \sum_{i=1}^n (x_i - x_i^0) \underline{a}_i = o(d(x, x^0)) \\ \overline{F}(x) - \overline{F}(x^0) - \sum_{i=1}^n (x_i - x_i^0) \overline{a}_i = o(d(x, x^0)) \end{cases}$$

$$\Leftrightarrow \begin{cases} \nabla \underline{F}(x^0) = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n) \\ \nabla \overline{F}(x^0) = (\overline{a}_1, \overline{a}_2, \dots, \overline{a}_n) \end{cases}. \quad \square$$

Corollary 3.8 *Let $F : M \to ([\mathbb{R}], D_H)$ be an interval valued function. If F is D-differentiable at $x^0 \in M$, then*

$$F(x) + \langle \nabla F^D(x^0), (x - x^0)^- \rangle = F(x^0) + \langle \nabla^D F(x^0), (x - x^0)^+ \rangle + \tilde{o}(\|x - x^0\|),$$

where $\tilde{o}(\|x - x^0\|) = [o(\|x - x^0\|), o(\|x - x^0\|)]$.

Proof Let F be D-differentiable at $x^0 \in M$. According to Definition 3.1, there exists

$$a = (a_1, a_2, \dots, a_n) \in [\mathbb{R}]^n$$

such that

$$\lim_{x \to x^0} \frac{D_H(F(x) + \langle a, (x - x^0)^- \rangle, F(x^0) + \langle a, (x - x^0)^+ \rangle)}{d(x, x^0)} = 0.$$

According to the proof of Theorem 3.7, we have

$$\underline{F}(x) + \langle \underline{a}, (x - x^0)^- \rangle = \underline{F}(x^0) + \langle \underline{a}, (x - x^0)^+ \rangle + o(d(x, x^0)),$$

$$\overline{F}(x) + \langle \overline{a}, (x - x^0)^- \rangle = \overline{F}(x^0) + \langle \overline{a}, (x - x^0)^+ \rangle + o(d(x, x^0)).$$

Therefore,

$$F(x) + \langle \nabla F^D(x^0), (x - x^0)^- \rangle = F(x^0) + \langle \nabla F^D(x^0), (x - x^0)^+ \rangle + \tilde{o}(d(x, x^0)). \quad \square$$

Theorem 3.9 *Let $M \subset \mathbb{R}^n$ be an convex open set in R^n, and $F : M \to ([\mathbb{R}], D_H)$ be a D-differentiable interval valued function. Then F is convex function if and only if*

$$F(x) + \langle \nabla F^D(y), (x - y)^- \rangle \geq F(y) + \langle \nabla F^D(y), (x - y)^+ \rangle,$$

for any $x, y \in M$.

Proof Necessary. Let $F : M \to ([\mathbb{R}], D_H)$ be a D-differentiable convex interval valued function. For $x \in M$, $\nabla F^D(x) = (a_1, a_2, \ldots, a_n) \in [\mathbb{R}]^n$, according to Definition 2.5 we can obtain that $\underline{F}(x)$ and $\overline{F}(x)$ are both differentiable convex real valued functions on M. And by Theorem 3.7 we have

$$\nabla \underline{F}(x) = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n),\ \nabla \overline{F}(x) = (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n).$$

Thus by using the properties of convex real valued function, we can obtain that

$$\underline{F}(x) \geq \underline{F}(y) + \langle \nabla \underline{F}(x), x - y \rangle,\ \overline{F}(x) \geq \overline{F}(y) + \langle \nabla \overline{F}(x), x - y \rangle.$$

This implies that

$$\begin{aligned} &\underline{F}(x) + \langle \nabla \underline{F}(x), (x-y)^- \rangle \geq \underline{F}(y) + \langle \nabla \underline{F}(x), (x-y)^+ \rangle, \\ &\overline{F}(x) + \langle \nabla \overline{F}(x), (x-y)^- \rangle \geq \overline{F}(y) + \langle \nabla \overline{F}(x), (x-y)^+ \rangle. \end{aligned}$$

Therefore

$$F(x) + \langle \nabla F^D(x), (x-y)^- \rangle \geq F(y) + \langle \nabla F^D(y), (x-y)^+ \rangle.$$

