A Conceivable Inequality for Analytic Functions and Its Application

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Abstract: This brief note presents an easily conceivable inequality for analytic functions. As an application, a function-theoretic proposition involving the Fundamental Theorem of Algebra (FTA) is deduced immediately.

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Without loss of generality, a non-constant analytic function \( f(z) \) defined for \( |z| < \rho \) in the Gaussian plane \( \mathbb{C} \) may be written in the form
\[
f(z) = a + bz^m + cz^{m+1} + \cdots, \tag{1}
\]
where \( b \neq 0 \) and \( m \geq 1 \).

**Proposition 1** Let \( f(t) \) be defined by (1) with \( f(0) = a \neq 0 \), then for every sufficiently small \( \delta > 0 \) there holds the inequality
\[
|f((-a\delta/b)^{1/m})| < |f(0)|, \tag{2}
\]
where \((-a\delta/b)^{1/m}\) may take any of the \( m \) distinct roots for \( m \geq 2 \).

**Proof** Solving the equation \( bz^m = -a\delta \), we get \( z = (-a\delta/b)^{1/m} \). Rewrite \( f(z) \) in the form
\[
f(z) = a + bz^m + bz^m g(z), \tag{1}^*
\]
where \( g(z) = (c/b)z + \cdots \) is also an analytic function so that \( |g(z)| \to 0 \) as \( |z| \to 0 \). Thus for every sufficiently small \( \delta \) with \( 0 < \delta < 1 \), we could have
\[
|g((-a\delta/b)^{1/m})| < \frac{1}{2}.
\]
Consequently, the following estimation holds via (1)*
\[
|f((-a\delta/b)^{1/m})| \leq |a - a\delta| + |(-a\delta)g((-a\delta/b)^{1/m})| < |a|(1 - \delta) + |a|\delta \cdot \frac{1}{2}
= |a|(1 - \frac{\delta}{2}) < |a| = |f(0)|.
\]

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Hence the Inequality (2) is always valid for small $\delta > 0$.

Evidently, Inequality (2) may be stated in a more general form: If $f(z)$ is non-constant and analytic in a neighborhood of $z_0 \in \mathbb{C}$, with $f(z_0) = a \neq 0$, then $f(z_0 + z)$ may be written in a similar form, as that of (1),

$$f(z_0 + z) = f(z_0) + bz^m + cz^{m+1} + \cdots, \quad (m \geq 1),$$

and consequently the following inequality

$$|f(z_0 + (-a\delta/b)^{1/m})| < |f(z_0)|$$

holds for every sufficiently small $\delta (0 < \delta < 1)$.

In what follows suppose that $F(z)$ is non-constant and analytic in a domain $D \subset \mathbb{C}$. Then the form of Inequality (4) shows that for every $z_0 \in D$ with $F(z_0) \neq 0$, the value $|F(z_0)| > 0$ can never become an absolute minimum $\min_z |F(z)|$. This implies that if $\min_z |F(z)|$ really exists and is attained at $z = z_0 \in D$, then it must be that $\min_z |F(z)| = |F(z_0)| = 0$. Accordingly, we get the following useful and well-known proposition as a consequence of (4).

**Proposition 2** Let $F(z)$ be a non-constant analytic function in $D \subset \mathbb{C}$, and let $|F(z)|$ attain an absolute minimum at $z_0 \in D$. Then it must be that

$$\min_z |F(z)| = |F(z_0)| = 0.$$

In words, $z_0$ must be a zero of $F(z)$.

In particular, if $F(z)$ is a polynomial in $z$ of degree $n(n \geq 1)$, then the obvious fact that $|F(z)| \to \infty (|z| \to \infty)$ implies that $|F(z)|$ should attain $\min_z |F(z)|$ at certain $z_0 \in \mathbb{C}$. Thus by Proposition 2 we must have $F(z_0) = 0$. This is what so-called the well-known existence theorem first proved by Gauss (1799):

FTA Any polynomial equation $F(z) = 0$ of degree $n(n \geq 1)$ has at least a root $z_0 \in \mathbb{C}$, viz.

$$F(z_0) = 0.$$

Note that for the example $F(z) = e^z$ we have $|e^z| = |e^{x+iy}| = e^x > 0$ for $z = x + iy \in \mathbb{C}$. This shows that the second condition in Proposition 2 cannot be omitted.

**Remark 1** Recall that in the complex analysis a limit process such as $f(z) \to A \quad (z \to a \in \mathbb{C})$ involves that both $z$ and $f(z)$ could tend to their limits in various possible directions in $\mathbb{C}$. In particular, if $f(z) \to f(a) \neq 0 \quad (z \to a)$ and if the mode of passage $z \to a$ could be so chosen that $|f(z)| \uparrow |f(a)|$, then we shall have $|f(z)| < |f(a)|$ for all those $z$ sufficiently close to $a$. Observing in this way, we see that (2) and (4) are geometrically comprehensible.

**Remark 2** For the polynomial equation $F(z) = 0$ of degree $n(\geq 1)$, the truth of (FTA) is just based on the fact that $\min_z |F(z)|$ really exists and cannot take any positive value such as $\min_z |F(z)| = |F(z_0)| > 0$. Looking in this way, we may say that the truth of (4) or (2) is the basic source for the truth of (FTA).
Remark 3  As usual, for the $n$th degree polynomial $|F(z)|$ we may denote $|F(z)| = |F(x+iy)| = |u(x, y) + iv(x, y)| = \sqrt{u(x, y)^2 + v(x, y)^2}$. Thus the assertion of (FTA), $\min_z |F(z)| = |F(z_0)| = |F(x_0 + iy_0)| = 0$, just means that $(x_0, y_0)$ is the intersection point of the two plane curves defined by $u(x, y)=0$ and $v(x, y)=0$, respectively. As is known, Gauss' famous doctoral thesis (1799) first proved this fact. Certainly, the existence of such a point $(x_0, y_0) \leftrightarrow x_0 + iy_0 = z_0$ can also be inferred from Proposition 2 or (4) directly.

References: