Note on Convergence Theorems of Iterative Sequences for Asymptotically Non-Expansive Mapping in a Uniformly Convex Banach Space

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Abstract: In this paper, we approximate fixed point of asymptotically nonexpansive mapping \( T \) on a closed, convex subset \( C \) of a uniformly convex Banach space. Our argument removes the boundedness assumption on \( C \), generalizing theorems of Liu and Xue.

Key words: asymptotically nonexpansive mapping; modified Ishikawa iterative sequence; fixed points.
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1. Introduction and preliminaries

Let \( E \) be a real normed linear space, \( E^* \) its dual, and \( \langle \cdot, \cdot \rangle \) the generalized duality pairing between \( E \) and \( E^* \). Let \( J : E \to 2^{E^*} \) be the normalized duality mapping defined for each \( x \in E \) by

\[
J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}.
\]

It is well known that if \( E \) is smooth then \( J \) is single-valued.

Definition 1.1 Let \( T : D(T) \subset E \to E \) be a mapping. \( T \) is said to be asymptotically nonexpansive if there exists a sequence \( \{k_n\} \subset [1, +\infty) \) with \( \lim_{n \to \infty} k_n = 1 \) such that

\[
\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in D(T), n = 1, 2, 3, \ldots.
\]

It is well known that if \( T \) is nonexpansive, then \( T \) is asymptotically nonexpansive with a constant sequence \( \{1\} \).

Definition 1.2 Let \( C \) be a nonempty convex subset of \( E \), \( T : C \to C \) be a mapping and \( x_1 \in C \) be a given point. If sequences \( \{x_n\}, \{y_n\} \subset C \) are defined by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, n \geq 1,
\]

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then \( \{x_n\} \) is called the modified Ishikawa iterative sequence of \( T \), where \( \{\alpha_n\}, \{\beta_n\} \subset [0, 1] \), with \( 0 < \delta \leq \alpha_n, \beta_n \leq 1 - \delta < 1 \).

In (1.1) if \( \beta_n = 0 \) for all \( n \geq 0 \), then \( y_n = x_n \). The sequence \( \{x_n\} \) defined by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^n x_n, \quad n \geq 1
\]

is called the modified Mann iterative sequence of \( T \), where \( \{\alpha_n\} \subset [0, 1] \), with \( 0 < \delta \leq \alpha_n \leq 1 - \delta < 1 \).

The concept of asymptotically nonexpansive mapping was first introduced and studied by Goebel and Kirk\(^1\) in 1972. They proved that if \( D \) is a nonempty bounded closed convex subset of a uniformly convex Banach space \( E \), then every asymptotically nonexpansive selfmapping \( T \) defined on \( D \) has a fixed point.

In 2000, Liu and Xue\(^2\) proved the convergence of iterative sequence in a uniformly convex Banach space for asymptotically nonexpansive mappings. They got the following main theorem.

**Theorem LX** Let \( T \) be a completely continuously asymptotically nonexpansive mapping with sequence \( \{k_n\} \) in a bounded closed convex subset \( C \) of a uniformly convex Banach space and \( k_n \geq 1, \sum_{n=1}^{\infty} (k_n - 1) < +\infty, x_1 \in C \), \( \{x_n\} \) defined by (1.1), where \( \{\alpha_n\} \) and \( \{\beta_n\} \) satisfy \( 0 < \alpha \leq \alpha_n \leq \frac{1}{2}, 0 \leq \beta_n \leq \frac{1}{2} \) and \( \lim_{n \to \infty} \beta_n = 0 \), then the iterative sequence \( \{x_n\} \) converges to a fixed point of \( T \).

In this paper, our results generalize Theorem LX, in that we remove the assumption that \( C \) is bounded and we approximate fixed point of asymptotically nonexpansive mapping \( T \) on a closed, convex subset \( C \) of a uniformly convex Banach space.

We shall need the following results.

**Lemma 1.1**\(^3,4\) Let \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) be sequences of nonnegative real numbers satisfying the inequality

\[
a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq 1.
\]

If \( \sum_{n=1}^{\infty} b_n < +\infty \) and \( \sum_{n=1}^{\infty} c_n < +\infty \), then \( \lim_{n \to \infty} a_n \) exists. In particular, if \( c_n \equiv 0 \), then \( \lim_{n \to \infty} a_n \) also exists.