Sufficiency. Let

$$F(x^2) + \langle \nabla F^D(x^1), (x^2 - x^1)^- \rangle \geq F(x^1) + \langle \nabla F^D(x^1), (x^2 - x^1)^+ \rangle,$$

for any $x^1, x^2 \in M$. And for $\lambda \in (0,1)$, taking $y = \lambda x^1 + (1-\lambda)x^2$, we know $y \in M$. Thus for $x^1, x^2, y \in M$, we have

$$F(x^1) + \langle \nabla F^D(y), (x^1 - y)^- \rangle \geq F(y) + \langle \nabla F^D(y), (x^1 - y)^+ \rangle, \tag{3.1}$$

$$F(x^2) + \langle \nabla F^D(y), (x^2 - y)^- \rangle \geq F(y) + \langle \nabla F^D(y), (x^2 - y)^+ \rangle. \tag{3.2}$$

By (3.1) and (3.2), we have

$$\underline{F}(x^1) \geq \underline{F}(y) + \langle \nabla \underline{F}(y), x^1 - y \rangle, \tag{3.3}$$

$$\overline{F}(x^1) \geq \overline{F}(y) + \langle \nabla \overline{F}(y), x^1 - y \rangle, \tag{3.4}$$

and

$$\underline{F}(x^2) \geq \underline{F}(y) + \langle \nabla \underline{F}(y), x^2 - y \rangle, \tag{3.5}$$

$$\overline{F}(x^2) \geq \overline{F}(y) + \langle \nabla \overline{F}(y), x^2 - y \rangle. \tag{3.6}$$

Hence, considering the sum of formula (3.3) multiplied by λ and formula (3.5) multiplied by $(1-\lambda)$, we have

$$\begin{aligned} \lambda \underline{F}(x^1) + (1-\lambda)\underline{F}(x^2) &\geq \underline{F}(y) + \langle \nabla \underline{F}(y), \lambda x^1 + (1-\lambda)x^2 - y \rangle \\ &= \underline{F}(y) = \underline{F}(\lambda x^1 + (1-\lambda)x^2). \end{aligned} \tag{3.7}$$

Similarly, considering the sum of formula (3.4) multiplied by λ and formula (3.6) multiplied by $1-\lambda$, we obtain

$$\lambda \overline{F}(x^1) + (1-\lambda)\overline{F}(x^2) \geq \overline{F}(\lambda x^1 + (1-\lambda)x^2). \tag{3.8}$$

According to (3.7) and (3.8),

$$\lambda F(x^1) + (1-\lambda)F(x^2) \geq F(\lambda x^1 + (1-\lambda)x^2).$$

So F is a convex interval valued function on M. $\square$

4. Optimality conditions for unconstrained interval valued programming

If $F : M \to ([\mathbb{R}], D_H)$ is an interval valued function, the following problem

$$\text{(INP)}\ \min F(x), \quad x \in M$$

is called the unconstrained interval valued programming problems. Set M is called the feasible set, and point $x \in M$ is called the feasible solution.

Since "$\leq$ " and "$<$" are both partial ordering on $[\mathbb{R}]$, we may quote some concepts of solution in multi-objective programming problems.

If $\overline{x} \in M$ and there exists no $x(\neq \overline{x}) \in M$ such that $F(x) \leq F(\overline{x})$, we call $\overline{x}$ the global optimal solution of interval valued programming problem (INP) on M. If there exists an $\varepsilon-$ neighborhood $N(\overline{x}, \varepsilon)$ around $\overline{x}$ such that there exists no $x(\neq \overline{x}) \in N(\overline{x}, \varepsilon) \cap M$ which allows $F(x) \leq F(\overline{x})$, we call $\overline{x}$ the local optimal solution of interval valued programming problem (INP) on M.

Theorem 4.1 *Let $F : M \to ([\mathbb{R}], D_H)$ be a D-differentiable interval valued function at $\overline{x}$. If there exists direction $d \in R^n$ such that*

$$\langle \nabla F^D(\overline{x}), d^+\rangle < \langle \nabla F^D(\overline{x}), d^-\rangle,$$

then there exists $\delta > 0$ such that $F(\overline{x} + \lambda d) < F(\overline{x})$ for any $\lambda \in (0, \delta)$.

