A Priori Estimates for a Quasilinear Elliptic PDE

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Abstract In the recent past many results have been established on positive solutions to boundary value problems of the form

\[
- \operatorname{div}( |Du(x)|^{p-2}Du(x)) = \lambda f(u(x)) \quad \text{in} \quad \Omega,
\]

\[
u(x) = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \lambda > 0 \), \( \Omega \) is a bounded smooth domain and \( f(s) \geq 0 \) for \( s \geq 0 \). In this paper we study a priori estimates of positive radial solutions of such problems when \( N > p > 1 \), \( \Omega = B_1 = \{x \in \mathbb{R}^N; |x| < 1\} \) and \( f \in C^1(0, \infty) \cap C^0([0, \infty)) \), \( f(0) = 0 \).

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1. Introduction

In this paper, we consider the set of positive radial solution to the following quasilinear elliptic PDE

\[
\operatorname{div}( |Du|^{p-2}Du) + \lambda f(u) = 0 \quad \text{in} \quad \Omega,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \Omega \) denotes the unit ball in \( \mathbb{R}^N (N \geq p) \), and \( \lambda > 0 \). Here \( f : [0, \infty) \to \mathbb{R} \) satisfies

\[
f(0) = 0, \quad f'(u) \geq 0, \quad \lim_{u \to \infty} \frac{f(u)}{u^q} = L_0 > 0, \quad \lim_{u \to \infty} \frac{f(u)}{u^q} = L_0^* \]

for some \( q \) with \( p - 1 < q < ((p - 1)N + p)/(N - p) \). For \( N = p \) we have \( p - 1 < q < \infty \).

Problems (1.1) - (1.2) arises from many branches of mathematics and applied mathematics. The existence and uniqueness of the positive solutions of (1.1) - (1.2) have been studied by many authors. See, for example, [1 - 12] and the references therein.

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A natural question which then arises is to determine how $\lambda$ and $d = \max_\Omega u$ are related. When $p = 2$, the related results have been obtained by [13]. In this paper, we will prove the following theorem.

**Theorem 1** If $u$ is a solution of (1.1), (1.2) with $d = \max_\Omega u$, and $f$ satisfies (1.3) and (1.4), then for small $\lambda$ we have

$$
\left( \frac{p-1}{p} \right)^{p-1} (N - 1) \leq \frac{\lambda f(d)}{d^{p-1}} \leq C,
$$

where $C$ is a constant that is independent of $\lambda$ and $d$.

Assuming now that (1.1), (1.2) has a positive radial solution, let us denote it as $\lambda$. Letting $d = \max_\Omega u$, we then define $\lambda = u/\lambda$. Then $\lambda$ satisfies

$$
\text{div}(|Dv|^{p-2}Dv) + \frac{\lambda}{d^{p-1}} f(v) = 0 \text{ in } \Omega,
$$

$$
v = 0 \text{ on } \partial\Omega,
$$

$$
0 < v \leq 1 \text{ in } \Omega
$$

with the aid of theorem 1, we will also prove the following theorem.

**Theorem 2** If $\lambda$ satisfies (1.1), (1.2) and $f$ satisfies (1.3) and (1.4) then (for some subsequence) $\lim_{\lambda \to 0} \lambda = v$ and $v$ is a positive solution of

$$
\text{div}(|Dv|^{p-2}Dv) + L_1 v = 0 \text{ in } \Omega,
$$

$$
v = 0 \text{ on } \partial\Omega,
$$

where $L_1 = \lim_{\lambda \to 0} \lambda f(v)/\lambda$.

**Theorem 3** Assume that $f$ satisfies (1.3) and (1.4). Let $\lambda_n \to 0$ and $u_n$ be positive radial solutions of (1.1) - (1.2) for $\lambda = \lambda_n$ such that $d = \|u_n\|_{\infty} \to \infty$ as $n \to \infty$. Then $(L_{\lambda_n}^{1/(q-p+1)} u_n \to \omega_q$ in $C^1(\Omega) \text{ as } n \to \infty$, where $\omega_q$ is a positive radial solution of the problem

$$
\text{div}(|D\omega|^{p-2}D\omega) = \omega^q \text{ in } \Omega, \quad \omega = 0 \text{ on } \partial\Omega.
$$

