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Embedding Tensors on 3-Hom-Lie Algebras

Wen TENG[1,*], **Jiulin JIN**[2], **Yu ZHANG**[1]

1. *School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou* 550025*, P. R. China;*
2. *School of Science, Guiyang University, Guizhou* 550005*, P. R. China*

Abstract In this paper, we introduce the notion of embedding tensors on 3-Hom-Lie algebras and show that embedding tensors induce naturally 3-Hom-Leibniz algebras. Moreover, the cohomology theory of embedding tensors on 3-Hom-Lie algebras is defined. As an application, we show that if two linear deformations of an embedding tensor on a 3-Hom-Lie algebra are equivalent, then their infinitesimals belong to the same cohomology class in the first cohomology group.

Keywords 3-Hom-Lie algebra; embedding tensor; representation; cohomology; deformation

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1. Introduction

The concept of embedded tensors provides a useful tool on the construction of supergravity theories [1] and higher gauge theories [2]. The embedding tensor is called the average operator in mathematics. Aguiar [3] studied the average operator on the associative algebra and Lie algebra. Later, the deformation and cohomology theory of embedding tensors of associative algebra, Lie algebra and 3-Lie algebra were given in [4–6]. Recently, Das and Makhlouf [7] introduced the embedding tensor on Hom-Lie algebra, and studied the related properties.

The aim of this paper is to extend the concept of embedded tensors of 3-Lie algebras to Hom-type algebras. Hom-Lie algebras were introduced by Hartwig, Larsson and Silvestrov [8] in the study of q-deformations of the Witt and Virasoro algebra. In the last fifteen years, Hom-type algebras have attracted extensive attention from scholars [7, 9–15]. In addition, Filippov [16] introduced 3-Lie algebra and more general n-Lie algebra, which can be regarded as a generalization of Lie algebra to higher algebra. In particular, 3-Lie algebras play an important role in string theory [17]. In [18], n-Hom-Lie algebras and various generalizations of n-ary algebras

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* Corresponding author
E-mail address: tengwen@mail.gufe.edu.cn (Wen TENG)

are considered, and the representation and cohomology of n-Hom-Lie algebras are first studied in [19].

This paper is organized as follows. In Section 2, we recall the definitions of 3-Hom-Lie algebras. Then we introduce the notion of an embedding tensor on a 3-Hom-Lie algebra, which naturally induces a 3-Hom-Leibniz algebra. In Section 3, we introduce the representation and cohomology theory of embedding tensor on 3-Hom-Lie algebra. In Section 4, we study linear deformations of embedding tensors on 3-Hom-Lie algebras, and show that if two linear deformations of an embedding tensor on a 3-Hom Lie algebra are equivalent, then their infinitesimals belong to the same cohomology class in the first cohomology group.

In this paper, all vector spaces are considered over a field $\mathbb{K}$ of characteristic 0.

2. Embedding tensors on 3-Hom-Lie algebras

In this section, we recall some basic definitions of 3-Hom-Lie algebras, Hom-Leibniz algebras and 3-Hom-Leibniz algebras. Then we introduce embedding tensors on 3-Hom-Lie algebras. We show that an embedding tensor naturally gives rise to a 3-Hom-Leibniz algebra structure. Finally, we provide some examples of embedding tensors on 3-Hom-Lie algebras.

Definition 2.1 ([18]) *A 3-Hom-Lie algebra is a triple $(L,[\cdot,\cdot,\cdot],\alpha)$ consisting of a vector space L, a trilinear skew-symmetric mapping $[\cdot,\cdot,\cdot]: L\times L\times L\to L$, and a linear map $\alpha: L\to L$ satisfying $\alpha([x,y,z])=[\alpha(x),\alpha(y),\alpha(z)]$ and the Hom-Filippov-Jacobi identity:*

$$[\alpha(a),\alpha(b),[x,y,z]]=[[a,b,x],\alpha(y),\alpha(z)]+[\alpha(x),[a,b,y],\alpha(z)]+[\alpha(x),\alpha(y),[a,b,z]], \quad (2.1)$$

for any $a,b,x,y,z\in L$. Furthermore, if $\alpha: L\to L$ is a vector space automorphism of L, then the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ is called a regular 3-Hom-Lie algebra.

A homomorphism between two 3-Hom-Lie algebras $(L,[\cdot,\cdot,\cdot],\alpha)$ and $(L',[\cdot,\cdot,\cdot]',\alpha')$ is a linear map $\psi: L\to L'$ satisfying $\psi\circ\alpha=\alpha'\circ\psi$ and

$$\psi([x,y,z])=[\psi(x),\psi(y),\psi(z)]', \quad \forall\, x,y,z\in L.$$

In particular, if ψ is nondegenerate, then ψ is called an isomorphism from L to L'.

Definition 2.2 ([12]) *A Hom-Leibniz algebra is a vector space L together with a bracket operation $[\cdot,\cdot]: L\times L\to L$ and a linear map $\alpha: L\to L$ satisfying $\alpha([x,y])=[\alpha(x),\alpha(y)]$ such that*

$$[\alpha(x),[y,z]]=[[x,y],\alpha(z)]+[\alpha(y),[x,z]],$$

for any $x,y,z\in L$.

Let $(L,[\cdot,\cdot,\cdot],\alpha)$ be a 3-Hom-Lie algebra. Then the elements in $\wedge^2 L$ are called fundamental objects of the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$. There is a bilinear operation $[\cdot,\cdot]'$ on $\wedge^2 L$, which is given by

$$[X,Y]'=[x_1,x_2,y_1]\wedge\alpha(y_2)+\alpha(y_1)\wedge[x_1,x_2,y_2], \quad \forall\, X=x_1\wedge x_2,\ Y=y_1\wedge y_2\in\wedge^2 L,$$

and a linear map $\tilde{\alpha}$ on $\wedge^2 L$ is defined by $\tilde{\alpha}(X) = \alpha(x_1) \wedge \alpha(x_2)$. Clearly, $(\wedge^2 L, [\cdot,\cdot]', \tilde{\alpha})$ is a Hom-Leibniz algebra [20].

