Drazin Spectrum and Weyl’s Theorem for Operator Matrices

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Abstract: A ∈ B(H) is called Drazin invertible if A has finite ascent and descent. Let σ̃(A) = {λ ∈ C : A − λI is not Drazin invertible } be the Drazin spectrum. This paper shows that if MC = (A C)
0 B
is a 2 × 2 upper triangular operator matrix acting on the Hilbert space H ⊕ K, then the passage from σ̃(A) \ σ̃(B) to σ̃(MC) is accomplished by removing certain open subsets of σ̃(A) \ σ̃(B) from the former, that is, there is equality
σ̃(A) \ σ̃(B) = σ̃(MC) \ G
where G is the union of certain holes in σ̃(MC) which happen to be subsets of σ̃(A) \ σ̃(B).

Weyl’s theorem and Browder’s theorem are liable to fail for 2 × 2 operator matrices. By using Drazin spectrum, it also explores how Weyl’s theorem, Browder’s theorem, a-Weyl’s theorem and a-Browder’s theorem survive for 2 × 2 upper triangular operator matrices on the Hilbert space.

Key words: Weyl’s theorem; a-Weyl’s theorem; Browder’s theorem; a-Browder’s theorem; Drazin spectrum.

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1. Introduction

Let H and K be infinite dimensional Hilbert spaces, let B(H, K) denote the set of bounded linear operators from H to K, and abbreviate B(H, H) to B(H). If A ∈ B(H), write σ(A) for the spectrum of A and σa(A) for the approximate point spectrum of A, ρ(A) = C \ σ(A). If A ∈ B(H), we use N(A) for the null space of A and R(A) for the range of A. For A ∈ B(H), if R(A) is closed and dimN(A) < ∞, we call A upper semi-Fredholm operator, and if dimH/R(A) < ∞, then A is called lower semi-Fredholm operator. Let Φ+(H) (Φ−(H)) be the set of all upper (lower) semi-Fredholm operators. A is called Fredholm operator if dimN(A) < ∞ and dimH/R(A) < ∞. Let A be semi-Fredholm and let n(A) = dimN(A) and d(A) = dimH/R(A), then we define the index of A by ind(A) = n(A) − d(A). An operator A is called Weyl if it is a Fredholm operator of index zero, and is called Browder if it is Fredholm “of finite ascent and descent”. We write α(A) and β(A) for the ascent and the descent for A ∈ B(H) respectively. The essential spectrum
\( \sigma_e(A) \), the Weyl spectrum \( \sigma_w(A) \) and the Browder spectrum \( \sigma_b(A) \) of \( A \) are defined respectively by: \( \sigma_e(A) = \{ \lambda \in C : A - \lambda I \) is not Fredholm\}, \( \sigma_w(A) = \{ \lambda \in C : A - \lambda I \) is not Weyl\} and \( \sigma_b(A) = \{ \lambda \in C : A - \lambda I \) is not Browder\}.

Following [1, Definition 4.1] we say that \( A \in B(H) \) is Drazin invertible (with a finite index) if there exist \( B, U \in B(H) \) such that \( U \) is nilpotent and

\[
AB = BA, \quad BAB = B, \quad ABA = A + U.
\]

Recall that the concept of Drazin invertibility was originally introduced by Drazin in [2] where elements of an associative semigroup satisfying an equivalent relation were called pseudo-invertible.

It is well known that \( A \) is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that \( A = A_1 \oplus A_2 \), where \( A_1 \) is invertible and \( A_2 \) not nilpotent (see [3, Proposition A] and [4, Corollary 2.2]). It is also well known that \( A \) is Drazin invertible if and only if \( A^* \) is Drazin invertible, where \( A^* \) is the conjugate of \( A \). The Drazin spectrum of \( A \) is defined by:

\[
\sigma_D(A) = \{ \lambda \in C : A - \lambda I \) is not Drazin invertible \}.
\]

If \( G \) is a compact subset of \( C \), write \( int G \) for the interior points of \( G \); \( iso G \) for the isolated points of \( G \); \( acc G \) for the accumulation points of \( G \); and \( \partial G \) for the topological boundary of \( G \). When \( A \in B(H) \) and \( B \in B(K) \) are given we denote by \( M_C \) an operator acting on \( H \oplus K \) of the form \( M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \), where \( C \in B(K,H) \).

In Section 2, we will characterize the Drazin spectrum of \( M_C \). Our result is: For a given pair \( (A, B) \) of operators, there is equality, for every \( C \in B(K,H) \), \( \sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup G \), where \( G \) is the union of certain holes in \( \sigma_D(M_C) \) which happen to be subsets of \( \sigma_D(A) \cap \sigma_D(B) \).