2. Preliminaries

We consider positive solution of (1.1), (1.2) are radial solutions, thus, $u = u(r, \lambda)$ satisfies

$$
(\Phi_p(u'))' + \frac{N-1}{r} \Phi_p(u') + \lambda f(u) = 0,
$$

$$
u(0) = d, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(r) < 0 \text{ for } 0 < r < 1.
$$

Multiplying (2.1) by $u'$ and integrating on $(0,1)$ gives

$$
(p-1)/p \int_0^1 u'(1) \quad p + \int_0^1 (N-1)/r \quad u' \quad p dr = \lambda F(d),
$$

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where \( F(u) = \int_0^u f(s) \, ds \).

Another useful identity is obtained by multiplying (2.1) by \( r^{N-1} \) and integrating on \((0, r)\). This gives

\[
- r^{N-1} \Phi_p(u') = \lambda \int_0^r s^{N-1} f(u(s)) \, ds. \tag{2.4}
\]

Now using the fact that \( f \) is increasing, \( u \) is decreasing and \( u \geq 0 \), we obtain

\[
- u' \geq \left( \frac{\lambda f(u)}{N} \right)^{1/(p-1)}. \tag{2.5}
\]

3. Proof of Theorem

We first recall a Pohozaev identity which was obtained by Ni and Serrin\[^8\].

**Lemma 3.1** Let \( u(r) \) be a solution of (2.1) in \((r_1, r_2) \subset (0, \infty)\) and let \( a \) be an arbitrary constant. Then, for each \( r \in (r_1, r_2) \) we have

\[
\frac{d}{dr} \left[ r^n (1 - a/ p) | u' |^p + F(u) + \frac{a}{r} uu' | u' |^{p-2} \right] = r^{n-1} \left[ nF(u) - aF(u) + (a + 1) - n/a \right] | u' |^p. \tag{2.3}
\]

**Proof of Theorem 1** The left hand side of the inequality mentioned in theorem 1 above is now established with identity (2.3). First, using Hölder’s inequality we have

\[
d = u(0) - u(1) = \int_0^1 - u' \, ds = \int_0^1 \frac{u'}{s} \, dF(r) \leq (\int_0^1 (- u')^p / r^d r)^{1/p} \left( \frac{p-1}{p} \right)^{(p-1)/p}.
\]

Using (2.3) we now obtain

\[
d^p \leq (\frac{p-1}{p})^{p-1} \int_0^1 (- u')^p / r \, dF \leq (\frac{p-1}{p})^{p-1} \frac{\lambda F(d)}{(N-1)}.
\]

Finally, since \( f' \geq 0 \) we have

\[
F(d) = \int_0^d f(s) \, ds = df(d) - \int_0^d sf'(s) \, ds \leq df(d).
\]

Therefore,

\[
((p-1)/p)^{p-1} (N-1) \leq \frac{\lambda f(d)}{d^{p-1}}. \tag{3.1}
\]

Thus, the left hand side of the inequality in theorem is established.

To obtain the other half of theorem 1, we begin with inequality (2.5)

\[
- u' \geq \left( \frac{\lambda f(u)}{N} \right)^{1/(p-1)}.
\]

Next, using (1.3) and (1.4) we see that there exists a \( C > 0 \) such that

\[
- u' \geq \left( \frac{\lambda f(u)}{N} \right)^{1/(p-1)} \geq \left( \frac{\Delta ru^q}{N} \right)^{1/(p-1)}.
\]
Dividing by \( u^{(q'-p')/(p'-1)} \) and integrating this inequality we obtain
\[
\frac{1}{u^{(q'-p')/(p'-1)}} - \frac{1}{d^{(q'-p')/(p'-1)}} \geq \partial^1/ (p'-1) r^{p'/ (p'-1)}.
\]

Therefore,
\[
\left( \frac{u}{d} \right)^{(q'-p')/(p'-1)} \leq \frac{1}{1 + \partial^1/ (p'-1) d^{(q'-p')/(p'-1)} r^{p'/ (p'-1)}}.
\] (3.2)

We will now estimate \( |u' (1)/d| \) from above and below. This will in turn lead to an upper bound for \( \lambda d^{q'-p'+1} \| f (d) \| / d^{p'-1} \) for small \( \lambda \) (and thus large \( d \) by (3.1)).

