An Unsolved Problem by Feller

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In Feller’s well-known book, Volume II, second edition [1], there is a very interesting problem:
No.29 on p.288 with a large Hint, as follows. If a probability distribution $F$ has the property:
for some $\alpha > 0$,
$$\lim_{x \to -\infty} x^\alpha \{F(-x) + 1 - F(x)\} = C, \quad 0 < C < \infty,$$
we say that $F$ and any random variable with distribution $F$ have the “Pareto Property”. Feller
proved an important result (on p.279) as follows.

Let $\{X_k, k \in N\}$ be a sequence of independent and identically distributed random variables,
and $S_n = \sum_{k=1}^{n} X_k$. If $X_1$ has the Pareto property, then the normed sum $S_n/n^{1/\alpha}$
for each $n$ also has the same property with the same $\alpha$. The proof is not hard and uses induction on $n$.

Now it is possible to use a very simple distribution $F$ with the Pareto property for the
common distribution of the $X_k$'s, e.g. let $F$ have the density function

$$p(x) = \begin{cases} \frac{\alpha}{\alpha x}, & \text{if } |x| > 1, \\ 0, & \text{if } |x| \leq 1, \end{cases}$$

(1)

for a fixed $\alpha, 0 < \alpha < 2$. Then by using characteristic function (Fourier transforms of prob-
bility distribution function) we can prove that the normed sum $S_n/n^{1/\alpha}$ above will converge in
distribution to the STABLE distribution with ch.f. $e^{-C_{\alpha} |t|^\alpha}$ where the constant $C_\alpha$
can be made explicit. The reader is asked to do this.

For the proof see my book, A Course In Probability Theory, 2nd or 3rd edition [2], Theorem
6.5.4. It seems to follow from this and Feller’s lemma cited above that the limiting stable
distribution should have the Pareto property with of course the same $\alpha$. That is a famous
theorem by Paul Lévy who discovered the stable laws and proved that every stable distribution
has the Pareto property for some $\alpha, 0 < \alpha < 2$. When I made the announcement in JMRE, Vol.23,
No.4, I was not aware of Feller’s problem 29 and 30, which are aimed at a new proof of Lévy’s
theorem. Actually, it was an active young probabilist who told me Feller’s lemma, although it is
not clear from Feller’s Hint whether Feller would solve the problem in the way suggested here.
The amazing thing is that the argument given above fails to prove Lévy’s theorem, and it is
possible Feller had a different idea. Nobody I have consulted can do this problem 29, not to
mention the harder problem 30.

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In any case, a direct proof of Lévy’s theorem will be valuable. It is clear that Feller (1904–1970) worked hard to do so in his last years. The second edition of his volume II was published posthumously in 1971.

Let me re-state the PROBLEM in a most direct way, as follows. The \( X_n \) and \( S_n \) are as before.

**Hypothesis** There is a number \( \alpha \in (0, 2) \) such that all

\[
\frac{S_n}{n^{1/\alpha}}, \quad n \geq 2
\]

have the same distributions \( X_1 \equiv \frac{S_1}{1^{1/\alpha}} \).

**Conclusion** That common distribution \( F \) has the property in (1):

\[
\lim_{x \to +\infty} x^\alpha \{1 - F(x)\} = C > 0.
\]

N.B. Here \( C > 0 \).

What is the trouble? I hope the reader can find out by him/her/self. It is not hard but apparently has fooled some “experts”.

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**References:**