In Section 3, we will use Drazin spectrum to study Weyl’s theorem. Our result is: If \( \sigma_D(A) \cap \sigma_D(B) \) has no interior points and if \( A \) is an isoloid operator for which Weyl’s theorem holds, then for every \( C \in B(K,H) \), Weyl’s theorem holds for \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Rightarrow \) Weyl’s theorem holds for \( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \).

2. Drazin Spectrum for \( 2 \times 2 \) upper triangular operator matrices

**Lemma 2.1** Suppose \( A \in B(H) \) and \( B \in B(K) \). If both \( A \) and \( B \) are Drazin invertible, then for every \( C \in B(K,H) \), \( M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) is Drazin invertible. Hence for every \( C \in B(K,H) \), \( \sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B) \).

**Proof** Suppose \( \alpha(A) = \beta(A) = p \) and \( \alpha(B) = \beta(B) = q \). Let \( n = \max\{p, q\} \).

1) First we will prove that for any \( C \in B(K,H) \), \( \alpha(M_C) < \infty \). If we have \( N(M_C^{2n+1}) = N(M_C^n) \), we get the result. So we only need to prove \( N(M_C^{2n+1}) \subseteq N(M_C^n) \).

If \( u_0 \in N(M_C^{2n+1}) \) and \( u_0 = (x_0, y_0) \), then:

\[
0 = M_C^{2n+1}(x_0, y_0) = (A^{2n+1}x_0 + A^nC y_0 + A^{n-1}CBy_0 + \cdots + A^nCB^ny_0 + \cdots + CB^n y_0, B^{2n+1}y_0).
\]
It follows that $B^{2n+1}x_0 = 0$ and
\[ A^{2n+1}x_0 + A^{2n}Cy_0 + A^{2n-1}CBy_0 + \cdots + A^nCB^n y_0 + \cdots + CB^{2n}y_0 = 0. \]

Then $y_0 \in N(B^{2n+1}) = N(B^n)$ and hence
\[ A^{2n+1}x_0 + A^{2n}Cy_0 + A^{2n-1}CBy_0 + \cdots + A^nCB^{n-1}y_0 = 0, \]
which means that $A^{n+1}[A^{n}x_0 + A^{n-1}Cy_0 + A^{n-2}CBy_0 + \cdots + CB^{n-1}y_0] = 0$, and hence
\[ A^{n}x_0 + A^{n-1}Cy_0 + A^{n-2}CBy_0 + \cdots + CB^{n-1}y_0 \in N(A^{n+1}) = N(A^n). \]

Then $A^{2n}x_0 + A^{2n-1}Cy_0 + A^{2n-2}CBy_0 + \cdots + A^nCB^{n-1}y_0 = 0$.

Now we get that
\[ (A^{2n}x_0 + A^{2n-1}Cy_0 + \cdots + A^nCB^{n-1}y_0 + A^{n-1}CB^n y_0 + \cdots + CB^{2n-1}y_0, B^{2n}y_0) = 0, \]
that is, $M^2_Cu_0 = 0$ and hence $u_0 \in N(M^2_C)$. Then $N(M^{2n+1}_C) = N(M^{2n}_C)$, and hence $M_C$ has finite ascent.

2) Secondly, we will prove that for any $C \in B(K, H)$, $M_C$ has finite descent. We will prove that $R(M^{2n}_C) = R(M^{2n+1}_C)$, so we need to prove that $R(M^{2n}_C) \subseteq R(M^{2n+1}_C)$.

For any $u_0 \in R(M^{2n}_C)$, there exist $x \in H$ and $y \in K$ such that $u_0 = M^{2n}_C(x, y)$, that is,
\[ u_0 = (A^{2n}x + A^{2n-1}Cy + A^{2n-2}CBy + \cdots + CB^{2n-1}y, B^{2n}y). \]

By $R(B^{2n}) = R(B^{2n+1})$, there exists $y_0 \in K$ such that $B^{2n}y = B^{2n+1}y_0$, then $y - By_0 \in N(B^{2n}) = N(B^n)$. Suppose $y = By_0 + y_1$, where $y_1 \in N(B^n)$. Then
\[
u_0 = (A^{2n}x + A^{2n-1}Cy_0 + A^{2n-2}CBy_0 + \cdots + A^nCB^{n-1}y_0 + A^{n-1}CB^n y_0 + A^{n-2}CB^{n+1}y_0 + \cdots + CB^{2n}y_0, B^{2n+1}y_0) = (A^{2n}x + A^{2n-1}Cy_0 + A^{2n-2}CBy_0 + \cdots + CB^{2n}y_0, B^{2n+1}y_0),
\]
\[ = A^{2n-1}Cy_1 + A^{2n-2}CBy_1 + \cdots + A^nCB^{n-1}y_1 - A^nCy_0 + A^{n}Cy_0 = A^n(A^{n}x + A^{n-1}Cy_1 + A^{n-2}CBy_1 + \cdots + CB^{n-1}y_1 - A^nCy_0) \in R(A^n) = R(A^{2n+1}). \]