Proof If F is D-differentiable at $\overline{x}$, according to Corollary 3.8, we have

$$F(\overline{x} + \lambda d) + \langle \nabla F^D(\overline{x}), (\lambda d)^-\rangle = F(\overline{x}) + \langle \nabla F^D(\overline{x}), (\lambda d)^+\rangle + \tilde{o}(\|\lambda d\|).$$

Thus we have

$$\underline{F}(\overline{x} + \lambda d) + \lambda\langle \nabla \underline{F}(\overline{x}), d^-\rangle = \underline{F}(\overline{x}) + \lambda\langle \nabla \underline{F}(\overline{x}), d^+\rangle + o(\|\lambda d\|),$$

$$\overline{F}(\overline{x} + \lambda d) + \lambda\langle \nabla \overline{F}(\overline{x}), d^-\rangle = \overline{F}(\overline{x}) + \lambda\langle \nabla \overline{F}(\overline{x}), d^+\rangle + o(\|\lambda d\|),$$

i.e.,

$$\underline{F}(\overline{x} + \lambda d) = \underline{F}(\overline{x}) + \lambda[\langle \nabla \underline{F}(\overline{x}), d\rangle + \frac{o(\|\lambda d\|)}{\lambda}], \tag{4.1}$$

$$\overline{F}(\overline{x} + \lambda d) = \overline{F}(\overline{x}) + \lambda[\langle \nabla \overline{F}(\overline{x}), d\rangle + \frac{o(\|\lambda d\|)}{\lambda}]. \tag{4.2}$$

According to $\langle \nabla F^D(\overline{x}), d^+\rangle < \langle \nabla F^D(\overline{x}), d^-\rangle$, we have

$$\langle \nabla \underline{F}(\overline{x}), d^+\rangle - \langle \nabla \underline{F}(\overline{x}), d^-\rangle = \langle \nabla \underline{F}(\overline{x}), d^+ - d^-\rangle = \langle \nabla \underline{F}(\overline{x}), d\rangle < 0,$$

$$\langle \nabla \overline{F}(\overline{x}), d^+\rangle - \langle \nabla \overline{F}(\overline{x}), d^-\rangle = \langle \nabla \overline{F}(\overline{x}), d^+ - d^-\rangle = \langle \nabla \overline{F}(\overline{x}), d\rangle < 0.$$

And by $\lim_{\lambda\to 0}\frac{\|\lambda d\|}{\lambda}=0$, we can obtain that there exists $\delta>0$ such that

$$\lambda[\langle \nabla \underline{F^D(\overline{x})}, d\rangle + \frac{o(\|\lambda d\|)}{\lambda}] < 0,\ \lambda[\langle \nabla \overline{F^D(\overline{x})}, d\rangle + \frac{o(\|\lambda d\|)}{\lambda}] < 0$$

for any $\lambda\in(0,\delta)$. Thus by (4.1) and (4.2) we have

$$\underline{F}(\overline{x}+\lambda d) < \underline{F}(\overline{x}),\ \overline{F}(\overline{x}+\lambda d) < \overline{F}(\overline{x}).$$

So there exists $\delta>0$ such that $F(\overline{x}+\lambda d) < F(\overline{x})$ for any $\lambda\in(0,\delta)$. □

Theorem 4.2 *Let $F: M\to([\mathbb{R}], D_H)$ be a D-differentiable convex interval valued function, and $\overline{x}\in M$. If $\nabla F^D(\overline{x})=0$, then $\overline{x}$ is the global optimal solution.*

Proof Let $F: M\to([\mathbb{R}], D_H)$ be a D-differentiable interval valued function and $\nabla F^D(\bar{x})=0$. Then for any $x\in M$, we have

$$\langle \nabla F^D(\overline{x}), (x-\overline{x})^-\rangle = \langle \nabla F^D(\overline{x}), (x-\overline{x})^+\rangle = 0.$$

According to Theorem 3.9 we have

$$F(x) = F(x) + \langle \nabla F^D(\overline{x}), (x-\overline{x})^-\rangle \geq F(\overline{x}) + \langle \nabla F^D(\overline{x}), (x-\overline{x})^+\rangle = F(\overline{x}).$$

So $\overline{x}$ is the global optimal solution. □

Definition 4.3 *Let $F: M\to([\mathbb{R}], D_H)$ be a D-differentiable interval valued function, and $d\in\mathbb{R}^n$ be a nonzero vector. We say that d is the descent direction of F at $\overline{x}$ if there exists $\delta>0$ such that $F(\overline{x}+\lambda d)<F(\overline{x})$ for $\lambda\in(0,\delta)$.*