Definition 2.3 ([11]) *A representation of a 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ on a vector space V with respect to $\beta \in \mathrm{End}(V)$ is a skew-symmetric linear map $\rho : \wedge^2 L \to \mathrm{End}(V)$ such that*

$$\rho(\alpha(x), \alpha(y)) \circ \beta = \beta \circ \rho(x, y), \tag{2.2}$$

$$\rho(\alpha(x), \alpha(y))\rho(a, b) - \rho(\alpha(a), \alpha(b))\rho(x, y) = \big(\rho([x, y, a], \alpha(b)) - \rho([x, y, b], \alpha(a))\big) \circ \beta, \tag{2.3}$$

$$\rho([x, y, a], \alpha(b)) \circ \beta - \rho(\alpha(y), \alpha(a))\rho(x, b) = \rho(\alpha(a), \alpha(x))\rho(y, b) + \rho(\alpha(x), \alpha(y))\rho(a, b), \tag{2.4}$$

for any $x, y, a, b \in L$. Furthermore, if $\beta : V \to V$ is a vector space automorphism of V, then $(V; \rho, \beta)$ is called a regular representation of $(L, [\cdot,\cdot,\cdot], \alpha)$.

It follows from the above definition that any 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ can be regarded as a representation of itself, where $\rho = \mathrm{ad} : \wedge^2 L \to \mathrm{End}(L)$ is given by $\mathrm{ad}(x, y)(z) := [x, y, z]$, for $x, y, z \in L$. This is called the adjoint representation.

Definition 2.4 ([13]) *A 3-Hom-Leibniz algebra is a triple $(\mathcal{L}, [\cdot,\cdot,\cdot]_{\mathcal{L}}, \alpha_{\mathcal{L}})$ consisting of a vector space $\mathcal{L}$, a trilinear mapping $[\cdot,\cdot,\cdot]_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, and a linear map $\alpha_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ satisfying $\alpha_{\mathcal{L}}([x, y, z]_{\mathcal{L}}) = [\alpha_{\mathcal{L}}(x), \alpha_{\mathcal{L}}(y), \alpha_{\mathcal{L}}(z)]_{\mathcal{L}}$ such that*

$$\begin{aligned}&[\alpha_{\mathcal{L}}(a), \alpha_{\mathcal{L}}(b), [x, y, z]_{\mathcal{L}}]_{\mathcal{L}} \\ &\quad= [[a, b, x]_{\mathcal{L}}, \alpha_{\mathcal{L}}(y), \alpha_{\mathcal{L}}(z)]_{\mathcal{L}} + [\alpha_{\mathcal{L}}(x), [a, b, y]_{\mathcal{L}}, \alpha_{\mathcal{L}}(z)]_{\mathcal{L}} + [\alpha_{\mathcal{L}}(x), \alpha_{\mathcal{L}}(y), [a, b, z]_{\mathcal{L}}]_{\mathcal{L}},\end{aligned} \tag{2.5}$$

for any $a, b, x, y, z \in L$.

Proposition 2.5 *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie algebra, V be a vector space, $\beta \in \mathrm{End}(V)$ and $\rho : \wedge^2 L \to \mathrm{End}(V)$ be a skew-symmetric linear map. Then $(V; \rho, \beta)$ is a representation of 3-Hom-Lie algebra L if and only if $L \oplus V$ is a 3-Hom-Leibniz algebra under the following maps:*

$$\begin{aligned}&(\alpha \oplus \beta)(x + u) := \alpha(x) + \beta(u), \\ &[x + u, y + v, z + w]_\rho := [x, y, z] + \rho(x, y)w,\end{aligned}$$

for any $x, y, z \in L$ and $u, v, w \in V$. $(L \oplus V, [\cdot,\cdot,\cdot]_\rho, \alpha \oplus \beta)$ is called the hemisemidirect product 3-Hom-Leibniz algebra, and denoted by $L \ltimes_\rho V$.

Proof For all $x, y, z, a, b \in L, u, v, w, s, t \in V$, by Eqs. (2.1)–(2.3) , we have

$$\begin{aligned}&(\alpha \oplus \beta)([x + u, y + v, z + w]_\rho) = (\alpha \oplus \beta)([x, y, z] + \rho(x, y)w) \\ &\quad= \alpha([x, y, z]) + \beta(\rho(x, y)w) = [\alpha(x), \alpha(y), \alpha(z)] + \rho(\alpha(x), \alpha(y))\beta(w) \\ &\quad= [\alpha(x) + \beta(u), \alpha(y) + \beta(v), \alpha(z) + \beta(w)]_\rho \\ &\quad= [(\alpha \oplus \beta)(x + u), (\alpha \oplus \beta)(y + v), (\alpha \oplus \beta)(z + w)]_\rho,\end{aligned}$$

$$\begin{aligned}&[[a + s, b + t, x + u]_\rho, (\alpha \oplus \beta)(y + v), (\alpha \oplus \beta)(z + w)]_\rho + \\ &[(\alpha \oplus \beta)(x + u), [a + s, b + t, y + v]_\rho, (\alpha \oplus \beta)(z + w)]_\rho + \\ &[(\alpha \oplus \beta)(x + u), (\alpha \oplus \beta)(y + v), [a + s, b + t, z + w]_\rho]_\rho -\end{aligned}$$

$$
\begin{aligned}
&[(\alpha\oplus\beta)(a+s),(\alpha\oplus\beta)(b+t),[x+u,y+v,z+w]_\rho]_\rho\\
&=[[a,b,x]+\rho(a,b)u,\alpha(y)+\beta(v),\alpha(z)+\beta(w)]_\rho+\\
&\quad[\alpha(x)+\beta(u),[a,b,y]+\rho(a,b)v,\alpha(z)+\beta(w)]_\rho+\\
&\quad[\alpha(x)+\beta(u),\alpha(y)+\beta(v),[a,b,z]+\rho(a,b)w]_\rho-\\
&\quad[\alpha(a)+\beta(s),\alpha(b)+\beta(t),[x,y,z]+\rho(x,y)w]_\rho\\
&=[[a,b,x],\alpha(y),\alpha(z)]+\rho([a,b,x],\alpha(y))\beta(w)+[\alpha(x),[a,b,y],\alpha(z)]+\\
&\quad\rho(\alpha(x),[a,b,y])\beta(w)+[\alpha(x),\alpha(y),[a,b,z]]+\rho(\alpha(x),\alpha(y))\rho(a,b)w-\\
&\quad[\alpha(a),\alpha(b),[x,y,z]]-\rho(\alpha(a),\alpha(b))\rho(x,y)w=0.
\end{aligned}
$$