Then there exists $x_0 \in H$ such that
\[ A^{2n}x + A^{2n-1}Cy_1 + A^{2n-2}CBy_1 + \cdots + A^nCB^{n-1}y_1 - A^nCy_0 = A^{2n+1}x_0. \]

Thus
\[ u_0 = (A^{2n+1}x_0 + A^{2n}Cy_0 + A^{2n-1}CBy_0 + \cdots + CB^{2n}y_0, B^{2n+1}y_0) = M^{2n+1}_C(x_0, y_0) \in R(M^{2n+1}_C). \]
So $R(M_C^{2n}) = R(M_C^{2n+1})$ and hence $M_C$ has finite descent. The proof is completed.

\begin{lemma}
\textbf{Lemma 2.2} For a given pair $(A, B)$ of operators, if $M_C$ is Drazin invertible for some $C \in B(K, H)$, then:
\begin{itemize}
  \item[(a)] $\alpha(A) < \infty$ and $\beta(A^*) < \infty$;
  \item[(b)] $\beta(B) < \infty$ and $\alpha(B^*) < \infty$.
\end{itemize}
\end{lemma}

\begin{proof}
Without loss of generality, we suppose that $0 \in \sigma(M_C)$. Suppose $\alpha(M_C) = \beta(M_C) = n < \infty$, then $\alpha(M_C^*) = \beta(M_C^*) = n$. Since $N(A^n) \oplus \{0\} \subseteq N(M_C^n)$, we get $\alpha(A) < \infty$. In order to prove $\beta(A^*) < \infty$, we only need to prove that $R(A^{*n}) = R(A^{*n+1})$. So we only need to prove that $R(A^{*n}) \subseteq R(A^{*n+1})$. For any $u \in R(A^{*n})$, let $u = A^{*n}x$, then $M_C^{*n}(x, 0) \in R(M_C^{*n}) = R(M_C^{*n+1})$. Thus there exists $(x_0, y_0) \in H \oplus K$ such that $(A^{*n}x, B^{*n-1}C^*x + \cdots + C^*A^{*-1}x) = M_C^{*n+1}(x_0, y_0) = (A^{*n+1}x_0, B^{*n+1}y_0 + B^{*n}C^*x_0 + \cdots + C^*A^{*n}x_0)$, then $u = A^{*n}x = A^{*n+1}x_0 \in R(A^{*n+1})$. Hence $\beta(A^*) < \infty$. By the same way, we can prove that $\beta(B) < \infty$ and $\alpha(B^*) < \infty$.
\end{proof}

\begin{lemma}
\textbf{Lemma 2.3} For a given pair $(A, B)$ of operators, if $M_C$ is Drazin invertible for some $C \in B(K, H)$, then $A$ is Drazin invertible if and only if $B$ is Drazin invertible.
\end{lemma}

\begin{proof}
Suppose that $A$ is Drazin invertible. Then there exists $\varepsilon > 0$ such that $A - \lambda I$ and $M_C - \lambda I$ is invertible if $0 < |\lambda| < \varepsilon$. Thus we get that $B - \lambda I$ is invertible if $0 < |\lambda| < \varepsilon$. [5, P332, Theorem 10.5] asserts that $B$ is Drazin invertible because $\beta(B) < \infty$. Conversely, if $B$ is Drazin invertible, similarly, we know that $A^*$ is Drazin invertible and hence $A$ is Drazin invertible.
\end{proof}

\begin{remark}
\textbf{Remark 2.4} In Lemma 2.1, for every $C \in B(K, H)$, we have $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$. Sometimes, this inclusion is proper for given $A$ and $B$. For example, let $A, B, C \in B(\ell^2)$ be defined by
\begin{align*}
A(x_1, x_2, x_3, \cdots) &= (0, x_1, x_2, x_3, \cdots), \\
B(x_1, x_2, x_3, \cdots) &= (x_2, x_3, x_4, \cdots), \\
C(x_1, x_2, x_3, \cdots) &= (x_1, 0, 0, \cdots).
\end{align*}
Then $\sigma(A) = \sigma(B) = \sigma_D(A) = \sigma_D(B) = \{ \lambda \in C : |\lambda| \leq 1 \}$. Since $M_C$ is a unitary operator, then $\sigma_D(M_C) \subseteq \{ \lambda \in C : |\lambda| = 1 \}$. Thus $\sigma_D(M_C)$ is proper subset in $\sigma_D(A) \cup \sigma_D(B)$.

