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On the Strong Law of Large Numbers for END Sequences

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Abstract We investigate the strong law of large numbers (SLLN) for a large class of mean based on the Extended Negatively Dependent (END) sequences. The sufficient conditions are obtained for the mean of SLLN in this paper. As an important application, the SLLN of Marcinkiewicz mean and logarithmic mean are presented immediately. In addition, we do some simulations for the mean of SLLN based on END random variables.

Keywords strong law of large numbers; Marcinkiewicz mean law; logarithmic mean law; END sequences

MR(2010) Subject Classification 60F15

1. Introduction

It is known that the strong law of large number (SLLN) is an important research in the limit theory. The SLLN of Kolmogorov mean, Marcinkiewicz mean and logarithmic mean can be found in many references [1–6]. Jajte [7] studied the SLLN for large class of mean for independent and identically distributed (i.i.d.) random variables.

Theorem 1.1 ([7]) *Let $g(\cdot)$ be a positive, increasing function and $h(\cdot)$ a positive function such that $\phi(y)=g(y)h(y)$ satisfies the following Assumption A:*

(i) For some $d\ge 0$, $\phi(\cdot)$ is strictly increasing on $[d,\infty)$ with range $[0,\infty)$.

(ii) There exist a positive constant C and a positive integer k_0 such that $\phi(y+1)/\phi(y)\le C$, $y\ge k_0$.

(iii) There exist positive constants a and b such that

$$\phi^2(s)\int_s^{\infty}\frac{1}{\phi^2(x)}\mathrm{d}x\le as+b,\quad s>d.$$

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Let $\{X, X_n, n \geq 1\}$ be an i.i.d. sequence of random variables. Then

$$\frac{1}{g(n)} \sum_{i=1}^{n} \frac{X_i - m_i}{h(i)} \to 0, \quad a.s. \tag{1.1}$$

if and only if

$$E(\phi^{-1}(|X|)) < \infty, \tag{1.2}$$

where $m_i = E[X_i I(|X_i| \leq \phi(i))]$ and $\phi^{-1}(\cdot)$ is the inverse of function $\phi(\cdot)$.

Remark 1.2 If $g(x) = x^{1/p}$ $(0 < p < 2)$ and $h(x) = 1$, then $\phi(x) = x^{1/p}$, $x \in [0, \infty)$, satisfies Assumption A. So one can get the SLLN of Marcinkiewicz mean. In addition, denote $\log x = \log(\max(x, e))$. If $h(x) = x$ and $g(x) = \log x$, then $\phi(x) = x \log x$, $x \in [e, \infty)$, also satisfies Assumption A. Thus, one can get SLLN of logarithmic mean. For the details, see Jajte [7].

Many researchers extended the result of Jajte [7] from independent case to dependent cases. For example, Jing and Liang [8] and Wang [9] studied Theorem 1.1 under the Negatively Associated (NA) case, Meng and Lin [10] considered the $\tilde{\rho}$-mixing case and Sung [11] considered the three series convergence of $\sum_{n=1}^{\infty} P(|X| > \phi(n))$, $\sum_{n=1}^{\infty} \frac{E[|X| I(|X| > \phi(n))]}{\phi(n)}$ and $\sum_{n=1}^{\infty} \frac{E[X^2 I(|X| \leq \phi(n))]}{\phi^2(n)}$, and obtained some equivalence conditions for the convergence of (1.1).

In this paper, we would like to consider the Theorem 1.1 under the case of Extended Negatively Dependent (END). In the following, let us recall the concept of END random variables.

Definition 1.3 *The random variables $\{X_n, n \geq 1\}$ are said to be END if there exists a positive constant M such that both*

$$P(X_i > x_i, i = 1, 2, \ldots, n) \leq M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_i \leq x_i, i = 1, 2, \ldots, n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i)$$

hold for each $n \geq 1$ and all real numbers $x_1, x_2, \ldots, x_n$.

The concept is introduced by Liu [12]. If the dominating coefficient $M = 1$, then END random variables reduce to Negatively Orthant Dependent (NOD), which contain NA random variables and Negatively Superadditive Dependent (NSD) random variables. It is pointed out that END sequence is a pairwise Negative Quadrant Dependent (NQD) sequence. For the definitions of pairwise NQD, NOD, NA and NSD, one can refer to Lehmann [13], Joag-Dev and Proschan [14], Hu [15], Chen [16], Zhao et al. [17], etc. There are many researchers who pay attention on the studying of END random variables. For the research of END sequences, Liu [18] studied the heavy tails; Chen et al. [19] obtained the SLLN; Shen [20] obtained some moment inequalities; Wang et al. [21], Hu et al. [22] and Shen et al. [23] investigated the complete convergence; Deng et al. [24] studied the Hajek-Renyi-type inequality and SLLN, etc. Under the END errors, Wang et al. [25] and Yang et al. [26] investigated the nonparametric regression model; Yang et al. [27] obtained some large deviation results of nonlinear regression models, etc.