According to Theorem 4.1, d is the descent direction of F at x if F is D-differentiable and $\langle\nabla F^D(x), d^+\rangle < \langle\nabla F^D(x), d^-\rangle$. And the set of all descent directions of F at x is denoted as

$$M_F = \left\{ d |\ \langle\nabla F^D(x), d^+\rangle < \langle\nabla F^D(x), d^-\rangle, d\in\mathbb{R}^n, d\neq 0\right\}. \tag{4.3}$$

Definition 4.4 *Let $M\subset\mathbb{R}^n$ be a closed set and $d\in\mathbb{R}^n$ be a nonzero vector, $\overline{x}\in M$. Then d is the feasible direction of M at $\overline{x}$ if there exists $\delta>0$ such that $\overline{x}+\lambda d\in M$ for any $\lambda\in(0,\delta)$.*

Set of all the feasible directions of M at $\overline{x}$ is denoted as

$$D_M = \{ d |\ d\neq 0, \exists\delta>0, \forall\lambda\in(0,\delta), \overline{x}+\lambda d\in M\}. \tag{4.4}$$

We call it the cone of feasible direction of M at $\overline{x}$.

Theorem 4.5 *Let F be a D-differentiable interval valued function at $\overline{x}$ in an interval valued programming problem (INP). If $\overline{x}$ is local optimal solution, then $M_F\cap D_M=\emptyset$.*

Proof Suppose that there exists nonzero vector $d\in M_F\cap D_M$, then $d\in M_F$ and $d\in D_M$. According to (4.3) we have

$$\langle\nabla F^D(x), d^+\rangle < \langle\nabla F^D(x), d^-\rangle.$$

So according to Theorem 4.1, there exists $\delta>0$ such that

$$F(\overline{x}+\lambda d) < F(\overline{x}) \text{ for any } \lambda\in(0,\delta_1). \tag{4.5}$$

On the other hand, according to (4.4), there exists $\delta_2 > 0$ such that

$$\overline{x} + \lambda d \in M \texttt{ for any } \lambda \in (0, \delta_2). \tag{4.6}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. When $\lambda \in (0, \delta)$, both (4.5) and (4.6) are established, which contradicts that $\overline{x}$ is the local optimal solution. So $M_F \cap D_M = \emptyset$. □

5. Optimality condition for constrained interval valued programming

Let $F : M \to ([\mathbb{R}], D_H)$ be an interval valued function, $G_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ be real valued function. Then

$$\text{(MINP)} \begin{cases} \min F(x) \\ G_i(x) \le 0, \quad i = 1, 2, \ldots, m \end{cases}$$

is called the constrained interval valued programming problem. Set

$$M = \{\, x|\, G_i(x) \le 0, \quad i = 1, 2, \ldots, m\}$$

is called the feasible set or the feasible field. Point $x \in M$ is called the feasible solution.

The constrained conditions which satisfy $G_i(\overline{x}) < 0$ is called the inactive constraint at $\overline{x}$. On the other hand, those which satisfy $G_i(\overline{x}) = 0$ is called the active constraint at $\overline{x}$. Let $I = \{\, i|\, G_i(\overline{x}) = 0\}$. Then when G_i is differentiable real valued function,

$$G_I = \{d | \langle G_i(\overline{x}), d\rangle < 0, \quad i \in I\},$$

which can take place of the cone of directions D_M in Theorem 4.5.

Theorem 5.1 *Suppose that $\overline{x} \in M$, F is D-differentiable, G_i $(i \in I)$ is differentiable at $\overline{x}$, and G_i $(i \notin I)$ is continuous at $\overline{x}$. If $\overline{x}$ is the optimal solution of interval valued programming problem (MINP), then $M_F \cap G_I = \emptyset$.*

Proof According to Theorem 4.5, $M_F \cap D_M = \emptyset$ at $\overline{x}$.

Next we prove that $G_I \subset D_M$. Let the direction $d \in G_I$. Then we have

$$\langle \nabla G_i(\overline{x}), d\rangle < 0. \tag{5.1}$$

Take $\underline{H}_i(\overline{x}) = G_i(\overline{x})$, $\overline{H}_i(\overline{x}) = G_i(\overline{x})$. Then interval valued function

$$H_i : M \to ([R], D_H)$$

is D-differentiable at $\overline{x}$, and

$$\nabla \underline{H}_i(\overline{x}) = \nabla \overline{H}_i(\overline{x}) = \nabla G_i(\overline{x}).$$