First, we estimate \( |u' (1)/d| \) from above. From (2.4) and (1.3) and (1.4) we have
\[
|u' (1)|^{p'-1} = \lambda \int_0^1 s^{N-1} f (u) \, ds \leq \partial \int_0^1 s^{N-1} u^q \, ds.
\]

Using (3.2), we obtain
\[
|\frac{u'}{d} (1)|^{p'-1} \leq \partial d^{q'-p'+1} \int_0^1 s^{N-1} (u/d)^q \, dr
\leq \partial d^{q'-p'+1} \int_0^1 \frac{s^{N-1} \, dr}{1 + \partial d^{q'-p+1} d^{p'-1} r^{p'/ (p'-1)}}.
\]

Letting
\[
s = (\lambda d^{q'-p+1})^{1/p} r,
\]
\[
ds = (\lambda d^{q'-p+1})^{1/p} d r,
\]
we obtain
\[
|\frac{u'}{d} (1)|^{p'-1} \leq \frac{1}{C(\lambda d^{q'-p+1})^{(N-1)/p}} \int_0^{\lambda d^{q'-p+1}} \frac{s^{N-1} \, ds}{1 + Cs^{p'/ (q'-p+1)}}.
\] (3.3)

Now we estimate \( |u' (1)/d| \) from below.

We return to (2.1), (2.2)
\[
(\Phi_p (u'))' + \frac{N}{r} \Phi (\Phi) + \lambda f (u) = 0 \quad 0 < r < 1,
\]
\[
u (0) = d > 0, \quad u' (0) = 0, \quad u (1) = 0, \quad u' (r) < 0.
\]

We define
\[
E (r) = (p-1)/p \mid u' \mid^p (r) + \lambda F (u (r))
\]
and
\[
H (r) = r E (r) + (N-1)/p uu' (r) \mid u' \mid^{p-2}.
\]

By Lemma 3.1, we obtain on \( (r_0, r_1) \) after a lengthy computation
\[
r_1^{N-1} H (r_1) - r_0^{N-1} H (r_0) = \int_{r_0}^{r_1} \lambda r^{N-1} [ NF (u) - (N-1)/p uf (u) ] \, d r.
\] (3.4)
Let $t_0$ be such that $d \geq u(r) \geq kd$ for all $0 \leq r \leq t_0$ and $u(t_0) = kd$ for $(\frac{N-p}{Np})^{1/(q+1)} < k < 1$.

Then from (2.4) we have

$$(( - u')^{p-1} = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(u) \, ds$$

and hence

$$\lambda f(kd) \, r \leq N ( - u')^{p-1} \leq \lambda f(d) \, r$$
on $[0, t_0]$. Integrating on $[0, t_0]$ gives

$$C_1 \left( \lambda f(d) \right)^{N/p} \leq t_0 \leq C_1 \left( \lambda f(kd) \right)^{N/p}, \quad (3.5)$$

where $C_1 = \left( \frac{p}{(p-1)} \right)^{1/(p-1)} (1-k)^{p-1} N^{1/p} > 0$. Substituting $r_0 = 0$ and $r_1 = t_0$ in (3.4) and using (3.5) gives

$$\int_0^{t_0} \lambda r^{N-1} (NF(u) - (N-p)/p) \, dr$$

$$\leq \int_0^{t_0} \lambda r^{N-1} (NF(kd) - (N-p)/p) \, dr$$

$$\geq \lambda \left( \frac{NF(kd) - (N-p)/p}{p} \right) \left( \frac{d^{p-1}}{f(d)} \right)^{N/p} \left( \frac{1}{\lambda^{N/p}} \right). \quad (3.6)$$

Since $f$ satisfies (1.3) and (1.4), we then have that there exists $A \geq 0$ such that

$$NF(u) - (N-p)/p \geq u$$

for all $u$. Thus, we have by (1.3), (1.4), (3.4) and (3.6)

$$(p-1)/p \left| u' \right|^{p-1} = H(1) \geq \frac{C}{\lambda} \left( \frac{NF(kd) - (N-p)/p}{p} \right) \int_0^{t_0} \left( \frac{d^{p-1}}{f(d)} \right)^{N/p}$$

$$(1 - t_0^N)/N \geq \lambda \left( \frac{p-N}{p} \right) d^{(N(p-1)+p-q(N-p))/p} - A \lambda$$