Inspired by Jajte [7], Jing and Liang [8], Meng and Lin [10], Sung [11], etc, we consider the END sequences and extend Theorem 1.1 to this case, and obtain the sufficient conditions for the SLLN of mean. As an important application, the SLLN of Marcinkiewicz mean and logarithmic mean are obtained. Our results are presented in Section 2 and the simulations for the SLLN are illustrated in Section 3. Later, some lemmas and the proofs of main results are presented in Section 4. Throughout the paper, we denote $C, C_1, \ldots$ to be some positive constants not dependent on n.

2. Main results

Similar to Theorem 1.1, we consider the SLLN of mean for END sequences.

Theorem 2.1 *Let $\{X, X_n, n \geq 1\}$ be an identically distributed END sequence with $E|X| < \infty$. Let $\phi^{-1}(\cdot)$ be the inverse of function $\phi(\cdot)$ and Assumption A hold. If*

$$E(\phi^{-1}(|X|)) < \infty, \tag{2.1}$$

then

$$\frac{1}{g(n)\log n}\sum_{i=1}^{n}\frac{X_i - m_i}{h(i)} \to 0, \quad \text{a.s.} \tag{2.2}$$

where $m_i = E[X_i I(|X_i| \leq \phi(i))]$, $1 \leq i \leq n$.

As an important application of Theorem 2.1, we have the following SLLN of Marcinkiewicz mean for END sequences.

Corollary 2.2 *Let $\{X, X_n, n \geq 1\}$ be an identically distributed END sequence. Then we have the following results:*

(i) If $0 < p < 1$ and $E|X|^p < \infty$, then

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}(X_i - a) \to 0, \quad \text{a.s.}, \tag{2.3}$$

where a is an arbitrary real number.

(ii) If $1 \leq p < 2$, $E|X|^p < \infty$ and $EX = 0$, then

$$\frac{1}{n^{1/p}\log n}\sum_{i=1}^{n}X_i \to 0, \quad \text{a.s.} \tag{2.4}$$

Similarly, by Theorem 2.1, we have the SLLN of logarithmic mean for END sequences.

Corollary 2.3 *Let $\{X, X_n, n \geq 1\}$ be an identically distributed END sequence with $EX = 0$. Then*

$$\frac{1}{\log^2 n}\sum_{i=1}^{n}\frac{X_i}{i} \to 0, \quad \text{a.s.} \tag{2.5}$$

Remark 2.4 On the one hand, the proof of Theorem 2.1 is based on the convergence theorem for END sequences [24, Corollary 1.1]. If the sequences of $\{X, X_n, n \geq 1\}$ are the ones of independent, NA, or $\tilde{\rho}$-mixing, then the factor $\log n$ in (2.2) and (2.4) can be removed, and the factor $\log^2 n$ in (2.5) can be improved by $\log n$. For the details, see the theorem and remark of Jajte [7], Theorem 2.3 and Remark 2.2 of Jing and Liang [8], Theorem 2.1 and Corollary 2.1 of Meng and Lin [10] and Theorem 3.1 and Corollary 3.2 of Wang [9]. On the other hand, we study the sufficient conditions for the SLLN of mean based on the END sequences in this paper. It is interesting to study its necessary conditions in future. Let $\{X, X_n, n \geq 1\}$ be an identically distributed END sequence with $E(|X|^p \log^2 |X|) < \infty$ for some $0 < p < 2$. In addition, it is assumed that $EX = 0$ if $1 \leq p < 2$. Then Deng et al. [24] obtained the following SLLN of Marcinkiewicz

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} X_i \to 0, \quad \text{a.s,} \tag{2.6}$$

(see Theorem 3.2 of Deng et al. [24] for details). Comparing with our Corollary 2.2, (2.4) has the factor $\log n$ while (2.6) does not have the factor $\log n$, but our moment condition $E|X|^p < \infty$ is weaker than the one of $E(|X|^p \log^2 |X|) < \infty$. Last, some simulations for (2.4) and (2.5) are presented in Section 3.