**Lemma 1.2**\(^5\) Let \( \{\rho_n\}_{n=1}^{\infty} \) and \( \{\sigma_n\}_{n=1}^{\infty} \) be sequences of nonnegative real numbers satisfying the inequality

\[
\rho_{n+1} \leq \rho_n + \sigma_n, \quad n \geq 1.
\]

If \( \sum_{n=1}^{\infty} \sigma_n < +\infty \), then \( \lim_{n \to \infty} \rho_n \) exists. In particular, if \( \{\rho_n\}_{n=1}^{\infty} \) has a subsequence which converges strongly to zero, then \( \lim_{n \to \infty} \rho_n = 0 \).

**Lemma 1.3**\(^6\) Suppose that \( E \) is a uniformly convex Banach space and \( 0 < p \leq t_n \leq q < 1 \) for all positive integers \( n \). Also suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences of \( E \) such that \( \limsup_{n \to \infty} \|x_n\| \leq r \), \( \limsup_{n \to \infty} \|y_n\| \leq r \), \( \lim_{n \to \infty} \|tx_n + (1 - t_n)y_n\| = r \) hold for some \( r \geq 0 \). Then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).
2. Main results

Lemma 2.1 Let $E$ be a real normed linear space, $C$ be a nonempty convex subset of $E$, and $T : C \to C$ be an asymptotically nonexpansive mapping with a real sequence $\{k_n\}$ in $[1, +\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. Let $x_1 \in C$, $\{x_n\}$ be the modified Ishikawa iterative sequence defined by (1.1). If $F(T) \neq \emptyset$, then for any given $q \in F(T)$, $\lim_{n \to \infty} \|x_n - q\|$ exists.

Proof For any given $q \in F(T)$, using iterates (1.1), we have

$$
\|x_{n+1} - q\| = \|(1 - \alpha_n)(x_n - q) + \alpha_n(T^n y_n - q)\|
\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n k_n \|y_n - q\|. \quad (2.1)
$$

Otherwise,

$$
\|y_n - q\| = \|(1 - \beta_n)(x_n - q) + \beta_n(T^n x_n - q)\|
\leq (1 - \beta_n)\|x_n - q\| + \beta_n k_n \|x_n - q\|
= [1 + (k_n - 1)\beta_n]\|x_n - q\|
\leq k_n \|x_n - q\|. \quad (2.2)
$$

Substituting (2.2) into (2.1), we have

$$
\|x_{n+1} - q\| \leq (1 - \alpha_n)\|x_n - q\| + \alpha_n k_n^2 \|x_n - q\|
= [1 + \alpha_n(k_n^2 - 1)]\|x_n - q\|. \quad (2.3)
$$

Set $a_n = \|x_n - q\|, b_n = \alpha_n(k_n^2 - 1) = \alpha_n(k_n + 1)(k_n - 1)$. Then Inequality (2.3) is equal to

$$
a_{n+1} \leq (1 + b_n)a_n.
$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$, $\{k_n\}$ is a bounded sequence and $\alpha_n \in [0, 1]$, so $\sum_{n=1}^{\infty} b_n < +\infty$. By Lemma 1.1, we know $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \|x_n - q\|$ exists.

Lemma 2.2 Let $E$ be a uniformly convex Banach space, $C$ be a nonempty convex subset of $E$, and $T : C \to C$ be an asymptotically nonexpansive mapping with a real sequence $\{k_n\}$ in $[1, +\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. Let $x_1 \in C$, $\{x_n\}$ be the modified Ishikawa iterative sequence defined by (1.1). If $F(T) \neq \emptyset$, then $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$.

Proof Since $F(T) \neq \emptyset$, let $q \in F(T)$. By Lemma 2.1, we know $\lim_{n \to \infty} \|x_n - q\|$ exists. Let $\lim_{n \to \infty} \|x_n - q\| = c, c \geq 0$.

Step 1. We prove $\lim_{n \to \infty} \|x_n - T^n x_n\| = 0$.