Therefore,

$$\left| u' \right|^{p-1} \left( \frac{1}{d^p} \right) \geq \frac{C}{(\lambda d^{q-p+1})^{(N-p)/p}}. \quad (3.7)$$

Hence, combining (3.3) and (3.7) gives

$$\frac{C}{(\lambda d^{q-p+1})^{(N-p)/p}} \leq \left| u' \right| \left( \frac{1}{d^p} \right) \quad (3.8)$$

Now we let $T = (\lambda d^{q-p+1})^{1/p}$, and by rewriting (3.8) we see that we have

$$CT^{(N-p)/(p-1)} \leq \left( \int_0^T \frac{s^{N-1} ds}{1 + Cs^{p/(q-p+1)}} \right)^{p/(p-1)}. \quad (3.9)$$
We want to show now that $T$ is bounded as $\lambda \to 0$. We need to consider separately the cases $N > p$ and $N = p$.

**Case I $\square N > p$**

Suppose now that $T \to 0$ as $\lambda \to 0$. We will show that this is impossible. First, if the term on the right is bounded then we are done (this is the case if $q < \frac{(p - 1) N}{(N - p)}$).

Assuming, therefore, that term goes to infinity as $\lambda \to 0$ we divide both sides by $T^{(N - p)}$ and get

$$0 < C^{p - 1} \leq \left( \int_0^T \frac{s^{N - 1} ds}{1 + C s^{pq/(q - p + 1)}} \right)^p.$$

We will now show that if $T \to \infty$ as $\lambda \to 0$ then the right hand side of the above goes to zero and thus contradicts the above inequality. Applying L’Hôpital’s rule we obtain

$$0 < C^{p - 1} \leq \left( \lim_{T \to \infty} \frac{T^{N - 1}}{(1 + CT^{pq/(q - p + 1)})^p} \right)^p$$

$$= \left( \lim_{T \to \infty} \frac{T^{N - 1}}{(1 + CT^{pq/(q - p + 1)})^p} \right)^p$$

$$= \left( \frac{p}{N - p} \right)^p \left( \lim_{T \to \infty} \frac{T^{1 - (p - 1)N + p}}{1 + CT^{pq/(q - p + 1)}} \right)^p$$

$$= \left( \frac{p}{N - p} \right)^p \left( \lim_{T \to \infty} \frac{T^{1 - (p - 1)N + p}}{1 + CT^{pq/(q - p + 1)}} \right)^p.$$

As $T \to \infty$ this goes to zero for $p - 1 < q < \frac{(p - 1) N}{(N - p)}$ and $N > p$. Hence we obtain the desired contradiction.

**Case II $\square N = p$**

For $N = p$, we can only conclude from (3.3) ad (3.7) (since $A\lambda / d^p \to 0$ as $\lambda \to 0$) that

$$0 < c_1 < | \frac{\mu A(1)}{d} | \leq \left( \int_0^T \frac{s^{N - 1} ds}{1 + C s^{pq/(q - p + 1)}} \right)^{1/(p - 1)}$$

$$\leq \left( \int_0^\infty \frac{s^{N - 1} ds}{1 + C s^{pq/(q - p + 1)}} \right)^{1/(p - 1)} \leq c_2 < \infty,$$

where $c_1, c_2$ are independent of $\lambda$. We now define

$$\lambda = \frac{A\lambda}{\lambda}.$$

Assuming that $T \to \infty$ as $\lambda \to 0$ we conclude from (3.2) that

$$\lambda \to 0 \quad (3.10)$$

uniformly say for $1/2 \leq r \leq 1$.