The following is our main theorem in this section. It says that the passage from $\sigma_D(A) \cup \sigma_D(B)$ to $\sigma_D(M_C)$ is accomplished by removing certain open subsets of $\sigma_D(A) \cap \sigma_D(B)$ from the former.

\begin{theorem}
\textbf{Theorem 2.5} For a given pair $(A, B)$ of operators there is equality, for every $C \in B(K, H)$,
\begin{equation*}
\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup \mathcal{G},
\end{equation*}
where $\mathcal{G}$ is the union of certain holes in $\sigma_D(M_C)$ which happen to be subsets of $\sigma_D(A) \cap \sigma_D(B)$.
\end{theorem}
Proof} We first claim that, for every $C \in B(K, H)$,
\[ (\sigma_D(A) \cup \sigma_D(B)) \setminus (\sigma_D(A) \cap \sigma_D(B)) \subseteq \sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B). \tag{1} \]
Indeed the second inclusion in (1) follows from Lemma 2.1. For the first inclusion, let $\lambda \in (\sigma_D(A) \cup \sigma_D(B)) \setminus (\sigma_D(A) \cap \sigma_D(B))$. Then $\lambda \in \sigma_D(A) \setminus \sigma_D(B)$ or $\lambda \in \sigma_D(B) \setminus \sigma_D(A)$. Lemma 2.3 asserts that $\lambda \in \sigma_D(M_C)$ for every $C \in B(K, H)$.

Next we claim that, for every $C \in B(K, H)$,
\[ \eta(\sigma_D(M_C)) = \eta(\sigma_D(A) \cup \sigma_D(B)), \tag{2} \]
where $\eta K$ denote the “ polynomially convex hull ” of the compact set $K \subseteq C$. Since $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$ for every $C \in B(K, H)$, we need to prove that $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \partial \sigma_D(M_C)$. But since $\text{int } \sigma_D(M_C) \subseteq \text{int } (\sigma_D(A) \cup \sigma_D(B))$, it suffices to show that $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \sigma_D(M_C)$.

Let $\rho_D^+(A) = \{ \lambda \in C : \alpha(A - \lambda I) < \infty \text{ and } \beta(A^* - \lambda I) < \infty \}$ and $\rho_D^-(B) = \{ \lambda \in C : \beta(B - \lambda I) < \infty \text{ and } \alpha(B^* - \lambda I) < \infty \}$ and let $\sigma_D^+(A) = C \setminus \rho_D^+(A)$ and $\sigma_D^-(B) = C \setminus \rho_D^-(B)$. Then there are inclusions
\[ \partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \partial \sigma_D(A) \cup \partial \sigma_D(B) \subseteq \sigma_D^+(A) \cup \sigma_D^-(B) \subseteq \sigma_D(M_C), \tag{3} \]
where the last inclusion follows from Lemma 2.2. For the second inclusion, if there exists $\lambda_0 \in (\partial \sigma_D(A) \cup \partial \sigma_D(B)) \setminus (\sigma_D^+(A) \cup \sigma_D^-(B))$, then there are two cases to consider.

**Case 1.** Suppose $\lambda_0 \in \partial \sigma_D(A)$. Then for any neighborhood of $\lambda_0$, there exists $\lambda$ such that $A - \lambda I$ is Drazin invertible. Thus for any neighborhood of $\lambda_0$, there exists $\lambda$ such that $A - \lambda I$ is invertible. And hence for any neighborhood of $\lambda_0$, there exists $\mu$ such that $A^* - \mu I$ is invertible. Since $\beta(A^* - \lambda_0 I) < \infty$, [5, P332, Theorem 10.5] tells us that $A^* - \lambda_0 I$ is Drazin invertible and hence $A - \lambda_0 I$ is Drazin invertible. It is in contradiction to the fact that $\lambda_0 \in \sigma_D(A)$.

**Case 2.** Suppose $\lambda_0 \in \partial \sigma_D(B)$. Similarly as in case 1, we induce a contradiction.

Then the second inclusion is true. Consequently, (2) asserts that the passage from $\sigma_D(M_C)$ to $\sigma_D(A) \cup \sigma_D(B)$ is the filling in certain holes in $\sigma_D(M_C)$. But since, by (1), $(\sigma_D(A) \cup \sigma_D(B)) \setminus \sigma_D(M_C)$ is contained in $\sigma_D(A) \cap \sigma_D(B)$, it follows that the filling in certain holes in $\sigma_D(M_C)$ should occur in $\sigma_D(A) \cap \sigma_D(B)$. The proof is completed.