Thus by (4.7) we have

$$\begin{aligned} &\langle \underline{\nabla H_i(\overline{x})}, d^+\rangle - \langle \underline{\nabla H_i(\overline{x})}, d^-\rangle = \langle \nabla \underline{H}_i(\overline{x}), d^+\rangle - \langle \nabla \underline{H}_i(\overline{x}), d^-\rangle \\ &\quad = \langle \nabla G_i(\overline{x}), d^+\rangle - \langle \nabla G_i(\overline{x}), d^-\rangle = \langle \nabla G_i(\overline{x}), d^+ - d^-\rangle = \langle \nabla G_i(\overline{x}), d\rangle > 0. \end{aligned}$$

That is $\langle \underline{\nabla H_i(\overline{x})}, d^+\rangle > \langle \underline{\nabla H_i(\overline{x})}, d^-\rangle$.

Likewise we can obtain $\langle \overline{\nabla H_i(\overline{x})}, d^+\rangle > \langle \overline{\nabla H_i(\overline{x})}, d^-\rangle$. So $\langle \nabla H_i^D(\overline{x}), d^+\rangle > \langle \nabla H_i^D(\overline{x}), d^-\rangle$.

On the other hand, by Theorem 4.1, there exists $\delta_1 > 0$ such that

$$H_i(\overline{x} + \lambda d) < H_i(\overline{x}),\ i \in I \text{ for any } \lambda \in (0, \delta_1).$$

So $G_i(\overline{x} + \lambda d) < G_i(\overline{x}) = 0$ $(i \in I)$. Since $G_i(\overline{x}) < 0$ when $i \notin I$, by the continuity of G_i $(i \notin I)$ at $\overline{x}$, there exists $\delta_2 > 0$ such that $G_i(\overline{x} + \lambda d) < 0$ $(i = 1, 2, \dots, n)$ for $\lambda \in (0, \delta_2)$.

Take $\delta = \min\{\delta_1, \delta_2\}$. Then $G_i(\overline{x} + \lambda d) < 0$ $(i = 1, 2, \dots, m)$ for $\lambda \in (0, \delta)$. That is $\overline{x} + \lambda d \in M$. According to Definition 4.4 we have $d \in D_M$. So $G_I \subset D_M$. Thus $M_F \cap G_I = \emptyset$. □

Theorem 5.2 *Let $\overline{x} \in M$, F be D-differentiable, G_i $(i \in I)$ be differentiable at $\overline{x}$, and G_i $(i \notin I)$ be continuous at $\overline{x}$. If $\overline{x}$ is local optimal solution of interval valued programming problem (MINP), then there exists non-negative real number families $\underline{\omega}_0, \underline{\omega}_i, \overline{\omega}_0, \overline{\omega}_i, i \in I$ which are not all zero such that*

$$\underline{\omega}_0 \nabla \underline{F}(\overline{x}) + \sum_{i \in I} \underline{\omega}_i \nabla G_i(\overline{x}) = 0,\ \overline{\omega}_0 \nabla \overline{F}(\overline{x}) + \sum_{i \in I} \overline{\omega}_i \nabla G_i(\overline{x}) = 0.$$

Proof Let $\overline{x}$ be a local optimal solution of (MINP). Then according to Theorem 5.1 we have $M_F \cap G_I = \emptyset$, i.e., the following inequality systems

$$\begin{cases} \langle \nabla G_i(\overline{x}), d \rangle < 0 \\ \langle \nabla \underline{F}(\overline{x}), d \rangle < 0 \end{cases} \text{ and } \begin{cases} \langle \nabla G_i(\overline{x}), d \rangle < 0 \\ \langle \nabla \overline{F}(\overline{x}), d \rangle < 0 \end{cases}$$

are both unsolvable. According to Lemma 2.1 we can obtain that there exists nonzero vector

$$\underline{\omega} = (\underline{\omega}_0, \underline{\omega}_i, i \in I) \geq 0,\ \overline{\omega} = (\overline{\omega}_0, \overline{\omega}_i, i \in I) \geq 0$$

such that

$$\underline{\omega}_0 \nabla \underline{F}(\overline{x}) + \sum_{i \in I} \underline{\omega}_i \nabla G_i(\overline{x}) = 0,\ \overline{\omega}_0 \nabla \overline{F}(\overline{x}) + \sum_{i \in I} \overline{\omega}_i \nabla G_i(\overline{x}) = 0. \ \square$$