3. Simulation

In this section, we will do some simulations for the SLLN of END sequences. For example, we use MATLAB software to do the simulation for the convergence of (2.4) and (2.5) in Corollaries 2.2 and 2.3, respectively. For $n \geq 2$, let $(X_1, X_2, \ldots, X_n)$ be a normal random vector such as $(X_1, X_2, \ldots, X_n) \sim N_n(0, \Sigma)$, where 0 is zero vector,

$$\Sigma = \begin{bmatrix} 1 & -\rho & -\rho^2 & 0 & 0 & \ldots & 0 \\ -\rho & 1 & -\rho & -\rho^2 & 0 & \ldots & 0 \\ -\rho^2 & -\rho & 1 & -\rho & -\rho^2 & \ldots & 0 \\ 0 & -\rho^2 & -\rho & 1 & -\rho & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & -\rho^2 & -\rho & 1 & -\rho & -\rho^2 \\ 0 & \ldots & 0 & -\rho^2 & -\rho & 1 & -\rho \\ 0 & \ldots & 0 & 0 & -\rho^2 & -\rho & 1 \end{bmatrix}_{n \times n},$$

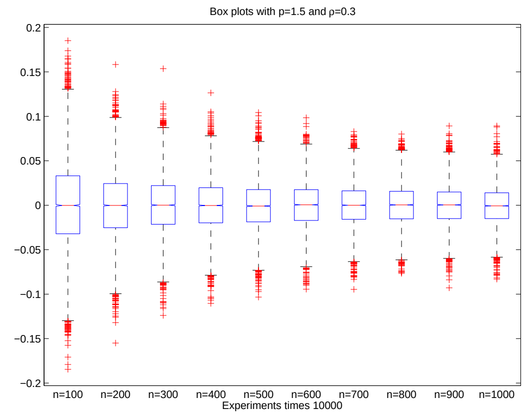
and $\rho \in [0, 1)$. By Joag-Dev and Proschan [14], it can be seen that $(X_1, X_2, \ldots, X_n)$ is an NA vector, which implies that it is also an END vector with dominating coefficient $M = 1$. For $p = 1.5$ $\rho = 0.3$ and $n = 100, 200, \ldots, 1000$, we draw the Box plot for

$$\frac{1}{n^{1/p} \log n} \sum_{i=1}^{n} X_i \tag{3.1}$$

by 10000 replications and obtain Figure 1. Similarly, for $\rho = 0.2$ and $n = 100, 200, \ldots, 1000$, we draw the Box plot for

$$\frac{1}{\log^2 n} \sum_{i=1}^{n} \frac{X_i}{i} \tag{3.2}$$

by 10000 times and obtain Figure 2. The pictures are presented as follows



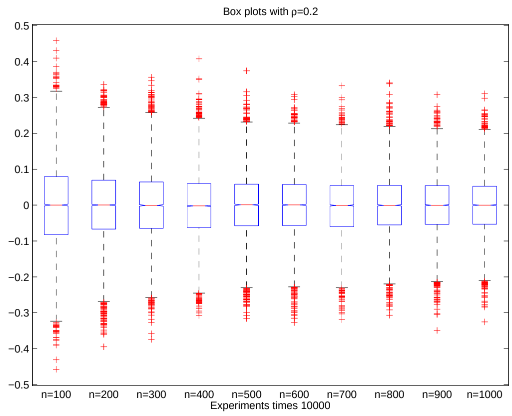


Figure 1 The SLLN of Marcinkiewicz mean in (3.1)   Figure 2 The SLLN of logarithmic mean in (3.2)

Obviously, by Figures 1 and 2, they are agreed with (2.4) in Corollary 2.2 and (2.5) in Corollary 2.3, respectively.

4. Some lemmas and the proofs of main results

Before the proofs of main results, we give some important lemmas.

Lemma 4.1 ([18]) *Let random variables $\{X_n, n \geq 1\}$ be a sequence of END random variables. If $\{f_n, n \geq 1\}$ is a sequence of all nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is also a sequence of END random variables.*

Lemma 4.2 ([24]) *Let $\{X_n, n \geq 1\}$ be a sequence of END sequence. If*

$$\sum_{n=1}^{\infty} \log^2 n \mathrm{Var}(X_n) < \infty,$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges almost surely.

Lemma 4.3 ([16]) *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD sequence with $\mathrm{Var}(X_n) < \infty$ for all $n \geq 1$. Let $\{a_n\}$ be a nondecreasing and positive sequence of real numbers with $a_n \to \infty$ as $n \to \infty$. Assume that the following conditions hold:*

(i) $\sup_{n\geq 1} a_n^{-1} \sum_{i=1}^{n} E|X_i - EX_i| < \infty$;

(ii) $\sum_{i=1}^{\infty} \mathrm{Var}(X_i)/a_i^2 < \infty$.