Thus, $(L\oplus V,[\cdot,\cdot,\cdot]_\rho,\alpha\oplus\beta)$ is a 3-Hom-Leibniz algebra. □

Definition 2.6 *Let $(V;\rho,\beta)$ be a representation of the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$. Then the linear map $T:V\to L$ is called an embedding tensor on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$ if T meets the following equations:*

$$T\circ\beta=\alpha\circ T, \tag{2.6}$$

$$[Tu,Tv,Tw]=T(\rho(Tu,Tv)w), \tag{2.7}$$

for any $u,v,w\in V$.

Theorem 2.7 *A linear map $T:V\to L$ is an embedding tensor on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$ if and only if the graph $Gr(T)=\{Tu+u|u\in V\}$ is a 3-Hom-Leibniz subalgebra of the hemisemidirect product 3-Hom-Leibniz algebra $L\ltimes_\rho V$.*

Proof Let $T:V\to L$ be a linear map. Then for all $u,v,w\in V$, we have

$$
\begin{aligned}
&(\alpha\oplus\beta)(Tu+u)=\alpha(Tu)+\beta(u),\\
&[Tu+u,Tv+v,Tw+w]_\rho=[Tu,Tv,Tw]+\rho(Tu,Tv)w.
\end{aligned}
$$

Thus, the graph

$$Gr(T)=\{Tu+u\mid u\in V\}$$

is a subalgebra of the hemisemidirect product 3-Hom-Leibniz algebra $L\ltimes_\rho V$ if and only if T meets Eqs. (2.6) and (2.7), which implies that T is an embedding tensor on the 3-Hom-Lie algebra L with respect to the representation $(V;\rho,\beta)$. □

Clearly, the algebraic structure underlying an embedding tensor on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ is a 3-Hom-Leibniz algebra. Thereby, we have the following proposition.

Proposition 2.8 *Let $T:V\to L$ be an embedding tensor on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$. If a linear map $[\cdot,\cdot,\cdot]_T:V\times V\times V\to V$ is given by*

$$[u,v,w]_T=\rho(Tu,Tv)w, \tag{2.8}$$

for any $u,v,w\in V$, then $(V,[\cdot,\cdot,\cdot]_T,\beta)$ is a 3-Hom-Leibniz algebra. Moreover, T is a homomor-

phism from the 3-Hom-Leibniz algebra $(V, [\cdot,\cdot,\cdot]_T, \beta)$ to the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$.

Proof For all $u, v, w, s, t \in V$, by Eqs. (2.2), (2.3), (2.6)–(2.8), we have

$$\begin{aligned}
&\beta([u,v,w]_T) = \beta(\rho(Tu,Tv)w)\\
&\quad = \rho(\alpha(Tu),\alpha(Tv))\beta(w) = \rho(T\beta(u),T\beta(v))\beta(w)\\
&\quad = [\beta(u),\beta(v),\beta(w)]_T,\\
&\ [[s,t,u]_T,\beta(v),\beta(w)]_T + [\beta(u),[s,t,v]_T,\beta(w)]_T + [\beta(u),\beta(v),[s,t,w]_T]_T\\
&\quad = [\rho(Ts,Tt)u,\beta(v),\beta(w)]_T + [\beta(u),\rho(Ts,Tt)v,\beta(w)]_T + [\beta(u),\beta(v),\rho(Ts,Tt)w]_T\\
&\quad = \rho(T\rho(Ts,Tt)u,T\beta(v))\beta(w) + \rho(T\beta(u),T\rho(Ts,Tt)v)\beta(w) + \rho(T\beta(u),T\beta(v))\rho(Ts,Tt)w\\
&\quad = \rho([Ts,Tt,Tu],\alpha(Tv))\beta(w) + \rho(\alpha(Tu),[Ts,Tt,Tv])\beta(w) + \rho(\alpha(Tu),\alpha(Tv))\rho(Ts,Tt)w\\
&\quad = \rho(\alpha(Ts),\alpha(Tt))\rho(Tu,Tv)w = \rho(T\beta(s),T\beta(t))\rho(Tu,Tv)w\\
&\quad = [\beta(s),\beta(t),\rho(Tu,Tv)w]_T = [\beta(s),\beta(t),[u,v,w]_T]_T.
\end{aligned}$$

Thus, $(V, [\cdot,\cdot,\cdot]_T, \beta)$ is a 3-Hom-Leibniz algebra. By Eqs. (2.6) and (2.7), T is a homomorphism from the 3-Hom-Leibniz algebra $(V, [\cdot,\cdot,\cdot]_T, \beta)$ to the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$. □

Definition 2.9 *Let T and T' be two embedding tensors on the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ with respect to the representation $(V;\rho,\beta)$. Then a homomorphism from T' to T consists of a 3-Hom-Lie algebra homomorphism $\psi_L : L \to L$ and a linear map $\psi_V : V \to V$ such that*

$$\beta\circ\psi_V = \psi_V\circ\beta, \tag{2.9}$$

$$T\circ\psi_V = \psi_L\circ T', \tag{2.10}$$

$$\psi_V(\rho(x,y)u) = \rho(\psi_L(x),\psi_L(y))\psi_V(u). \tag{2.11}$$

In particular, if both ψ_L and ψ_V are invertible, (ψ_L,ψ_V) is called an isomorphism from T' to T.

The association of a 3-Hom-Leibniz algebra from an embedding tensor enjoys the functorial property.