**Claim $\square \lambda, v\lambda, (\phi_p(v\lambda))'$ are uniformly bounded on $[1/2, 1]$**.

Once the claim is proven, we can then conclude that there is a subsequence with $\lambda \to v$ and $v\lambda \to v'$ uniformly on $[1/2, 1]$. From (3.10), we have that $v \equiv 0$ on $[1/2, 1]$. On the other hand, we have that

$$0 < c_1 \leq | v\lambda(1) | \leq c_2.$$
Therefore, \( v'(1) = \lim_{\lambda \to 0} v \lambda(1) \leq -c_1 < 0 \). Thus, \( v > 0 \) on \((1 - \varepsilon, 1)\). This contradicts that \( v \equiv 0 \) on \([1/2, 1]\). Thus, we only need to prove the claim.

**Proof of Claim**
Recalling (2.4) and (3.2) gives

\[
\begin{align*}
\lambda^{q' \cdot p + 1} \int_0^r s^{q' \cdot p + 1} v d s & \leq \lambda^{d^q \cdot p + 1} r^{N' \cdot p} \int_0^r s^{p' - 1} v d s \\
& \leq \lambda^{d^q \cdot p + 1} r^{N' \cdot p} \int_0^r \frac{s^{p' - 1} v d s}{1 + C(\lambda^{d^q \cdot p + 1}) q'(q' \cdot p + 1) s^{q'}(q' \cdot p + 1)}.
\end{align*}
\]

Letting

\[
\begin{align*}
t & = (\lambda^{d^q \cdot p + 1})^{1/p} s, \\
dt & = (\lambda^{d^q \cdot p + 1})^{1/p} ds,
\end{align*}
\]

gives

\[
(-v')^{p - 1} \leq \frac{\lambda^{d^q \cdot p + 1}}{1 + Ct^{q'/(q' \cdot p + 1)}} \int_0^r \frac{t^{p - 1} dt}{1 + Ct^{q'/(q' \cdot p + 1)}} = B < \infty.
\]

Thus, for \( r \in [1/2, 1] \) we have

\[
|v'| \leq B,
\]

where \( B \) is independent of \( \lambda \). From the differential equation we have

\[
|\Phi_p(v')'| = \frac{N - 1}{r} |v'|^{p - 1} + \frac{\lambda}{d^{p - 1}} f (vd).
\]

Using (3.2) again gives

\[
\begin{align*}
|\Phi_p(v')'| & \leq B^{p - 1} (N - 1) / r + \lambda v^q d^q \cdot p + 1 \\
& \leq B^{p - 1} (N - 1) / r + \frac{\lambda d^{q' \cdot p + 1}}{C(\lambda d^{q' \cdot p + 1}) q'(q' \cdot p + 1)} \\
& \leq B^{p - 1} (N - 1) / r + \frac{1}{(\lambda d^{q' \cdot p + 1}) (p - 1)/(q' \cdot p + 1)} \\
& \leq C
\end{align*}
\]

on \([1/2, 1]\) because by assumption \( \lambda d^{q' \cdot p + 1} \to \infty \) as \( \lambda \to 0 \). Thus

\[
v, |v'|, |\Phi_p(v')'| \leq C
\]

and this completes the proof of the claim.

**Proof of Theorem 2**
From (2.4) we have that

\[
\begin{align*}
\lambda^{d^q \cdot p + 1} \int_0^r s^{N' \cdot p} f (\lambda d) d s.
\end{align*}
\]

From theorem 1, we have that \( \lambda f (d) / d^{p - 1} \leq C \). Thus, since \( f'(u) \geq 0 \) we have that

\[
\begin{align*}
\lambda^{d^q \cdot p + 1} \int_0^r s^{N' \cdot p} f (\lambda d) d s & \leq C \frac{N}{N'}
\end{align*}
\]

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Hence,

\[ \frac{|v'|}{r} p^{-1} \leq C. \]

Substituting back into (2.1) we further obtain

\[ \left| (\Phi_p(v\lambda))' \right| \leq \frac{N-1}{r} \left| v\lambda \right| p^{-1} + \frac{\lambda f(d)}{dp^{-1}} \leq C. \]

so combining the above and recalling that \( 0 \leq \lambda \leq 1 \) we obtain that

\[ \lambda, \left| v\lambda \right|, \left| (\Phi_p(v'))' \right| \leq C. \]

