On Poincaré-Type Continuum and Certain of Its Basic Properties

HSU Leetsch Charles
(Department of Applied Mathematics, Dalian University of Technology, Liaoning 116024, China)

Abstract. This is an expositive paper reflecting certain viewpoint of intuitionists for the structure of continuum. What we provide is a descriptive theory for Poincaré-type continuum whose first hyperstandard model (PC) was given previously[3]. Here we construct a new hyperstandard model [PC] instead of (PC), and present three propositions related to the concept of semi-infinitesimal. Finally, an application of \( \mathbb{R} \) (non-Cantorian continuum) in Calculus is expounded.

Keywords. Poincaré’s intimate bond (IB); hyperstandard microinterval; semi-infinitesimal; non-punctiform element.

1. Introduction

The common term “continuum” has been used widely and frequently. In particular, we agree with the following equivalences between several technical terms:

\[
\text{Linear continuum} \equiv \text{Line continuum} \equiv \text{Real line} \equiv \text{Real continuum;}
\]

\[
\text{Cartor’s continuum} \equiv \text{Point-constituted continuum} \equiv \text{Real number continuum} \equiv \text{Real number-axis} \equiv \text{Set of all positional points.}
\]

We use \( L \) and \( R \) to denote the real line and Cantor’s continuum respectively. As a real line, \( L \) has no points, but it is divisible and even infinitely divisible, so that \( L \) may have positional points, and sets of positional points on it.

By a kind of one-to-one correspondence, the real number set \( R \) and the rational number set \( Q \) may be represented by certain point-sets on \( L \), respectively. In these cases we may say that \( L \) receives \( R \) and \( Q \) to be embedded on it. Moreover, by introducing certain conditions, Robinson’s nonstandard real number set \( \ast R \) can also be embedded on \( L \) as a nonstandard point-set[4].

Let us mention that some brief expositions of Poincaré’s remark[5], p347 on Cantor’s continuum had been presented in our previous paper [3], in which the meaning of Poincaré’s concept “intimate bond” (IB) was deeply explored. Of course Poincaré’s (IB) does not exist in \( R \), but also certainly, we think that Poincaré had a picture of the line continuum in mind, so that he could suppose that a sort of “intimate bond” should exist between the points of Cantor’s continuum (when it has been embedded on \( L \)), and it could connect all the positional points into the line.
As a matter of fact, our tentative formulation of Poincaré-type continuum (PC) as described in Ref. [3] was made on the basic idea of (IB).

In this expositive paper, certain basic concepts expounded in Ref. [3] will be refined and extended. A significant step to be taken is to postulate (via descriptive definition) the existence of the leap (gap) between neighboring monads with a certain hyperstandard quantity (so-called semi-infinitesimal) as a micro-distance. Of course, everything to be defined or postulated in this paper is a new kind of entity or abstract structure lying outside of the nonstandard real number field \( \ast R \), and all what we wish to propose or provide is a descriptive theory about (PC).

2. (PC) as a hyper-structure consisting of microintervals

We shall make our start from \( \ast R \), and any terminologies related to Non-standard Analysis (NSA) will be used without explanations.

As in Ref. [3], we may think that a kind of generalized Dedekind cut could be formulated on the ordered set of monads of \( \ast R \), namely, \( \ast R' := \{ m(a) | a \in R \} \). Here any two different monads \( m(a) \) and \( m(b) \) of \( \ast R' \) can be ordered as \( m(a) < m(b) \) according as \( a < b \). Thus, if \( \ast R' \) is divided into two disjoint non-empty subsets, say \( \ast R'_1 \) and \( \ast R'_2 \), such that

\[
\forall m(a) \in \ast R'_1, \forall m(b) \in \ast R'_2 \Rightarrow m(a) < m(b),
\]

then the bisection denoted by \( (\ast R'_1|\ast R'_2) \) may be called a hyperstandard Dedekind cut.

Denote

\[ R_1 = \{ x | m(x) \in \ast R'_1 \}, \quad R_2 = \{ y | m(y) \in \ast R'_2 \}. \]

Evidently \( (R_1|R_2) \) yields a classical Dedekind cut on \( R \), and may be called a companion cut of \( (\ast R'_1|\ast R'_2) \).