Then

$$\frac{1}{a_n}\sum_{i=1}^{n}(X_i - EX_i) \to 0, \quad \text{a.s.}$$

Proof of Theorem 2.1 For every $n \geq 1$, let

$$X_n' = -\phi(n)I(X_n < -\phi(n)) + X_n I(|X_n| \leq \phi(n)) + \phi(n)I(X_n > \phi(n))$$

and

$$Y_n = -\frac{1}{\log n} I(X_n < -\phi(n)) + \frac{X_n I(|X_n| \le \phi(n))}{\phi(n) \log n} + \frac{1}{\log n} I(X_n > \phi(n)).$$

By (2.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty} P(X_n \ne X_n') &\le \sum_{n=1}^{\infty} P(|X_n| > \phi(n)) = \sum_{n=1}^{\infty} P(\phi^{-1}(|X_n|) > n) \\ &\le CE[\phi^{-1}(|X|)] < \infty. \end{aligned} \tag{4.1}$$

By Lemma 4.1, $\{X_n'\}$ and $\{Y_n\}$ are also END sequences with the same dominating coefficient M. Then, by the proof of (4.1) and conditions (ii) and (iii) in Assumption A, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} \log^2 n \mathrm{Var}(Y_n) &\le \sum_{n=1}^{\infty} \log^2 n E Y_n^2 \le \sum_{n=1}^{\infty} \frac{EX^2 I(|X| \le \phi(n)}{\phi^2(n)} + \sum_{n=1}^{\infty} P(|X| > \phi(n)) \\ &\le C_1 k_0 + C_2 E \sum_{n=k_0+1}^{\infty} \frac{X^2 I(|X| \le \phi(n))}{\phi^2(n+1)} + C_3 \\ &\le C_4 + C_2 E X^2 \int_{\phi^{-1}(|X|)}^{\infty} \frac{1}{\phi^2(x)} dx \\ &\le C_4 + C_5 (aE(\phi^{-1}(|X|)) + b) < \infty. \end{aligned} \tag{4.2}$$

Combining Lemma 4.2 with (4.2), we have

$$\sum_{n=1}^{\infty} (Y_n - EY_n) = (I_1 + I_2) \quad \text{converges, a.s.} \tag{4.3}$$

where

$$I_1 = \sum_{n=1}^{\infty} \frac{X_n I(|X_n| \le \phi(n)) - m_n}{g(n) h(n) \log n}$$

$$I_2 = \sum_{n=1}^{\infty} \frac{1}{\log n} [I(X_n > \phi(n)) - I(X_n < -\phi(n)) - P(X_n > \phi(n)) + P(X_n < -\phi(n))].$$

Combining (4.1) with Borel-Cantelli Lemma,

$$\sum_{n=1}^{\infty} [I(X_n > \phi(n)) - I(X_n < -\phi(n)) - P(X_n > \phi(n)) + P(X_n < -\phi(n))], \quad \text{converges, a.s.,}$$

which implies I_2 converges, a.s.. Consequently, by (4.3), I_1 converges, a.s.. Then, by Kronecker lemma, it yields

$$\frac{1}{g(n) \log n} \sum_{i=1}^{n} \frac{X_i I(|X_i| \le \phi(i)) - m_i}{h(i)} \to 0, \quad \text{a.s.}$$

To complete the proof of (2.2), it is enough to show that

$$\frac{1}{g(n) \log n} \sum_{i=1}^{n} \frac{X_i I(|X_i| > \phi(i))}{h(i)} \to 0, \quad \text{a.s.}$$

which follows from (4.1) and Borel-Cantelli Lemma. □

Proof of Corollary 2.2 For $p>0$ and $n\ge 1$, let

$$X_n' = -n^{1/p} I(X_n < -n^{1/p}) + X_n I(|X_n| \le n^{1/p}) + n^{1/p} I(X_n > n^{1/p}).$$

For $0<p<1$, the condition $E|X|<\infty$ implies that $E|X|^p<\infty$. Thus,

$$\sum_{n=1}^{\infty} P(X_n \ne X_n') \le \sum_{n=1}^{\infty} P(|X_n| > n^{1/p}) = \sum_{n=1}^{\infty} P(|X|^p > n) \le CE|X|^p < \infty. \tag{4.4}$$