Thus, by the Arzela - Ascoli theorem, there is a subsequence of the \( \lambda \) (still denoted \( \lambda \)) such that

\[ \Phi_p(v\lambda) \to \omega \Rightarrow v\lambda \to \Phi_p^{-1}(\omega) = v' \text{ uniformly}, \]

\[ \lambda = \int_1^r v\lambda(s) \, ds \to \int_1^r \Phi_p^{-1}(\omega) \, ds = v \text{ uniformly}. \]

For some further subsequence (again denoted \( \lambda \)), we have

\[ \lim_{\lambda \to 0} \frac{\lambda f(d)}{dp^{-1}} = L_1 < \infty. \]

Since \( \lambda \to v \) uniformly on \([0, 1]\), we also have that \( v(0) = 1 \) and \( v(1) = 0 \). Now, since \( L_1 < \infty \) we have by (3.8) that \(- v'(1) = \lim_{\lambda \to 0} v\lambda(1) = c > 0 \). Thus, for all \( \varphi > 0 \) there exists \( \delta > 0 \) such that \( \lambda \geq \delta \) for \( 0 \leq r \leq 1 - \delta \). Returning to

\[ (- v\lambda) p^{-1} = \frac{\lambda}{r^{N-1} dp^{-1}} \int_0^r r^{N-1} f(\lambda d) \, dr, \]

for \( 0 < r \leq 1 - \delta < 1 \), this converges to

\[ (- v') p^{-1} = \frac{L_1}{r^{N-1}} \int_0^r r^{N-1} v^q \, dr. \]

Thus, \( v \) is a solution of

\[ (\Phi_p(v'))' + \frac{N-1}{r} \Phi_p(v') + L_1 v^q = 0 \]

on \( 0 < r \leq 1 - \delta \). This holds for all \( \delta > 0 \). Now as \( \varphi \to 0 \) also \( \delta \to 0 \), thus \( v \) is a solution on \( 0 < r < 1 \). Further, \( v > 0 \) on \([0, 1]\). This completes the proof of Theorem 2.

**Proof of Theorem 3** Let \( u_n = \| u_n \| \infty u_n \). Then, \( v_n(r) \) satisfies \( \| v_n \| \infty = 1 \) and

\[ - (r^{N-1} | v_n'' | p^{-2} r^{N-1} v_n')' = \lambda_n r^{N-1} (\| u_n \| \infty)^q p^{1-1} f(\| u_n \| \infty)/\| u_n \| \infty) \in (0, 1), \]

\[ v_n'(0) = 0, v_n(1) = 0. \]

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We shall show that the limit of \( \lambda n \| u_n \|_{q} \) is a non-zero number as \( n \to \infty \). In fact, we shall see that \( \lambda n \| u_n \|_{q} \to 0 \) as \( n \to \infty \). On the contrary, if \( \lambda n \| u_n \|_{q} \to 0 \) as \( n \to \infty \), then let \( \lambda n \| u_n \|_{q} = T_n \) and \( T_n \to 0 \) as \( n \to \infty \). Thus,

\[
- \text{div}(Dv_n | p-2 Dv_n) = T_n \left\{ f(\| u_n \|_{\infty}) / \| u_n \|_{q} \right\}.
\]

This and the regularity of \(- \text{div}(D \| p-2 D)\) implies that \( v_n \to 0 \) as \( n \to \infty \), since \( f(\| u_n \|_{\infty}) / \| u_n \|_{q} \) is uniformly bounded. This contradicts the fact that \( \| v_n \| = \infty = 1 \) for all \( n \). By proof of theorem 1, \( T_n \) is bounded. Let \( z_n = (L_0 T_n)^{1/(q-p+1)} v_n \) and \( T = \lim_{n \to \infty} T_n \). Then, the arguments above imply that \( z_n \to z \) in \( C^1(\Omega) \) and \( z = (L_0 T)^{1/(q-p+1)} v \). Here \( v \) satisfies

\[
- \left( r^{N-1} | v' | \right)^{\prime} + \lambda F(v) = 0 \quad \text{in} \quad (0,1),
\]

\[
v'(0) = 0, \quad v(1) = 0.
\]

Hence, \( z \) satisfies the problem

\[
- \left( r^{N-1} | z' | \right)^{\prime} + \lambda F(z) = 0 \quad \text{in} \quad (0,1)
\]

and

\[
z'(0) = 0, \quad z(1) = 0.
\]

This implies that

\[
(L_0 \lambda_n)^{1/(q-p+1)} u_n \to z \quad \text{in} \quad \Omega.
\]

This completes the proof of this theorem.