Proposition 2.10 *Let T and T' be two embedding tensors on the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ with respect to the representation $(V;\rho,\beta)$, and (ψ_L,ψ_V) be a homomorphism from T' to T. Then ψ_V is a homomorphism of 3-Hom-Leibniz algebras from $(V, [\cdot,\cdot,\cdot]_{T'}, \beta)$ to $(V, [\cdot,\cdot,\cdot]_T, \beta)$.*

Proof For all $u, v, w \in V$, by Eqs. (2.8), (2.10) and (2.11), we have

$$\begin{aligned}
&\psi_V([u,v,w]_{T'}) = \psi_V(\rho(T'u,T'v)w)\\
&\quad = \rho(\psi_L(T'u),\psi_L(T'v))\psi_V(w) = \rho(T\psi_V(u),T\psi_V(v))\psi_V(w)\\
&\quad = [\psi_V(u),\psi_V(v),\psi_V(w)]_T.
\end{aligned}$$

Using Eq. (2.9), we can get ψ_V is a homomorphism of 3-Hom-Leibniz algebras from $(V, [\cdot,\cdot,\cdot]_{T'}, \beta)$ to $(V, [\cdot,\cdot,\cdot]_T, \beta)$. □

Next, we present some examples of embedding tensors on 3-Hom-Lie algebras.

Example 2.11 Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie algebra. Then the identity map $\mathrm{Id} : L \to L$ is an embedding tensor on the 3-Hom-Lie algebra L with respect to the adjoint representation.

Example 2.12 Let $(L,[\cdot,\cdot,\cdot],\alpha)$ be a 3-Hom-Lie algebra. Then a linear map $D: L\to L$ is said to be a derivation for the 3-Hom-Lie algebra L if $\alpha\circ D = D\circ\alpha$ and $D[x,y,z] = [Dx,y,z]+[x,Dy,z]+[x,y,Dz]$, for all $x,y,z\in L$. If $D^2=0$, then D is an embedding tensor on L with respect to the adjoint representation.

Example 2.13 Let $(V;\rho,\beta)$ be a representation of a 3-Hom-Lie algebra$(L,[\cdot,\cdot,\cdot],\alpha)$. If a linear map $f: V\to L$ satisfies:

$$\alpha(f(u)) = f(\beta(u)),$$
$$f(\rho(x,f(u))v) = [x,f(u),f(v)],$$

for any $x\in L, u,v\in V$, then f is an embedding tensors on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$.

Example 2.14 Let $(L,[\cdot,\cdot,\cdot],\alpha)$ be a 3-Hom-Lie algebra. Then it can be easily checked that $(\oplus^n L;\rho,\oplus^n\alpha)$ is a representation of the 3-Hom-Lie algebra L, where

$$\rho:\wedge^2 L\to \mathrm{End}(\oplus^n L), \rho(x,y)((x_1,\ldots,x_n)) = ([x,y,x_1],\ldots,[x,y,x_n]),$$

for any $x\in L, (x_1,\ldots,x_n)\in\oplus^n L$. Moreover, $T:\oplus^n L\to L, T(x_1,\ldots,x_n)=x_1+\cdots+x_n$ is an embedding tensor on L with respect to the representation $(\oplus^n L;\rho,\oplus^n\alpha)$.

Example 2.15 With the notations of the previous example, then the i-th projection map $T_i:\oplus^n L\to L, T_i(x_1,\ldots,x_n)=x_i$ is an embedding tensor on $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(\oplus^n L;\rho,\oplus^n\alpha)$.

3. The cohomology of embedding tensors on 3-Hom-Lie algebras

In this section, we recall some basic results of representations and cohomologies of 3-Hom-Leibniz algebras. We construct a representation of the 3-Hom-Leibniz algebra $(V,[\cdot,\cdot,\cdot]_T,\beta)$ on the vector space L, and define the cohomologies of an embedding tensor on 3-Hom-Lie algebras.

Definition 3.1 *A representation of the 3-Hom-Leibniz algebra $(\mathcal{L},[\cdot,\cdot,\cdot]_{\mathcal{L}},\alpha)$ is a pair $(V;\beta)$ of vector space V and a linear map $\beta: V\to V$, equipped with 3 actions*

$$l:\mathcal{L}\otimes\mathcal{L}\otimes V\to V,$$
$$m:\mathcal{L}\otimes V\otimes\mathcal{L}\to V,$$
$$r: V\otimes\mathcal{L}\otimes\mathcal{L}\to V,$$

satisfying for any $x,y,z,a,b\in\mathcal{L}$ and $u\in V$

$$l(\alpha(x),\alpha(y),\beta(u)) = \beta(l(x,y,u)),$$
$$m(\alpha(x),\beta(u),\alpha(z)) = \beta(m(x,u,z)),$$
$$r(\beta(u),\alpha(y),\alpha(z)) = \beta(r(u,y,z))$$

and

$$
\begin{aligned}
& l([a,b,x]_{\mathcal{L}},\alpha(y),\beta(u))+l(\alpha(x),[a,b,y]_{\mathcal{L}},\beta(u))+l(\alpha(x),\alpha(y),l(a,b,u)) \\
& \quad = l(\alpha(a),\alpha(b),l(x,y,u)), && (3.1)\\
& m([a,b,x]_{\mathcal{L}},\beta(u),\alpha(z))+m(\alpha(x),l(a,b,u),\alpha(z))+m(\alpha(x),\beta(u),[a,b,z]_{\mathcal{L}}) \\
& \quad = l(\alpha(a),\alpha(b),m(x,u,z)), && (3.2)\\
& r(l(a,b,u),\alpha(y),\alpha(z))+r(\beta(u),[a,b,y]_{\mathcal{L}},\alpha(z))+r(\beta(u),\alpha(y),[a,b,z]_{\mathcal{L}}) \\
& \quad = l(\alpha(a),\alpha(b),r(u,y,z)), && (3.3)\\
& r(m(a,u,x),\alpha(y),\alpha(z))+m(\alpha(x),m(a,u,y),\alpha(z))+l(\alpha(x),\alpha(y),m(a,u,z)) \\
& \quad = m(\alpha(a),\beta(u),[x,y,z]_{\mathcal{L}}), && (3.4)\\
& r(r(u,b,x),\alpha(y),\alpha(z))+m(\alpha(x),r(u,b,y),\alpha(z))+l(\alpha(x),\alpha(y),r(u,b,z)) \\
& \quad = r(\beta(u),\alpha(b),[x,y,z]_{\mathcal{L}}). && (3.5)
\end{aligned}
$$