Let us recall a basic concept expounded in Ref. [3], namely

**Definition 1** Let \( a \in R \). Suppose that there is a hyperstandard Dedekind cut \( (\ast R'_1|\ast R'_2) \) such that \( m(a) \in \ast R'_1 \) with \( a = \sup R_1 \in R_1 \). Then \( m(a)^+ \) as defined by

\[ m(a)^+ := (\ast R'_1|\ast R'_2) \]

is called the right-leap of \( m(a) \). Similarly, the left-leap \( m(a)^- \) may be defined by

\[ m(a)^- := (\ast R'_1|\ast R'_2) \]

in which \( m(a) \in \ast R'_2 \) with \( a = \inf R_2 \in R_2 \).

Note that monads within \( \ast R \) are external to each other, or in other words, they do not touch each other. Thus, naturally, one may conceive of the sets \( \ast R'_1 \) and \( \ast R'_2 \) as having a certain “gap” between them, so that \( m(a)^+ \) and \( m(a)^- \) may be well interpreted as jumps over the gaps from either sides of \( m(a) \).

The above interpretation of the meaning of \( m(a)^+ \) and \( m(a)^- \) suggests that one may make use of certain geometrical language and terminology to introduce a pair of descriptive assumptions involving some new concepts and terms, as follows.
Assumption 1 For any given monad $m(a)(a \in R)$, the right-leap $m(a)^+$ links $m(a)$ to its right-neighboring monad $m^+(a) \equiv m(a^+)$ having center at $a^+$, and the left-leap $m(a)^-$ links $m(a)$ to its left-neighboring monad $m^-(a) \equiv m(a^-)$ having center at $a^-$, where $a^+$ and $a^-$ are assumed to be implicitly contained in $R$ and may be called invisible real numbers of $R$.

Assumption 2 For given $m(a)$ with its neighboring monads $m(a^+)$ and $m(a^-)$, the intervals $(a, a^+)$ and $(a^-, a)$ are called hyperstandard micro-intervals with a micro-length $\lambda$ defined as a hyperstandard micro-difference $\lambda := a^+ - a = a - a^-$, where $\lambda$ is also an invisible number in $R$.

Certainly the ordering relation “<” used for the elements of $^\ast R$ may be naturally extended to apply to the new entities $m(a)^+$, $m(a)^-$, $a^+$ and $a^-$, etc. Thus in accordance with Assumptions 1 and 2, we have

$$m(a^-) < m(a) < m(a)^+ < m(a^+),$$

$$a^- = a - \lambda < a < a + \lambda < a^+.$$

Remark 1 Note that for any two distinct real numbers $a$ and $b$, the difference $d = a - b$ (with $|d| > 0$) is again a real number of $R$. Here $a, b$ and $d$ may be called visible real numbers of $R$. Thus this means that the usual meaning for the distinctness of two real numbers is restricted to the case where the difference between two numbers should be a visible number. Actually, the 4 basic arithmetical operations are usually and only applied to all the visible numbers of $R$. However, Assumptions 1 and 2 imply that two real numbers of $R$ may differ by an invisible micro-difference $\lambda$. This also suggests a possible extension that the basic arithmetical operations could be assumed to be still applicable to the invisible elements of $R$. This will be made more precise in the next section of this paper.

Observe that $(a, a^+)$ consists of $m(a)^+$ and the non-standard elements $a + \varepsilon$ and $a^+ - \varepsilon$ for all $\varepsilon > 0$ with $m(\varepsilon) = 0$. Also, $(a^-, a)$ consists of $m(a)^-$ and $a^- + \varepsilon, a - \varepsilon$ for $\varepsilon > 0$ with $m(\varepsilon) = 0$. However, both $m(a)^+$ and $m(a)^-$ are empty-interior micro-intervals within $^\ast R$, just linking pairs of neighboring monads.

Considering $m(a)^+$ and $m(a)^-$ as extra-elements added to $^\ast R$, and using the set-theoretic operation $U$ (union), we may define the following concepts.

Definition 2 For given $m(a)$, the set-theoretic union given by

$$\overline{m(a)} := m(a) \cup \{m(a)^+, m(a)^-\}$$

is called “Poincaré’s non-punctiform element” with center at $a$. Moreover, the collection given by

$$\Lambda := \{m(a)^+, m(a)^- | a \in R\}$$

is called the leap-structure.