4. The \( N = 1 \) Case

For \( N = 1 \), equations (1.1) and (1.2) become

\[
(\Phi_p(u'))' + \lambda f(u) = 0 \quad \text{in} \quad 0 < r < 1,
\]

\[
u(0) = 0, \quad u(1) = 0, \quad u'(0) = A.
\]

Note that \( u \) must be symmetric about \( r = 1/2 \). We therefore denote \( u(1/2) = d = \max_{\Omega} u \). Multiplying by \( u' \) and integrating on \( (0, r) \) gives

\[
\frac{p-1}{p} | u' | \left( r \right)^{p-2} + \lambda F(u) = \frac{p-1}{p} A^p,
\]

where \( F(u) = \int_0^u f(s) \, ds \). Thus, at \( r = 1/2 \) we obtain

\[
p/ (p - 1) \lambda F(d) = A^p.
\]

Thus,

\[
| u' |^p = p/ (p - 1) \lambda \left[ F(d) - F(u) \right].
\]

Therefore, integrating on \( (0, 1/2) \), we have

\[
\int_0^{1/2} \frac{u' \, dr}{\sqrt{F(d) - F(u)}} = 1/2 \left( \frac{p}{\lambda} \right)^{1/p} N p - 1.
\]
Letting $s = u(r)$, then $ds = u'(r) \, dr$, gives

$$\int_0^d \frac{ds}{\sqrt{F'(d) - F(s)}} = 1/2 \sqrt{N} \lambda.$$

Thus,

$$\int_0^d \frac{f(d) \, ds}{N F'(d) - F(s)} = 1/2 \sqrt{N} \lambda f(d).$$

Letting $t = s/d$ gives $dt = ds/d$ and, hence

$$\int_0^1 \frac{f(d) \, dt}{N F'(d) - F(td)} = 1/2 \sqrt{N} \lambda f(d).$$

Thus, to show that $\lambda f(d) / d^{p-1}$ is bounded, we only need to consider the left hand side of the above equation. Using (1.3) it is straightforward to see that

$$\lim_{d \to \infty} \frac{f(d) \, dt}{N F'(d) - F(td)} = \frac{q+1}{1 - t^{q+1}}.$$

Thus, we would like to such that

$$\lim_{d \to \infty} \int_0^1 \frac{f(d) \, dt}{N F'(d) - F(td)} = \sqrt{p+1} \int_0^1 \frac{dt}{1 - t^{q+1}}.$$

In order to do this, we will break the integral into two pieces and show that each is finite. For $0 \leq t \leq 1/2$ we have

$$\lim_{d \to \infty} \sup \int_0^{1/2} \frac{f(d) \, dt}{N F'(d) - F(td)} \leq \lim_{d \to \infty} \sup \int_0^{1/2} \frac{f(d) \, dt}{2 N p (F'(d) - F(1/2) d))} = \frac{p^{q+1}(q+1)}{2 p^{q+1} - 2^p}.$$

For $1/2 \leq t \leq 1$ we have $F(d) - F(td) = f(c) (1 - t) d$, where $td \leq c \leq d$. Since $f$ is increasing we have $f(c) \geq f(td)$. Thus,

$$\lim_{d \to \infty} \sup \int_{1/2}^1 \frac{f(d) \, dt}{N F'(d) - F(td)} \leq \lim_{d \to \infty} \sup \int_{1/2}^1 \frac{f(d) \, dt}{N f(d/2) \int_{1/2}^1 \frac{dt}{1 - t}} = p/ (p - 1) 2^{1(q - 1)p+1}.$$

Thus, by the dominated convergence theorem we have

$$\lim_{\lambda \to 0} \frac{\lambda f(d) \, dt}{N (p - 1) d^{p-1}} = \lim_{d \to \infty} \int_0^1 \frac{f(d) \, dt}{N F'(d) - F(td)} = \sqrt{p+1} \int_0^1 \frac{dt}{1 - t^{q+1}} < \infty.$$

The proof of Theorem 2 for the case $N = 1$ is similar to the proof for $N \geq p$ described earlier.
References