Theorem 5.3 (KKT condition) *Let $\overline{x} \in M$, F be D-differentiable, G_i $(i \in I)$ be differentiable at $\overline{x}$, G_i $(i \notin I)$ be continuous at $\overline{x}$, and $\{\nabla G_i(\overline{x}) | \, i \in I\}$ be linearly independent. If $\overline{x}$ is a local optimal solution of (MINP), then there exists two non-negative arrays $\underline{\omega}_i$ $(i \in I)$ and $\overline{\omega}_i$ $(i \in I)$ such that*

$$\nabla \underline{F}(\overline{x}) + \sum_{i \in I} \underline{\omega}_i \nabla G_i(\overline{x}) = 0,\ \nabla \overline{F}(\overline{x}) + \sum_{i \in I} \overline{\omega}_i \nabla G_i(\overline{x}) = 0.$$

Proof Let $\overline{x}$ be a local optimal solution of (MINP). Then according to Theorem 5.2 we can obtain that there exists two different non-negative real number families $\underline{\omega}_0, \underline{\omega'}_i$ $(i \in I)$ and $\overline{\omega}_0, \overline{\omega}'_i$ $(i \in I)$ such that

$$\underline{\omega}_0 \nabla \underline{F}(\overline{x}) + \sum_{i \in I} \underline{\omega'}_i \nabla G_i(\overline{x}) = 0,\ \overline{\omega}_0 \nabla \overline{F}(\overline{x}) + \sum_{i \in I} \overline{\omega}'_i \nabla G_i(\overline{x}) = 0.$$

Considering $\{\nabla G_i(\overline{x}) | \, i \in I\}$ is linearly independent, we know $\underline{\omega}_0 \neq 0$ and $\overline{\omega}_0 \neq 0$ (otherwise, $\{\nabla G_i(\overline{x}) | \, i \in I\}$ would be linearly dependent because $\underline{\omega'}_i$ $(i \in I)$ and $\overline{\omega}'_i$ $(i \in I)$ are not all zero). Therefore, take

$$\underline{\omega}_i = \frac{\underline{\omega'}_i}{\underline{\omega}_0},\ i \in I,\ \ \overline{\omega}_i = \frac{\overline{\omega}'_i}{\overline{\omega}_0},\ i \in I.$$

Then $\underline{\omega}_i$ $(i \in I)$ and $\overline{\omega}_i$ $(i \in I)$ are two non-negative real arrays which allow

$$\nabla \underline{F}(\overline{x}) + \sum_{i \in I} \underline{\omega}_i \nabla G_i(\overline{x}) = 0, \ \nabla \overline{F}(\overline{x}) + \sum_{i \in I} \overline{\omega}_i \nabla G_i(\overline{x}) = 0. \ \square$$

Note 5.4 In Theorem 5.3, if G_i $(i \notin I)$ is differentiable at $\overline{x}$, we can obtain the following KKT optimal conditions

$$\begin{cases} \nabla \underline{F}(\overline{x}) - \sum_{i=1}^m \underline{\omega}_i \nabla G_i(\overline{x}) = 0, \\ \nabla \overline{F}(\overline{x}) - \sum_{i=1}^m \overline{\omega}_i \nabla G_i(\overline{x}) = 0, \\ \underline{\omega}_i G_i(\overline{x}) = 0, \\ \overline{\omega}_i G_i(\overline{x}) = 0, \\ \underline{\omega}_i \geq 0, \quad i = 1, 2, \ldots, m, \\ \overline{\omega}_i \geq 0, \quad i = 1, 2, \ldots, m. \end{cases} \tag{5.2}$$

Example 5.5 We consider the following interval valued programming problem:

$$\begin{cases} \min F(x_1, x_2) = a^2 + b^2, \\ x_1 + x_2 \geq 4, \\ x_1 \geq 1, \ x_2 \geq 1, \end{cases}$$

where $a = [x_1 - 1, x_1 + 1]$, $b = [x_2 - 1, x_2 + 1]$ are interval numbers.