In addition, it can be checked that

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_n')}{n^{2/p}} \le \sum_{n=1}^{\infty} \frac{EX^2 I(|X| \le n^{1/p})}{n^{2/p}} + \sum_{n=1}^{\infty} P(|X| > n^{1/p}) \le CE|X|^p < \infty.$$

Since END sequence is a pairwise NQD sequence [28] and $\{X_n', n\ge 1\}$ is an END sequence, one can find that the conditions of Lemma 4.3 for $\{X_n', n\ge 1\}$ are satisfied. So, by $0<p<1$, we have

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} (X_i' - EX_i') \to 0, \quad \text{a.s.}$$

Obviously, it can be seen that for $0<p<1$,

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} |EX_i'| \le \frac{nE|X|}{n^{1/p}} \to 0, \quad \frac{1}{n^{1/p}} \sum_{i=1}^{n} a \to 0,$$

where a is an arbitrary real number. Combining (4.4) with Borel-Cantelli Lemma, we complete the proof of (2.3).

Next, we prove (2.4). For $1<p<2$, by $E|X|^p<\infty$, it can be checked that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{E[|X_n| I(|X_n| > n^{1/p})]}{n^{1/p}} &= \sum_{n=1}^{\infty} \frac{E[|X| I(|X| > n^{1/p})]}{n^{1/p}} \\
&= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{E[|X| I(m < |X|^p \le m+1)]}{n^{1/p}} \\
&= \sum_{m=1}^{\infty} E[|X| I(m < |X|^p \le m+1)] \sum_{n=1}^{m} \frac{1}{n^{1/p}} \\
&\le C_1 \sum_{m=1}^{\infty} m^{-1/p+1} E[|X| I(m < |X|^p \le m+1)] \\
&\le C_2 E|X|^p < \infty.
\end{aligned}$$

Then, for $1<p<2$, by Kronecker Lemma, we have

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} E[|X_i| I(|X_i| > i^{1/p})] \to 0, \quad n\to\infty. \tag{4.5}$$

Now, we consider the case of $p=1$. Obviously, from $EX=0$, it follows $E|X|<\infty$. So we have $E|X|I(|X|>n)\to 0$, as $n\to\infty$. Then, there exists some positive n_0 large enough such that

$$\frac{1}{n} \sum_{i=1}^{n} E[|X_i| I(|X_i| > i)] = \frac{1}{n} \sum_{i=1}^{n_0} E[|X_i| I(|X_i| > i)] + \frac{1}{n} \sum_{i=n_0+1}^{n} E[|X_i| I(|X_i| > i)]$$

$$= \frac{1}{n}\sum_{i=1}^{n_0} E[|X|I(|X| > i)] + \frac{1}{n}\sum_{i=n_0+1}^{n} E[|X|I(|X| > i)] \to 0, \quad n \to \infty. \tag{4.6}$$

Then, for $1 \le p < 2$, let $\phi(i) = i^{1/p}$ and $m_i = E[X_i I(|X_i| \le \phi(i))]$, $1 \le i \le n$. We apply Theorem 2.1 with $g(n) = n^{1/p}$ and $h(n) = 1$, and obtain that

$$\frac{1}{n^{1/p}\log n}\sum_{i=1}^{n}(X_i - m_i) \to 0, \quad \text{a.s.} \tag{4.7}$$

In addition, by (4.5), (4.6) and the assumption $EX = 0$, one can check that

$$\frac{1}{n^{1/p}\log n}\sum_{i=1}^{n}|m_i| = \frac{1}{n^{1/p}\log n}\sum_{i=1}^{n}|EXI(|X| > i^{1/p})| \to 0. \tag{4.8}$$

Combining (4.7) and (4.8), the proof of (2.4) is completed. □

Proof of Corollary 2.3 $EX = 0$ implies $E|X| < \infty$ and $E|XI(|X| > n\log n)| = o(1)$ as $n \to \infty$. Let $m_i = E[X_i I(|X_i| \le \phi(i))]$ and $\phi(i) = i\log i$. In addition, by $EX = 0$ and the proof of (4.8), we have

$$\frac{1}{\log n}\sum_{i=1}^{n}\frac{|m_i|}{i} = \frac{1}{\log n}\sum_{i=1}^{n}\frac{|EXI(|X| \ge i\log i)|}{i} = o(1).$$

Let $\phi(x) = g(x)h(x)$, where $g(x) = \log x$ and $h(x) = x$. In this case $\phi^{-1}(x) \sim \frac{x}{\log x}$ as $x \to \infty$. Thus, $E\phi^{-1}(|X|) \le E|X| < \infty$. Last, combining Theorem 2.1 with the above statement, we obtain (2.5) immediately. □

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