**Corollary 2.6** If $\sigma_D(A) \cap \sigma_D(B)$ has no interior points, then for every $C \in B(K, H)$,
\[ \sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B). \tag{4} \]
In particular if either $A \in B(H)$ or $B \in B(K)$ is a Riesz operator, then (4) holds.

**Corollary 2.7** If either $A^*$ or $B$ is hyponormal, then for every $C \in B(K, H)$, (4) holds.

Let $\rho_D^+(A) = \sigma(A) \setminus \sigma_D^+(A)$ and $\rho_D^+(B) = \sigma(B) \setminus \sigma_D^+(B)$. From Theorem 2.5, we can see that the holes in $\sigma_D(M_C)$ should lie in $\rho_D^+(A) \cap \rho_D^+(B)$. Thus we have:
Corollary 2.8  If $\rho^+_{D}(A) \cap \rho^-_{D}(B) = \emptyset$, then (4) holds for every $C \in B(K, H)$.

Lemma 2.9  If $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ or $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then (4) holds.

Proof  Suppose that $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$. If $M_C - \lambda_0I$ is Drazin invertible, then there exists $\varepsilon > 0$ such that $M_C - \lambda I$ is invertible and $B - \lambda I$ is surjective if $0 < |\lambda - \lambda_0| < \varepsilon$. Since $\lambda$ is not in $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, it follows that $B - \lambda I$ are Weyl. Then $B - \lambda I$ are invertible if $0 < |\lambda - \lambda_0| < \varepsilon$. Now we have proved that $\lambda_0 \in \iso \sigma(B) \cup \rho(B)$. [5, P332, Theorem 10.5] tells us that $B - \lambda_0I$ is Drazin invertible hence $B - \lambda_0I$ is Drazin invertible. Then $\lambda_0$ is not in $\sigma_D(A) \cup \sigma_D(B)$. If $\sigma(M_C) = \sigma(A) \cup \sigma(B)$, by the same way, we can prove the result. □

Corollary 2.10  If $\sigma_w(A) \cap \sigma_w(B)$ (or $\sigma(A) \cap \sigma(B)$) has no interior points, then (4) holds for every $C \in B(K, H)$.

Proof  By Lemma 2.9 and Corollary 8 in [6] and Corollary 7 in [7], we get the result. □

3. Weyl’s theorem for $2 \times 2$ upper triangular operator matrices

H.Weyl[8] has shown that every hermitian operator $A \in B(H)$ satisfies the equality

$$\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$$

where $\pi_{00}(A) = \{ \lambda \in \iso \sigma(A) : 0 < \dim N(A - \lambda I) < \infty \}$. Now we say Weyl’s theorem holds for $A \in B(H)$ if $A$ satisfies the equality (5). If $\sigma_w(A) = \sigma_b(A)$, we say that Browder’s theorem holds for $A$. Clearly, Weyl’s theorem implies Browder’s theorem.

Let $\Phi_+(H)$ be the class of all $A \in \Phi_+(H)$ with $\ind(A) \leq 0$, and for any $A \in B(H)$, and let

$$\sigma_{ea}(A) = \{ \lambda \in C : A - \lambda I \text{ is not in } \Phi_+(H) \}$$

and $\sigma_{ab}(A) = \{ \lambda \in C : A - \lambda I \text{ is not an upper semi-Fredholm operator with finite ascent}\}$. We call $\sigma_{ea}(A)$ and $\sigma_{ab}(A)$ the essential approximate point spectrum and Browder essential approximate point spectrum respectively.

Let $\pi_{00}^a(A) = \{ \lambda \in \iso \sigma_a(A), 0 < \dim N(A - \lambda I) < \infty \}$. Similarly, we say that a-Weyl’s theorem holds for $A$ if there is equality $\sigma_a(A) \setminus \sigma_{ea}(A) = \pi_{00}^a(A)$, and that a-Browder’s theorem holds for $A$ if there is equality $\sigma_{ea}(A) = \sigma_{ab}(A)$.

Weyl’s theorem may or may not hold for a direct sum of operators for which Weyl’s theorem holds. Thus Weyl’s theorem may fail for upper triangular operator matrices. So does a-Weyl’s theorem. Weyl’s theorem for upper triangular operator matrices is more delicate in comparison with the diagonal matrices. In this section, we consider this question: If Weyl’s (a-Weyl’s) theorem holds for \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \), when does it hold for \( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \)? We begin with

Theorem 3.1  If $\sigma_D(A) \cap \sigma_D(B)$ (or $\sigma(A) \cap \sigma(B)$) has no interior points, then for every $C \in B(K, H)$,

(a)  Browder’s theorem holds for \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) $\implies$ Browder’s theorem holds for \( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \);
(b) $a$-Browder’s theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ $\iff$ $a$-Browder’s theorem holds for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

**Proof** (a) Suppose $M_C - \lambda_0I$ is Weyl. Then there exists $\varepsilon > 0$ such that $M_C - \lambda I$ is Weyl and hence $A - \lambda I$ is upper semi-Fredholm operator and $B - \lambda I$ is lower semi-Fredholm operator, and $A - \lambda I$ is Weyl if and only if $B - \lambda I$ is Weyl if $|\lambda - \lambda_0| < \varepsilon$.