Then by using the addition and multiplication of interval numbers, we can obtain that

$$\begin{aligned} & F(x_1, x_2) = [(x_1 - 1)^2 + (x_2 - 1)^2, (x_1 + 1)^2 + (x_2 + 1)^2], \\ & G_1(x_1, x_2) = 4 - x_1 - x_2 \leq 0, \\ & G_2(x_1, x_2) = 1 - x_1 \leq 0, \\ & G_2(x_1, x_2) = 1 - x_2 \leq 0. \end{aligned}$$

So according to Theorem 3.7 we have

$$\begin{aligned} & \nabla F^D(x_1, x_2) = ([2(x_1 - 1), 2(x_1 + 1)], [2(x_2 - 1), 2(x_2 + 1)]), \\ & \nabla G_1(x_1, x_2) = (-1, -1), \\ & \nabla G_2(x_1, x_2) = (-1, 0), \\ & \nabla G_3(x_1, x_2) = (0, -1). \end{aligned}$$

By (5.2), we know

$$\begin{cases} (2(x_1 - 1), 2(x_2 - 1)) + \underline{\omega}_1(-1, -1) + \underline{\omega}_2(-1, 0) + \underline{\omega}_3(0, -1) = 0, \\ (2(x_1 + 1), 2(x_2 + 1)) + \overline{\omega}_1(-1, -1) + \overline{\omega}_2(-1, 0) + \overline{\omega}_3(0, -1) = 0, \\ \underline{\omega}_1(x_1 + x_2 - 4) = \overline{\omega}_1(x_1 + x_2 - 4) = 0, \\ \underline{\omega}_2(x_1 - 1) = \overline{\omega}_2(x_1 - 1) = 0, \\ \underline{\omega}_3(x_2 - 1) = \overline{\omega}_3(x_2 - 1) = 0, \\ \underline{\omega}_i \geq 0, \quad i = 1, 2, 3, \\ \overline{\omega}_i \geq 0, \quad i = 1, 2, 3, \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x_1 - 2 - \underline{\omega}_1 - \underline{\omega}_2 = 0, \\ 2x_2 - 2 - \underline{\omega}_1 - \underline{\omega}_3 = 0, \\ 2x_1 + 2 - \overline{\omega}_1 - \overline{\omega}_2 = 0, \\ 2x_1 + 2 - \overline{\omega}_1 - \overline{\omega}_3 = 0, \\ \underline{\omega}_1(x_1 + x_2 - 4) = \overline{\omega}_1(x_1 + x_2 - 4) = 0, \\ \underline{\omega}_2(x_1 - 1) = \overline{\omega}_2(x_1 - 1) = 0, \\ \underline{\omega}_3(x_2 - 1) = \overline{\omega}_3(x_2 - 1) = 0, \\ \underline{\omega}_i \geq 0, \quad i = 1, 2, 3, \\ \overline{\omega}_i \geq 0, \quad i = 1, 2, 3. \end{cases}$$

After some algebraic calculations, we can obtain that

(1) When $x_1 = 1$, $\underline{\omega}_1 = \underline{\omega}_2 = 0$, $\underline{\omega}_3 = 2(x_2 - 1) \geq 0$; $\overline{\omega}_1 = \overline{\omega}_2 = 0$, $\overline{\omega}_3 = 2(x_2 + 1) \geq 0$, $x_2 \geq 3$;

(2) When $x_2 = 1$, $\underline{\omega}_1 = \underline{\omega}_3 = 0$, $\underline{\omega}_2 = 2(x_1 - 1) \geq 0$; $\overline{\omega}_1 = \overline{\omega}_3 = 0$, $\overline{\omega}_2 = 2(x_1 + 1) \geq 0$, $x_1 \geq 3$;

(3) When $x_1 \neq 1$ and $x_2 \neq 1$,

$$\underline{\omega}_2 = \underline{\omega}_3 = 0,\ \underline{\omega}_1 = 2(x_1 - 1) \geq 0;\ \overline{\omega}_2 = \overline{\omega}_3 = 0,\ \overline{\omega}_1 = 2(x_1 + 1) \geq 0,\ x_1 = x_2, x_2 \geq 3.$$

i.e., the set of points which satisfy KKT conditions is three half-lines

$$x_1 = 1,\ x_2 \geq 3;\ x_2 = 1,\ x_1 \geq 3;\ x_1 = x_2,\ x_2 \geq 3.$$

Theorem 5.6 *Let G_i $(i = 1, 2, \ldots, m)$ be a convex real valued function and differentiable on M, F be a convex interval valued function and D-differentiable on M. If $\overline{x}$ satisfies the KKT conditions of (MINP), then $\overline{x}$ is the global optimal solution.*

Proof Let F be convex interval valued function and D-differentiable at $\overline{x} \in M$. Then according to Theorem 3.9, for $x \in M$ we have

$$F(x) + \langle \nabla F^D(\overline{x}), (x - \overline{x})^- \rangle \geq F(\overline{x}) + \langle \nabla F^D(\overline{x}), (x - \overline{x})^+ \rangle.$$