From (1.1), we have

$$
\|y_n - q\| = \|(1 - \beta_n)(x_n - q) + \beta_n(T^n x_n - T^n q)\|
\leq [1 + (k_n - 1)\beta_n]\|x_n - q\|
\leq k_n \|x_n - q\|.
$$
Taking \( \lim \sup \) on both sides of the above inequality, we have

\[
\lim \sup_{n \to \infty} \| y_n - q \| \leq c. \tag{2.4}
\]

Next, consider \( \| T^n y_n - q \| \leq k_n \| y_n - q \| \). Taking \( \lim \sup \) on both sides of the above inequality and then using (2.4), we get that

\[
\lim \sup_{n \to \infty} \| T^n y_n - q \| \leq c.
\]

Furthermore, \( \lim_{n \to \infty} \| x_{n+1} - q \| = c \) means that

\[
\lim_{n \to \infty} \| (1 - \alpha_n)(x_n - q) + \alpha_n(T^n y_n - q) \| = c.
\]

Hence applying Lemma 1.3, we obtain that

\[
\lim_{n \to \infty} \| x_n - T^n y_n \| = 0.
\]

Next,

\[
\| x_n - q \| \leq \| x_n - T^n y_n \| + \| T^n y_n - q \| \leq \| x_n - T^n y_n \| + k_n \| y_n - q \|
\]

gives that

\[
c \leq \lim \inf_{n \to \infty} \| y_n - q \| \leq \lim \sup_{n \to \infty} \| y_n - q \| \leq c.
\]

That is \( \lim_{n \to \infty} \| y_n - q \| = c \). Now \( \lim_{n \to \infty} \| y_n - q \| = c \) can be expressed as

\[
\lim_{n \to \infty} \| (1 - \beta_n)(x_n - q) + \beta_n(T^n x_n - q) \| = c.
\]

Observe that \( \| T^n x_n - q \| \leq k_n \| x_n - q \| \). Taking \( \lim \sup \) on both the sides in the above inequality, we have \( \lim \sup_{n \to \infty} \| T^n x_n - q \| \leq c \). So again by Lemma 1.3, we have \( \lim_{n \to \infty} \| x_n - T^n x_n \| = 0. \)

**Step 2.** We prove \( \lim_{n \to \infty} \| x_n - T x_n \| = 0. \)

For convenience, let \( \rho_n = \| x_n - T^n x_n \|. \)

Now consider

\[
\| x_n - x_{n+1} \| = \| \alpha_n(x_n - T^n y_n) \| \\
\leq \alpha_n \| x_n - T^n x_n \| + \alpha_n \| T^n x_n - T^n y_n \| \\
\leq \alpha_n \| x_n - T^n x_n \| + \alpha_n k_n \| x_n - y_n \| \\
\leq \alpha_n \| x_n - T^n x_n \| + \alpha_n \beta_n k_n \| x_n - T^n x_n \| \\
= \alpha_n \rho_n + \alpha_n \beta_n k_n \rho_n.
\]

That is

\[
\| x_n - x_{n+1} \| \leq \alpha_n \rho_n + \alpha_n \beta_n k_n \rho_n.
\]
Next consider
\[
\|x_{n+1} - Tx_{n+1}\| \leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\|
\]
\[
\leq \rho_{n+1} + k_1\|T^n x_{n+1} - x_{n+1}\|
\]
\[
\leq \rho_{n+1} + k_1\|T^n x_{n+1} - T^n x_n\| + k_1\|T^n x_n - x_n\| + k_1\|x_n - x_{n+1}\|
\]
\[
\leq \rho_{n+1} + k_1k_n\|x_{n+1} - x_n\| + k_1\|T^n x_n - x_n\| + k_1\|x_n - x_{n+1}\|
\]
\[
= \rho_{n+1} + k_1(k_n + 1)\|x_n - x_{n+1}\| + k_1\rho_n
\]
\[
\leq \rho_{n+1} + k_1(k_n + 1)\alpha_n\rho_n + k_1k_n(k_n + 1)\alpha_n\beta_n\rho_n + k_1\rho_n.
\]

From Step 1, we have \(\lim_{n \to \infty} \rho_n = 0\), \(k_n \to 1(n \to \infty)\) and \(0 \leq \alpha_n, \beta_n \leq 1\), so
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\]

**Theorem 2.1** Let \(E\) be a uniformly convex Banach space and \(C\) its nonempty closed convex subset of \(E\), and \(T : C \to C\) be a completely continuously asymptotically nonexpansive mapping with a real sequence \(\{k_n\}\) in \([1, +\infty)\) such that \(\sum_{n=1}^{\infty} (k_n - 1) < +\infty\). Let \(x_1 \in C\), \(\{x_n\}\) be the modified Ishikawa iterative sequence defined by (1.1). If \(F(T) \neq \emptyset\), then \(\{x_n\}\) converges strongly to a fixed point of \(T\).