An n-cochain on a 3-Hom-Leibniz algebra $(\mathcal{L},[\cdot,\cdot,\cdot]_{\mathcal{L}},\alpha)$ with coefficients in a representation $(V;l,m,r,\beta)$ is a linear map

$$
f:\overbrace{\wedge^2\mathcal{L}\otimes\cdots\otimes\wedge^2\mathcal{L}}^{n-1}\otimes\mathcal{L}\to V,\quad n\geq 1
$$

such that $\beta\circ f=f\circ(\tilde{\alpha}^{\otimes n-1}\otimes\alpha)$. The space generated by n-cochains is denoted as $\mathcal{C}^n_{\mathrm{3HLei}}(\mathcal{L},V)$. The coboundary map δ from n-cochains to $(n+1)$-cochains, for $X_i=x_i\wedge y_i\in\wedge^2\mathcal{L}, 1\leq i\leq n$ and $z\in\mathcal{L}$, is defined as

$$
\begin{aligned}
&(\delta f)(X_1,X_2,\ldots,X_n,z)\\
&\quad=\sum_{1\leq j<k\leq n}(-1)^j f(\tilde{\alpha}(X_1),\ldots,\widehat{X_j},\ldots,\tilde{\alpha}(X_{k-1}),\alpha(x_k)\wedge[x_j,y_j,y_k]_{\mathcal{L}}+\\
&\qquad [x_j,y_j,x_k]_{\mathcal{L}}\wedge\alpha(y_k),\ldots,\tilde{\alpha}(X_n),\alpha(z))+\\
&\qquad \sum_{j=1}^{n}(-1)^j f(\tilde{\alpha}(X_1),\ldots,\widehat{X_j},\ldots,\tilde{\alpha}(X_n),[x_j,y_j,z]_{\mathcal{L}})+\\
&\qquad \sum_{j=1}^{n}(-1)^{j+1} l(\tilde{\alpha}^{n-1}(X_j),f(X_1,\ldots,\widehat{X_j},\ldots,X_n,z))+\\
&\qquad (-1)^{n+1}(m(\alpha^{n-1}(x_n),f(X_1,\ldots,X_{n-1},y_n),\alpha^{n-1}(z))+\\
&\qquad r(f(X_1,\ldots,X_{n-1},x_n),\alpha^{n-1}(y_n),\alpha^{n-1}(z))).
\end{aligned}
$$

It was proved in [13] that $\delta^2=0$. Thus, $(\oplus_{n=1}^{+\infty}\mathcal{C}^n_{\mathrm{3HLei}}(\mathcal{L},V),\delta)$ is a cochain complex. We denote the set of n-cocycles by $\mathcal{Z}^n_{\mathrm{3HLei}}(\mathcal{L},V)$, the set of n-coboundaries by $\mathcal{B}^n_{\mathrm{3HLei}}(\mathcal{L},V)$ and the n-th cohomology group by $\mathcal{H}^n_{\mathrm{3HLei}}(\mathcal{L},V)=\mathcal{Z}^n_{\mathrm{3HLei}}(\mathcal{L},V)/\mathcal{B}^n_{\mathrm{3HLei}}(\mathcal{L},V)$.

Lemma 3.2 *Let $T:V\to L$ be an embedding tensor on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$. Define actions*

$$
l_T:V\otimes V\otimes L\to L, m_T:V\otimes L\otimes V\to L, r_T:L\otimes V\otimes V\to L,
$$

by

$$l_T(u,v,x) = [Tu,Tv,x],$$
$$m_T(u,x,v) = [Tu,x,Tv] - T\rho(Tu,x)v,$$
$$r_T(x,u,v) = [x,Tu,Tv] - T\rho(x,Tu)v,$$

for any $u,v \in V, x \in L$. *Then* $(L; l_T, m_T, r_T, \alpha)$ *is a representation of the 3-Hom-Leibniz algebra* $(V, [\cdot,\cdot,\cdot]_T, \beta)$.

Proof For all $u,v,w,s,t \in V$ and $x \in L$, by Eqs. (2.2) and (2.6), we have

$$\begin{aligned}
&l_T(\beta(u),\beta(v),\alpha(x)) = [T\beta(u),T\beta(v),\alpha(x)] = [\alpha(Tu),\alpha(Tv),\alpha(x)]\\
&\quad = \alpha([Tu,Tv,x]) = \alpha(l_T(u,v,x)),\\
&m_T(\beta(u),\alpha(x),\beta(v)) = [T\beta(u),\alpha(x),T\beta(v)] - T(\rho(T\beta(u),\alpha(x))\beta(v))\\
&\quad = [\alpha(Tu),\alpha(x),\alpha(Tv)] - T(\rho(\alpha(Tu),\alpha(x))\beta(v))\\
&\quad = \alpha([Tu,x,Tv]) - T\beta(\rho(Tu,x)v) = \alpha([Tu,x,Tv]) - \alpha(T\rho(Tu,x)v)\\
&\quad = \alpha(m_T(u,x,v)).
\end{aligned}$$

Similarly, we can show that $r_T(\alpha(x),\beta(u),\beta(v)) = \alpha(r_T(x,u,v))$ holds.