**Case 1.** Suppose that $\lambda_0 \in \partial \sigma_D(A)$ or $\lambda_0$ is not in $\sigma_D(A)$. Then in any neighborhood of $\lambda_0$, there exists $\lambda$ such that $A - \lambda I$ is Drazin invertible and hence in any neighborhood of $\lambda_0$, there exists $\mu$ such that $A - \mu I$ is invertible. Since $A - \lambda_0 I$ is upper semi-Fredholm operator, by perturbation theory of upper semi-Fredholm, it follows that $A - \lambda_0 I$ is Browder. Then $B - \lambda_0 I$ is Weyl and hence $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_0 I$ is Weyl. Browder’s theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_0 I$ is Browder. Thus $A - \lambda_0 I$ and $B - \lambda_0 I$ are Drazin invertible. Lemma 2.1 tells us that $M_C - \lambda_0 I$ is Drazin invertible. Since $M_C - \lambda_0 I$ is Weyl, we get that $M_C - \lambda_0 I$ is Browder.

**Case 2.** Suppose that $\lambda_0 \in \text{int} \sigma_D(A)$. Since $\sigma_D(A) \cap \sigma_D(B)$ has no interior points, we know that $\lambda_0 \in \partial \sigma_D(B)$ or $\lambda_0$ is not in $\sigma_D(B)$. The following proof is the same as the proof in Case 1.

Now we have proved that $\sigma_a(M_C) = \sigma_b(M_C)$ for every $C \in B(K,H)$, which means that Browder’s theorem holds for $M_C$ for every $C \in B(K,H)$.

(b) Suppose that $M_C - \lambda_0 I \in \Phi_+(H \oplus K)$. Then $A - \lambda_0 I \in \Phi_+(H)$.

**Case 1.** $\lambda_0$ is not in $\sigma_D(A)$ or $\lambda_0 \in \partial \sigma_D(A)$. Similarly to the proof in case 1 in (a), we know that $A - \lambda_0 I$ is Browder. By perturbation theory of semi-Fredholm operator, there exists $\varepsilon > 0$ such that $M_C - \lambda I \in \Phi_+(H \oplus K)$ with $N(M_C - \lambda I) \subseteq \cap_{n=1}^{\infty} R[(M_C - \lambda I)^n]$ and $A - \lambda I$ is invertible if $0 < |\lambda - \lambda_0| < \varepsilon$. Then $B - \lambda I \in \Phi_+(K)$ and hence $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda I \in \Phi_+(H \oplus K)$. a-Browder’s theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $\alpha(A - \lambda I) < \infty$ and $\alpha(B - \lambda I) < \infty$ hence $\alpha(M_C - \lambda I) < \infty$. [9, Lemma 3.4] asserts that $N(M_C - \lambda I) = N(M_C - \lambda I) \cap \cap_{n=1}^{\infty} R[(M_C - \lambda I)^n] = \{0\}$ if $0 < |\lambda - \lambda_0| < \varepsilon$. Now we have that $\lambda_0 \in \text{iso} \sigma_a(M_C)$. Then $M_C$ has single valued extension property in $\lambda_0$. [10, Theorem 15] tells us that $\alpha(M_C - \lambda_0 I) < \infty$.

**Case 2.** If $\lambda_0 \in \text{int} \sigma_D(A)$, then $\lambda_0$ is not in $\sigma_D(B)$ or $\lambda_0 \in \partial \sigma_D(B)$. By perturbation theory of upper semi-Fredholm, there exists $\varepsilon > 0$ such that $M_C - \lambda I \in \Phi_+(H \oplus K)$ with $N(M_C - \lambda I) \subseteq \cap_{n=1}^{\infty} R[(M_C - \lambda I)^n]$, $n(M_C - \lambda I)$ is constant, and $A - \lambda I \in \Phi_+(H)$ if $0 < |\lambda - \lambda_0| < \varepsilon$. There exists $\lambda_1 \in C$ such that $B - \lambda_1 I$ is invertible and $0 < |\lambda_1 - \lambda_0| < \varepsilon$. Then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_1 I \in \Phi_+(H \oplus K)$. Similarly to the case 1 in (b), $M_C - \lambda_1 I$ is bounded below. Therefore $M_C - \lambda I$ is bounded below because $n(M_C - \lambda I)$ is constant if $0 < |\lambda - \lambda_0| < \varepsilon$. It follows that $\alpha(M_C - \lambda_0 I) < \infty$.