Therefore,

$$\underline{F}(x) + \langle \nabla \underline{F}(\overline{x}), (x - \overline{x})^- \rangle \geq \underline{F}(\overline{x}) + \langle \nabla \underline{F}(\overline{x}), (x - \overline{x})^+ \rangle,$$

$$\overline{F}(x) + \langle \nabla \overline{F}(\overline{x}), (x - \overline{x})^- \rangle \geq \overline{F}(\overline{x}) + \langle \nabla \overline{F}(\overline{x}), (x - \overline{x})^+ \rangle.$$

So

$$\underline{F}(x) \geq \underline{F}(\overline{x}) + \langle \nabla \underline{F}(\overline{x}), x - \overline{x} \rangle, \tag{5.3}$$

$$\overline{F}(x) \geq \overline{F}(\overline{x}) + \langle \nabla \overline{F}(\overline{x}), x - \overline{x} \rangle. \tag{5.4}$$

Because $\overline{x}$ satisfies the KKT conditions, i.e., there exist two non-negative real valued arrays $\underline{\omega}_i$ $(i \in I)$ and $\overline{\omega}_i$ $(i \in I)$ such that

$$\nabla \underline{F}(\overline{x}) + \sum_{i \in I} \underline{\omega}_i \nabla G_i(\overline{x}) = 0, \tag{5.5}$$

$$\nabla \overline{F}(\overline{x}) + \sum_{i \in I} \overline{\omega}_i \nabla G_i(\overline{x}) = 0. \tag{5.6}$$

Using (5.3) and (5.5) gives

$$\underline{F}(x) \geq \underline{F}(\overline{x}) - \sum_{i \in I} \underline{\omega}_i \langle \nabla G_i(\overline{x}), x - \overline{x} \rangle. \tag{5.7}$$

By (5.4) and (5.6), we have

$$\overline{F}(x) \geq \overline{F}(\overline{x}) - \sum_{i \in I} \overline{\omega}_i \langle \nabla G_i(\overline{x}), x - \overline{x} \rangle. \tag{5.8}$$

Because G_i $(i = 1, 2, \ldots, m)$ is convex real valued function, for $i \in I$, we have

$$G_i(x) \geq G_i(\overline{x}) + \langle \nabla G_i(\overline{x}), x - \overline{x} \rangle.$$

Therefore,

$$\langle \nabla G_i(\overline{x}), x - \overline{x} \rangle \leq G_i(x) - G_i(\overline{x}), \quad i \in I.$$

Thus, by $G_i(\overline{x}) = 0$, $G_i(x) \leq 0$, we can obtain

$$\langle \nabla G_i(\overline{x}), x - \overline{x} \rangle \leq 0, \quad i \in I.$$

By (5.7) and (5.8), we have

$$\underline{F}(x) \geq \underline{F}(\overline{x}), \ \overline{F}(x) \geq \overline{F}(\overline{x}).$$

So $F(x) \geq F(\overline{x})$, i.e., $\overline{x}$ is global optimal solution of (MINP). $\square$

6. Conclusion

The concepts of the differentiability of interval valued function include H-derivative (H-partial derivative), Hg-derivative (Hg-partial derivative), H-directional derivative (Hg-directional partial derivative), D-directional derivative and so on. Because the H-difference does not always exist, the generalization of H-derivative (H-partial derivative) using Hg-difference is imported which are called Hg-derivative (Hg-partial derivative) and Hg-directional derivative.

In this paper, we introduce the concepts of D-differentiability and its gradient by using the method of total differential of real valued function. By discussing the relationship between D-directional differential and D-differential, we point out that the gradient under the condition of H-differential is equal to the gradient under the condition of D-differential, but its reverse is not always true. The optimal condition of unconstrained interval valued programming, KKT condition and the relevant example of interval valued programming whose constrained condition is real valued function are given under the condition of D-differential. These results are more general than similar results under the condition of H-differential (which are only discussed aiming at convex interval valued programming).

In this paper, we take no account of the concept of H-difference or Hg-difference, which provides a new method for further research on interval valued programming. At the same time, some conclusions in this paper build a good foundation for the research of KKT condition of interval valued programming whose constrained condition is interval valued function and the establishment of sub-differential theory of interval valued function under the condition of D-differential.

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