By Eqs. (2.1), (2.7) and (2.8), we have

$$\begin{aligned}
&l_T([u,v,s]_T,\beta(t),\alpha(x)) + l_T(\beta(s),[u,v,t]_T,\alpha(x)) + l_T(\beta(s),\beta(t),l_T(u,v,x))\\
&\quad = [T[u,v,s]_T,T\beta(t),\alpha(x)] + [T\beta(s),T[u,v,t]_T,\alpha(x)] + [T\beta(s),T\beta(t),[Tu,Tv,x]]\\
&\quad = [[Tu,Tv,Ts],\alpha(Tt),\alpha(x)] + [\alpha(Ts),[Tu,Tv,Tt],\alpha(x)] + [\alpha(Ts),\alpha(Tt),[Tu,Tv,x]]\\
&\quad = [\alpha(Tu),\alpha(Tv),[Ts,Tt,x]] = [T\beta(u),T\beta(v),[Ts,Tt,x]] = l_T(\beta(u),\beta(v),l_T(s,t,x)),
\end{aligned}$$

which indicates that Eq. (3.1) holds.

By Eqs. (2.1), (2.3), (2.6)–(2.8), we have

$$\begin{aligned}
&m_T([u,v,s]_T,\alpha(x),\beta(t)) + m_T(\beta(s),l_T(u,v,x),\beta(t)) + m_T(\beta(s),\alpha(x),[u,v,t]_T)\\
&\quad = [T[u,v,s]_T,\alpha(x),T\beta(t)] - T\rho(T[u,v,s]_T,\alpha(x))\beta(t) + [T\beta(s),[Tu,Tv,x],T\beta(t)] -\\
&\qquad T\rho(T\beta(s),[Tu,Tv,x])\beta(t) + [T\beta(s),\alpha(x),T[u,v,t]_T] - T\rho(T\beta(s),\alpha(x))[u,v,t]_T\\
&\quad = [[Tu,Tv,Ts],\alpha(x),\alpha(Tt)] - T\rho([Tu,Tv,Ts],\alpha(x))\beta(t) + [\alpha(Ts),[Tu,Tv,x],\alpha(Tt)] -\\
&\qquad T\rho(\alpha(Ts),[Tu,Tv,x])\beta(t) + [\alpha(Ts),\alpha(x),[Tu,Tv,Tt]] - T\rho(\alpha(Ts),\alpha(x))\rho(Tu,Tv)t\\
&\quad = [\alpha(Tu),\alpha(Tv),[Ts,x,Tt]] - T\rho(\alpha(Tu),\alpha(Tv))\rho(Ts,x)t\\
&\quad = [\alpha(Tu),\alpha(Tv),[Ts,x,Tt]] - [\alpha(Tu),\alpha(Tv),\rho(Ts,x)t]\\
&\quad = [T\beta(u),T\beta(v),[Ts,x,Tt] - \rho(Ts,x)t] = l_T(u,v,m_T(s,x,t)),\\
&r_T(l_T(u,v,x),\beta(s),\beta(t)) + r_T(\alpha(x),[u,v,s]_T,\beta(t)) + r_T(\alpha(x),\beta(s),[u,v,t]_T)\\
&\quad = [[Tu,Tv,x],T\beta(s),T\beta(t)] - T\rho([Tu,Tv,x],T\beta(s))\beta(t) + [\alpha(x),T\rho(Tu,Tv)s,T\beta(t)] -\\
&\qquad T\rho(\alpha(x),T\rho(Tu,Tv)s)\beta(t) + [\alpha(x),T\beta(s),T\rho(Tu,Tv)t] - T\rho(\alpha(x),T\beta(s))\rho(Tu,Tv)t\\
&\quad = [[Tu,Tv,x],\alpha(Ts),\alpha(Tt)] - T\rho([Tu,Tv,x],\alpha(Ts))\beta(t) + [\alpha(x),[Tu,Tv,Ts],\alpha(Tt)] -
\end{aligned}$$

$$
\begin{aligned}
&T\rho(\alpha(x),[Tu,Tv,Ts])\beta(t)+[\alpha(x),\alpha(Ts),[Tu,Tv,Tt]]-T\rho(\alpha(x),\alpha(Ts))\rho(Tu,Tv)t\\
=&[\alpha(Tu),\alpha(Tv),[x,Ts,Tt]]-T\rho(\alpha(Tu),\alpha(Tv))\rho(x,Ts)t\\
=&[\alpha(Tu),\alpha(Tv),[x,Ts,Tt]]-[\alpha(Tu),\alpha(Tv),T\rho(x,Ts)t]\\
=&[T\beta(u),T\beta(v),[x,Ts,Tt]-T\rho(x,Ts)t]=l_T(\beta(u),\beta(v),r_T(x,s,t)),
\end{aligned}
$$

which imply that Eqs. (3.2) and (3.3) hold. Similarly, we can prove that Eqs. (3.4) and (3.5) are true. Thus, $(L;l_T,m_T,r_T,\alpha)$ is a representation of the 3-Hom-Leibniz algebra $(V,[\cdot,\cdot,\cdot]_T,\beta)$. □

When $n\geq 1$, let $\delta_T:\mathcal{C}^n_{\mathrm{3HLei}}(V,L)\to\mathcal{C}^{n+1}_{\mathrm{3HLei}}(V,L)$ be the coboundary operator of the 3-Hom-Leibniz algebra $(V,[\cdot,\cdot,\cdot]_T,\beta)$ with coefficients in the representation $(L;l_T,m_T,r_T,\alpha)$. More precisely, for all $f\in\mathcal{C}^n_{\mathrm{3HLei}}(V,L), V_i=u_i\wedge v_i\in\wedge^2V, 1\leq i\leq n$ and $w\in V$, we have

$$
\begin{aligned}
&(\delta_T f)(V_1,V_2,\ldots,V_n,w)\\
&=\sum_{1\leq j<k\leq n}(-1)^j f(\tilde\beta(V_1),\ldots,\widehat{V_j},\ldots,\tilde\beta(V_{k-1}),\beta(u_k)\wedge\\
&[u_j,v_j,v_k]_T+[u_j,v_j,u_k]_T\wedge\beta(v_k),\ldots,\tilde\beta(V_n),\beta(w))+\\
&\sum_{j=1}^{n}(-1)^j f(\tilde\beta(V_1),\ldots,\widehat{V_j},\ldots,\tilde\beta(V_n),[u_j,v_j,w]_T)+\\
&\sum_{j=1}^{n}(-1)^{j+1}l_T(\tilde\beta^{n-1}(V_j),f(V_1,\ldots,\widehat{V_j},\ldots,V_n,w))+\\
&(-1)^{n+1}(m_T(\beta^{n-1}(u_n),f(V_1,\ldots,V_{n-1},v_n),\beta^{n-1}(w))+\\
&r_T(f(V_1,\ldots,V_{n-1},u_n),\beta^{n-1}(v_n),\beta^{n-1}(w))).
\end{aligned}
$$