Then $\sigma_{ea}(M_C) = \sigma_{ab}(M_C)$, which means that a-Browder’s theorem holds for $M_C$ for every $C \in B(K,H)$. $\square$

We call $A$ is isoloid if $\text{iso} \sigma(A) \subseteq \sigma_p(A)$, where $\sigma_p(A)$ is the set of all point spectrums. And
we call $A$ approximate isoloid (abbrev. a-isoloid) if $iso\, \sigma_{a}(A) \subseteq \sigma_{p}(A)$. Clearly, a-isoloid implies isoloid.

**Remark 3.2** If $\sigma_{w}(A) \cap \sigma_{w}(B)$ had no interior points, then (a) in Theorem 3.1 is also true. But Theorem 3.1 may fail for “a-Weyl’s theorem” even with the additional assumption that a-Weyl’s theorem holds for $A$ and $B$ and both $A$ and $B$ are a-isoloid. To see this, let $A$, $B$, $C \in B(\ell_{2})$ are defined by

$$A(x_{1}, x_{2}, x_{3}, \cdots) = (0, x_{1}, 0, x_{2}, 0, x_{3}, \cdots),$$

$$B(x_{1}, x_{2}, x_{3}, \cdots) = (0, x_{2}, 0, x_{4}, 0, x_{6}, \cdots),$$

$$C(x_{1}, x_{2}, x_{3}, \cdots) = (0, 0, 0, \frac{1}{3}x_{3}, 0, \frac{1}{5}x_{5}, \cdots).$$

Then $\sigma_{a}(A) = \sigma_{ea}(A) = T$, $\sigma_{D}(A) = D$, $\pi_{00}^{a}(A) = \emptyset$ and $\sigma_{a}(B) = \sigma_{ea}(B) = \{0, 1\}$, $\sigma_{D}(B) = \pi_{00}^{a}(B) = \emptyset$, which says that a-Weyl’s theorem holds for $A$ and $B$, both $A$ and $B$ are a-isoloid, and $\sigma_{D}(A) \cap \sigma_{D}(B)$ ( $\sigma_{w}(A) \cap \sigma_{w}(B)$ ) has no interior points. Also a straightforward calculation shows that

$$\sigma_{a}\left(\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right) = \sigma_{ea}\left(\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right) = T \cup \{0\}, \quad \pi_{00}^{a}\left(\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right) = \emptyset,$$

$$\sigma_{a}(M_{C}) = \sigma_{ea}(M_{C}) = T \cup \{0\}, \quad \pi_{00}^{a}(M_{C}) = \{0\}.$$

Then a-Weyl’s theorem holds for $\left(\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right)$, but fails for $\left(\begin{array}{cc} A & C \\ 0 & B \end{array}\right)$.

But for Weyl’s theorem, we have:

**Theorem 3.3** If $\sigma_{D}(A) \cap \sigma_{D}(B)$ (or $\sigma_{w}(A) \cap \sigma_{w}(B)$) has no interior points and if $A$ is an isoloid operator for which Weyl’s theorem holds, then for every $C \in B(K, H)$,

\[\text{Weyl’s theorem holds for } \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right) \implies \text{Weyl’s theorem holds for } \left(\begin{array}{cc} A & C \\ 0 & B \end{array}\right).\]

**Proof** Theorem 3.1 gives that $\sigma(M_{C}) \setminus \sigma_{w}(M_{C}) \subseteq \pi_{00}(M_{C})$. For the reverse inclusion, suppose that $\lambda_{0} \in \pi_{00}(M_{C})$. Then there exists $\varepsilon > 0$ such that $M_{C} - \lambda I$ is invertible and hence $A - \lambda I$ is bounded below and $B - \lambda I$ is surjective if $0 < |\lambda - \lambda_{0}| < \varepsilon$. $\sigma_{D}(A) \cap \sigma_{D}(B)$ (or $\sigma_{w}(A) \cap \sigma_{w}(B)$) has no interior points, then $\sigma_{D}(M_{C}) = \sigma_{D}(A) \cup \sigma_{D}(B)$. Since $\lambda$ is not in $\sigma_{D}(M_{C}) = \sigma_{D}(A) \cup \sigma_{D}(B)$, it follows that $A - \lambda I$ and $B - \lambda I$ are Drazin invertible. Thus $A - \lambda I$ and $B - \lambda I$ are invertible, which means that $\lambda_{0} \in iso\, \sigma\left(\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right)$. The following proof is same as the proof in Theorem 2.4 in [11].