**Proof** Since \(F(T) \neq \emptyset\), by Lemma 2.2, we know
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0. \tag{2.5}
\]
Since \(T\) is completely continuous, by Lemma 2.1, we have \(\{x_n\}\) is bounded and \(C\) is a closed subset, so there must exist \(\{Tx_{n_k}\}_{k=1}^{+\infty} \subset \{Tx_n\}_{n=1}^{+\infty}\). Set
\[
\lim_{k \to +\infty} Tx_{n_k} = q. \tag{2.6}
\]
It follows from (2.5) and (2.6), we get
\[
\lim_{k \to +\infty} x_{n_k} = q. \tag{2.7}
\]
Since \(T\) is completely continuous, \(T\) is obviously continuous. It follows from (2.6) and (2.7) that so \(\|q - Tq\| = 0\). Thus \(q\) is a fixed point of \(T\).

From (1.1), we have
\[
\|x_{n+1} - q\| = \|(1 - \alpha_n)x_n + \alpha_n T^n y_n - q\| = \|(1 - \alpha_n)(x_n - q) + \alpha_n (T^n y_n - q)\|
\]
\[
\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n k_n\|y_n - q\|. \tag{2.8}
\]
Next, consider
\[
\|y_n - q\| = \|(1 - \beta_n)x_n + \beta_n T^n x_n - q\| = \|(1 - \beta_n)(x_n - q) + \beta_n (T^n x_n - q)\|
\]
\[
\leq (1 - \beta_n)\|x_n - q\| + \beta_n k_n\|x_n - q\| = [1 + (k_n - 1)\beta_n]\|x_n - q\|. \tag{2.9}
\]
Using (2.9) in (2.8), we obtain
\[
\|x_{n+1} - q\| \leq (1 - \alpha_n)\|x_n - q\| + \alpha_n k_n [1 + (k_n - 1)\beta_n]\|x_n - q\|
\]
\[
= \|x_n - q\| + (k_n - 1)(1 + k_n\beta_n)\alpha_n\|x_n - q\|. \quad (2.10)
\]
By Lemma 2.1, for all \(n \geq 0\), \(\|x_n - q\|\) is bounded, \(k_n \to 1\) (\(n \to \infty\)) and \(0 \leq \alpha_n, \beta_n \leq 1\), so there exists \(M > 0\) such that
\[
(1 + k_n\beta_n)\alpha_n\|x_n - q\| \leq M. \quad (2.11)
\]
Submitting it into (2.10), we have
\[
\|x_{n+1} - q\| \leq \|x_n - q\| + M(k_n - 1).
\]
Since \(\sum_{n=1}^{\infty}(k_n - 1) < +\infty\), by Lemma 1.2, we also know \(\lim_{n \to \infty}\|x_n - q\|\) exists. Again by (2.7) and Lemma 1.2, we know \(\{x_n\}\) converges strongly to \(q\).

**Corollary 2.1** Let \(E\) be a uniformly convex Banach space and \(C\) its nonempty closed convex subset of \(E\), and \(T : C \to C\) be a completely continuously asymptotically nonexpansive mapping with a real sequence \(\{k_n\}\) in \([1, +\infty)\) such that \(\sum_{n=1}^{\infty}(k_n - 1) < +\infty\). Let \(x_1 \in C\), \(\{x_n\}\) be the modified Mann iterative sequence defined by (1.2). If \(F(T) \neq \emptyset\), then \(\{x_n\}\) converges strongly to a fixed point of \(T\).

**Proof** Taking \(\beta_n = 0\) in Theorem 2.1, we know Corollary 2.1 is true.

**References:**


关于一致凸的 Banach 空间上的渐近非扩张映象的迭代序列的收敛性定理的注记

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**摘要**：本文研究了在一敛凸 Banach 空间中定义在闭凸集 \(C\) 上渐近非扩张映象 \(T\) 不动点的迭代问题。我们的讨论去掉掉了在刘和薛 [2] 中 \(C\) 是有界的假设。

**关键词**：渐近非扩张映射；修改的 Ishikawa 迭代序列；不动点。