In particular, for $f\in\mathcal{C}^1_{\mathrm{3HLei}}(V,L):=\{g\in\mathrm{Hom}(V,L)\mid\alpha\circ g=g\circ\beta\}$ and $u,v,w\in V$, we have

$$
\begin{aligned}
&(\delta_T f)(u,v,w)=-f([u,v,w]_T)+l_T(u,v,f(w))+m_T(u,f(v),w)+r_T(f(u),v,w)\\
&=-f(\rho(Tu,Tv)w)+[Tu,Tv,f(w)]+[Tu,f(v),Tw]-T\rho(Tu,f(v))w+\\
&\quad[f(u),Tv,Tw]-T\rho(f(u),Tv)w.
\end{aligned}
$$

When $n=0$, for any $(a,b)\in\mathcal{C}^0_{\mathrm{3HLei}}(V,L):=\{(x,y)\in\wedge^2L\mid\alpha(x)=x,\alpha(y)=y\}$, we define

$$\delta_T:\mathcal{C}^0_{\mathrm{3HLei}}(V,L)\to\mathcal{C}^1_{\mathrm{3HLei}}(V,L),(a,b)\mapsto\wp(a,b)$$

by

$$\wp(a,b)v=T\rho(a,b)\beta^{-1}(v)-[a,b,T\beta^{-1}(v)],\quad\forall\, v\in V,$$

where $\beta:V\to V$ is a vector space isomorphism.

Proposition 3.3 *Let $T:V\to L$ be an embedding tensor on the regular 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the regular representation $(V;\rho,\beta)$. Then $\delta_T(\wp(a,b))=0$, that is the composition $\mathcal{C}^0_{\mathrm{3HLei}}(V,L)\xrightarrow{\delta_T}\mathcal{C}^1_{\mathrm{3HLei}}(V,L)\xrightarrow{\delta_T}\mathcal{C}^2_{\mathrm{3HLei}}(V,L)$ is the zero map.*

Proof For any $u,v,w\in V$, by Eqs. (2.1)–(2.3), (2.6) and (2.7) we have

$$(\delta_T\wp(a,b))(u,v,w)$$

$$
\begin{aligned}
&= -\wp(a,b)\rho(Tu,Tv)w + [Tu,Tv,\wp(a,b)w] + [Tu,\wp(a,b)v,Tw] - T\rho(Tu,\wp(a,b)v)w +\\
&\quad [\wp(a,b)u,Tv,Tw] - T\rho(\wp(a,b)u,Tv)w\\
&= -T\rho(a,b)\rho(\alpha^{-1}(Tu),\alpha^{-1}(Tv))\beta^{-1}(w) + [a,b,T\rho(\alpha^{-1}(Tu),\alpha^{-1}(Tv))\beta^{-1}(w)] +\\
&\quad [Tu,Tv,T\rho(a,b)\beta^{-1}(w)] - [Tu,Tv,[a,b,T\beta^{-1}(w)]] + [Tu,T\rho(a,b)\beta^{-1}(v),Tw] -\\
&\quad [Tu,[a,b,T\beta^{-1}(v)],Tw] - T\rho(Tu,T\rho(a,b)\beta^{-1}(v))w + T\rho(Tu,[a,b,T\beta^{-1}(v)])w +\\
&\quad [T\rho(a,b)\beta^{-1}(u),Tv,Tw] - [[a,b,T\beta^{-1}(u)],Tv,Tw] - T\rho(T\rho(a,b)\beta^{-1}(u),Tv)w +\\
&\quad T\rho([a,b,T\beta^{-1}(u)],Tv)w\\
&= -T\rho(a,b)\rho(\alpha^{-1}(Tu),\alpha^{-1}(Tv))\beta^{-1}(w) + [a,b,[T\beta^{-1}(u),T\beta^{-1}(v),T\beta^{-1}(w)]] +\\
&\quad T\rho(Tu,Tv)\rho(a,b)\beta^{-1}(w) - [Tu,Tv,[a,b,T\beta^{-1}(w)]] + T\rho(Tu,T\rho(a,b)\beta^{-1}(v))w -\\
&\quad [Tu,[a,b,T\beta^{-1}(v)],Tw] - T\rho(Tu,T\rho(a,b)\beta^{-1}(v))w + T\rho(Tu,[a,b,T\beta^{-1}(v)])w +\\
&\quad T\rho(T\rho(a,b)\beta^{-1}(u),Tv)w - [[a,b,T\beta^{-1}(u)],Tv,Tw] - T\rho(T\rho(a,b)\beta^{-1}(u),Tv)w +\\
&\quad T\rho([a,b,T\beta^{-1}(u)],Tv)w = 0.
\end{aligned}
$$

Therefore, $\delta_T(\wp(a,b)) = 0$. □

Next we define the cohomology theory of an embedding tensor T on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$. For $n \geq 0$, define the set of n-cochains of T by $\mathcal{C}^n_T(V,L) := \mathcal{C}^n_{\mathrm{3HLei}}(V,L)$. Then $(\oplus_{n=0}^{\infty}\mathcal{C}^n_T(V,L),\delta_T)$ is a cochain complex. For $n \geq 1$, we denote the set of n-cocycles by $\mathcal{Z}^n_T(V,L)$, the set of n-coboundaries by $\mathcal{B}^n_T(V,L)$ and the n-th cohomology group of the embedding tensor T by $\mathcal{H}^n_T(V,L) = \mathcal{Z}^n_T(V,L)/\mathcal{B}^n_T(V,L)$.

4. Deformations of embedding tensors on 3-Hom-Lie algebras

In this section, we discuss linear deformations of embedding tensors on 3-Hom-Lie algebras. we show that if two linear deformations of an embedding tensor on a 3-Hom Lie algebra are equivalent, then their infinitesimals belong to the same cohomology class in the first cohomology group.

Definition 4.1 *Let $T : V \to L$ be an embedding tensor on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$, and let $\mathfrak{I} : V \to L$ be a linear map. If $T_t = T + t\mathfrak{I}$ is an embedding tensor on $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$ for all t, we say that $\mathfrak{I}$ generates a linear deformation of the embedding tensor T.*

Suppose that $\mathfrak{I}$ generates a linear deformation of the embedding tensor T, then we have

$$
\begin{aligned}
&T_t \circ \beta = \alpha \circ T_t,\\
&[T_t u, T_t v, T_t w] = T_t\rho(T_t u, T_t v)w,
\end{aligned}
$$

for all $u,v,w \in V$. This is equivalent to the following conditions

$$\mathfrak{I} \circ \beta = \alpha \circ \mathfrak{I}, \tag{4.1}$$

$$[Tu,Tv,\mathfrak{I}w] + [Tu,\mathfrak{I}v,Tw] + [\mathfrak{I}u,Tv,Tw] = \mathfrak{I}\rho(Tu,Tv)w + T\rho(Tu,\mathfrak{I}v)w + T\rho(\mathfrak{I}u,Tv)w, \tag{4.2}$$

$$[Tu,\mathfrak{I}v,\mathfrak{I}w]+[\mathfrak{I}u,Tv,\mathfrak{I}w]+[\mathfrak{I}u,\mathfrak{I}v,Tw]=\mathfrak{I}\rho(Tu,\mathfrak{I}v)w+\mathfrak{I}\rho(\mathfrak{I}u,Tv)w+T\rho(\mathfrak{I}u,\mathfrak{I}v)w, \quad (4.3)$$

$$[\mathfrak{I}u,\mathfrak{I}v,\mathfrak{I}w]=\mathfrak{I}\rho(\mathfrak{I}u,\mathfrak{I}v)w, \quad (4.4)$$

for all $u,v,w\in V$. Thus, T_t is a linear deformation of T if and only if Eqs. (4.1)–(4.4) hold. From Eqs. (4.1) and (4.4) it follows that the map $\mathfrak{I}$ is an embedding tensor on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$.

Proposition 4.2 *Let $T_t=T+t\mathfrak{I}$ be a linear deformation of an embedding tensor T on a 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the representation $(V;\rho,\beta)$. Then $\mathfrak{I}\in\mathcal{C}^1_T(V,L)$ is a 1-cocycle of the embedding tensor T. Moreover, the 1-cocycle $\mathfrak{I}$ is called the infinitesimal of the linear deformation T_t of T.*

Proof We observe that Eq. (4.2) implies that $\delta_T\mathfrak{I}=0$. □

Next, we discuss equivalent linear deformations.

Definition 4.3 *Let T be an embedding tensor on the regular 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the regular representation $(V;\rho,\beta)$. Two linear deformations $T^1_t=T+t\mathfrak{I}_1$ and $T^2_t=T+t\mathfrak{I}_2$ are said to be equivalent if there exist two elements $a,b\in L$ such that $\alpha(a)=a,\alpha(b)=b$ and the pair $(\mathrm{Id}_L+t\alpha^{-1}(\mathrm{ad}(a,b)),\mathrm{Id}_V+t\beta^{-1}(\rho(a,b)))$ is a homomorphism from T^2_t to T^1_t.*

Let us recall from Definition 2.9 that the pair $(\mathrm{Id}_L+t\alpha^{-1}(\mathrm{ad}(a,b)),\mathrm{Id}_V+t\beta^{-1}(\rho(a,b)))$ is a homomorphism from T^2_t to T^1_t if the following conditions are true:

(1) The map $\mathrm{Id}_L+t\alpha^{-1}(\mathrm{ad}(a,b)):L\to L, x\mapsto x+t\alpha^{-1}(\mathrm{ad}(a,b)x)$ is a 3-Hom-Lie algebra homomorphism;

(2) $(T+t\mathfrak{I}_1)(u+t\beta^{-1}(\rho(a,b)u))=(\mathrm{Id}_L+t\alpha^{-1}(\mathrm{ad}(a,b)))(Tu+t\mathfrak{I}_2u)$;

(3) $\rho(x,y)u+t\beta^{-1}(\rho(a,b)\rho(x,y)u)=\rho\big(x+t\alpha^{-1}(\mathrm{ad}(a,b)x),y+ t\alpha^{-1}(\mathrm{ad}(a,b)y)\big)(u+t\beta^{-1}(\rho(a,b)u)),\ \forall\, x,y\in L,u\in V.$

Theorem 4.4 *Let T be an embedding tensor on a regular 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with respect to the regular representation $(V;\rho,\beta)$. If two linear deformations $T^1_t=T+t\mathfrak{I}_1$ and $T^2_t=T+t\mathfrak{I}_2$ of T are equivalent, then $\mathfrak{I}_1$ and $\mathfrak{I}_2$ define the same cohomology class in $\mathcal{H}^1_T(V,L)$.*

Proof Comparing coefficients of t^1 on both sides of equation in condition (2), we have

$$\begin{aligned}\mathfrak{I}_2u-\mathfrak{I}_1u&=T\beta^{-1}(\rho(a,b)u)-\alpha^{-1}([a,b,Tu])\\&=T\rho(a,b)\beta^{-1}(u)-[a,b,T\beta^{-1}(u)]\\&=\wp(a,b)u\in\mathcal{B}^1_T(V,L),\end{aligned}$$

which implies that $\mathfrak{I}_2$ and $\mathfrak{I}_1$ belong to the same cohomology class in $\mathcal{H}^1_T(V,L)$. □

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