**Remark 3.4** Theorem 3.3 in this paper is not compatible with Theorem 2.4 in [11]. For example:

(a) Let $A \in B(\ell_{2})$ be defined by

\[A(x_{1}, x_{2}, x_{3}, \cdots) = (x_{2}, x_{4}, x_{6}, \cdots),\]
and let $B = A - 2I$. Then

(I) $\sigma_D(A) = D$, $\sigma_D(B) = \{ \lambda \in C : |\lambda + 2| \leq 1 \}$. Then $\sigma_D(A) \cap \sigma_D(B)$ has no interior points;

(II) $\sigma_e(A) = D$, $\sigma_e(B) = \{ \lambda \in C : |\lambda + 2| = 1 \}$ and $\sigma_e(B) = \{ \lambda \in C : |\lambda + 2| \leq 1 \}$, where $\sigma_e(A) = \{ \lambda \in C : A - \lambda I$ is not lower semi-Fredholm operator $\}$. Then both $SP(A)$ and $SP(B)$ have pseudoholes;

(III) $\sigma(A) = \sigma_w(A) = D$ and $\pi_{00}(A) = \emptyset$, then $A$ is isoloid and Weyl’s theorem holds for $A$;

(IV) $\sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = D$ and $\pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset$, then Weyl’s theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

By Theorem 3.3 in this note, Weyl’s theorem holds for $M_C$ for every $C \in B(\ell_2, \ell_2)$. But using Theorem 2.4 in [11], we do not know whether Weyl’s theorem holds for $M_C$ for every $C \in B(K, H)$.

(b) Let $T_1, T_2, B \in B(\ell_2)$ are defined by

$$T_1(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, 0, \cdots),$$

$$T_2(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots),$$

and

$$B(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Let $A = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$.

Then

(I) $\sigma_D(A) = D$, $\sigma_D(B) = D$. Then $\sigma_D(A) \cap \sigma_D(B)$ has interior points;

(II) $\sigma_e(A) = \sigma_{SF_+}(A) = \sigma_{SF_+}(A) = D$, $\sigma_e(B) = \sigma_{SF_+}(B) = \sigma_{SF_+}(B) = T$, then both $SP(A)$ and $SP(B)$ have no pseudoholes;

(III) $\sigma(A) = \sigma_w(A) = D$, $\pi_{00}(A) = \emptyset$. Then $A$ is isoloid and Weyl’s theorem holds for $A$;

(IV) $\sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = D$, $\pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset$. Then Weyl’s theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Using Theorem 2.4 in [11], we know that for every $C \in B(\ell_2, \ell_2 \otimes \ell_2)$, Weyl’s theorem holds for $M_C$. But using Theorem 3.3 in this paper, we do not know whether Weyl’s theorem holds for $M_C$ for every $C \in B(\ell_2, \ell_2 \otimes \ell_2)$.

For a-Weyl’s theorem, similarly to the prove of Theorem 3.3, we have that:

**Theorem 3.5** If $\sigma_D(A)$ (or $\sigma(A)$) has no interior points, and if $A$ is an a-isoloid operator for which a-Weyl’s theorem holds, then for every $C \in B(K, H)$,

$$a\text{-Weyl’s theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Rightarrow a\text{-Weyl’s theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$
References:


**Drazin 谱和算子矩阵的 Weyl 定理**

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摘要: $A \in B(H)$ 称为是一个 Drazin 可逆的算子，若 $A$ 有有限的升标和降标，用 $\sigma_D(A) = \{ \lambda \in C : A - \lambda I$ 不是 Drazin 可逆的 $\}$ 表示 Drazin 谱集。本文证明了对于 Hilbert 空间的有限维子子空间 $C$，算子 $A$ 的 Drazin 谱是定义在 $C$ 上的有限维子空间 $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ 的子空间 $\sigma_D(M_C)$ 的道路需要从前面算子矩阵 $A$ 中的有限子空间移动 $\sigma_D(A) \cap \sigma_D(B)$ 中的一定子空间，即有等式:

$$\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup \mathcal{G},$$

其中 $\mathcal{G}$ 为 $\sigma_D(M_C)$ 中一定子空间的并，且为 $\sigma_D(A) \cap \sigma_D(B)$ 的子空间。对于 $\sigma_D(A) \cap \sigma_D(B)$ 的子空间 $\mathcal{G}$ 可以被选择为 $A$ 的 Weyl 定理，利用 Drazin 谱，我们研究了 $2 \times 2$ 上三角算子矩阵的 Weyl 定理。Browder 定理，a-Weyl 定理和 a-Browder 定理。

关键词: Weyl 定理；a-Weyl 定理；Browder 定理；a-Browder 定理；Drazin 谱。