Definition 3 Poincaré-type continuum (PC) and Poincaré’s intimate bond (IB) can be defined as follows

$$(PC) := \bigcup_{a \in R} \overline{m(a)}, \quad (IB) := (PC) \setminus R$$
where \( A \backslash B \) denotes the difference set of \( A \) minus \( B \).

Obviously, (PC) may be rewritten as

\[
(PC) \equiv ^*R \cup A = R \cup (IB).
\]

This means that, as a hyperstandard model of continuum, (PC) is the nonstandard model \(^*R\) enriched with the supplemental construction \( \Lambda \). Consequently, in accordance with Assumption 1 and 2, we may also denote (PC) by the following set-theoretic unions consisting of micro-intervals

\[
(PC) = \bigcup_{a \in R} (a^-, a^+) = \bigcup_{a \in R} [a, a^+ = \bigcup_{a \in R} (a^-, a]
\]

where \([\cdots]\) and \((\cdots)\) denote semi-closed intervals.

Remark 2 Notice that Assumption 2 involves \( \lambda = 0^+ - 0 = 0^+ \). In the next section (§3) we shall show that, \( \lambda \), as a “semi-infinitesimal” (the basic component for Poincaré’s (IB)), is larger than every infinitesimal \( \varepsilon \) in \(^*R\) with \( st(\varepsilon) = 0 \), and less than every \( \delta > 0 (\delta \in R) \). From certain viewpoint, \( \lambda \equiv 0^+ \) may be regarded as a common concept involving Cavalieri’s 1-dimensional indivisibles, Leibniz’s monads, Newton’s linelets (timelets), Weyl’s transitional point (time-point), etc., as historically original particular instances.

3. A few propositions relating to the [PC] model

As may be observed, (PC) as a hyperstandard model may reflect certain original idea of Poincaré for the structure of continuum. However, the definition described in §2 is not useful for analytic computations, since \( m(a)^+ \) and \( m(a)^- \) have been treated as abstract elements representing void micro-intervals without any computable elements. In order to remove this defect, we may define an enriched model [PC] instead of (PC) as follows.

Definition 4 Let \( \overline{R} \) be an extension of \( R \), consisting of all the real numbers \( x \in R \) and all the elements generated by \( x \) and \( \lambda \) via the 4 basic arithmetic operations (+, −, ·, ÷), so that \( \overline{R} \) is an extended number field containing, e.g., \( \alpha \lambda \) and \( \alpha / \lambda (\alpha \in R) \) as invisible elements. Then the hyperstandard set given by

\[
[PC] := ^*R \cup \overline{R}
\]

is called the second model of Poincaré-type continuum, or simply [PC]-model.

Obviously [PC] contains all the following micro-intervals with \( a \in R \):

\[
[a, a^+] := \{a + x \lambda | 0 \leq x \leq 1, x \in R\} \cup \{a + \varepsilon, a^+ - \varepsilon | \varepsilon > 0, \varepsilon \in m(0)\},
\]

\[
[a^-, a] := \{a - x \lambda | 0 \leq x \leq 1, x \in R\} \cup \{a^- + \varepsilon, a - \varepsilon | \varepsilon > 0, \varepsilon \in m(0)\},
\]

\[
[a^-, a^+] := \{a \pm x \lambda | 0 \leq x \leq 1, x \in R\} \cup m(a) \cup \{a^- + \varepsilon, a^+ - \varepsilon | \varepsilon > 0, \varepsilon \in m(0)\}.
\]

As regards the magnitude of \( \lambda \), we need a concept for the semi-infinitesimal.

Definition 5 A hyperstandard quantity \( \theta \) of [PC] is said to be a semi-infinitesimal if \( \theta \neq 0 \) and it satisfies the following two conditions
(i) $|\theta| < \delta$ for every $\delta > 0$ ($\delta \in \mathbb{R}$);
(ii) $|\varepsilon/\theta| < \delta$ for every given $\varepsilon \in m(0)$ and every $\delta > 0$ ($\delta \in \mathbb{R}$).

Clearly, condition (ii) means that the ratio $\varepsilon/\theta$ is also an infinitesimal. Let us now give the following

**Proposition 1** The hyperstandard quantity $\lambda = 0^+$ is a semi-infinitesimal.

**Proof** Condition (i) for $\lambda$ is obvious. To justify condition (ii) for $\lambda$, let us consider a pair of neighboring monads $m(a)$ and $m(a^+)$. Since the monads are external to each other, and of the same size, the mid-point of the micro-interval $[a, a^+]$, viz. $a + \frac{1}{2}(a^+ - a) = a + \frac{1}{2}\lambda$, does not belong to $m(a)$ and $m(a^+)$. Consequently, for every given $\varepsilon > 0$ with $\text{st}(\varepsilon) = 0$, i.e., $\varepsilon \in m(0)$, we have the inequality

$$a + \varepsilon < a + \frac{1}{2}\lambda.$$ 

As a next step, we take the mid-point of the micro-interval $[a, a + \frac{1}{2}\lambda]$, namely, $a + \frac{1}{4}\lambda$. Clearly this point still does not belong to the monad $m(a)$. Otherwise, it would imply that $a + \frac{1}{2}\lambda$ also belongs to $m(a)$, and a contradiction follows.

Repeating the above argument with successive bisecting process for the micro-intervals, we may get the inequalities

$$a + \varepsilon < a + \lambda/2^n, \quad n = 1, 2, 3, \ldots.$$ 

This implies that $\varepsilon/\lambda < 1/2^n$ ($n \in \mathbb{N}$) and shows that $\varepsilon/\lambda$ is also an infinitesimal. Hence the proposition is proved.

As is known, the “least difference problem” for distinct real numbers cannot be formulated in the classical real analysis. In the NSA, we have seen that any two distinct standard real numbers of $^*\mathbb{R}$, however close to each other, should have an absolute difference greater than any positive infinitesimal $\varepsilon \in m(0)$. Still, the least difference problem cannot be resolved in NSA insomuch as every monad having a unique standard real number as a center does not have a boundary. However, once we have had a hyperstandard model $[PC]$ in which $a^+, a^-$ and $\lambda \equiv 0^+$ have been defined descriptively, we could obtain a definite answer to the least difference problem for the standard real numbers. Indeed, we have the following immediate corollary of Proposition 1.

**Corollary** The difference between any two consecutive real numbers in $R \subset [PC]$ is the semi-infinitesimal $\lambda$. Moreover, the micro-distance between any two neighboring (consecutive) monads is also a semi-infinitesimal.

**Remark 3** As a summary, our descriptive theory follows the following lines. First, our basic assumption that “real numbers are external to each other” and the defining existence of a companion cut ($R_1|R_2$) associated with ($^*R_1|^*R_2$) for $m(a)\pm$ ($a \in R$) could imply the existence of a “gap” from either sides of $a$ ($a \in R$). Consequently, it is capable of assuming the existence of a pair of invisible real numbers $a^+$ and $a^-$ as the right and left consecutive numbers of $a$, respectively. Here it is also natural to suppose that $a^+, a^-$ and $\lambda = a^+ - a = a - a^-$ are included in $R$ as invisible elements. In this way $R$ should be called “non-Cantorian continuum” that has
two characters: As a number field, \( R \) is a field consisting of all ordinary standard real numbers as visible elements (following Archimedian axiom). As a point-set, \( R \) contains both visible and invisible elements, and arithmetical operations do not apply to invisible elements. The 3rd step is to define a genuine extension \( \overline{R} \) of \( R \) so that the hyper-standard model [PC] can be constructed via \( \ast R \) and \( R \).

Use \( \langle 1 \ 0 \rangle \) to indicate that it may be one or zero. Thus \( \langle 1 \ 0 \rangle \rho \) is a number that may be \( \rho \) or zero. Let \( \varepsilon \) and \( \rho \) denote unspecified infinitesimal (with \( \varepsilon \in m(0) \)) and semi-infinitesimal, respectively. Then any finite quantity \( x \) contained in [PC] may be expressed generally in the form

\[
x = 0x + \varepsilon + \langle 1 \ 0 \rangle \rho,
\]

in which \( 0x = \text{st}(x) \in R \) is called the standard part of \( x \). Here “st” means “taking standard part”, and as an operation, it applies to every computational expression involving elements of [PC] whenever the result has a meaning.

As is known, Lebesgue’s measure theory for the point-sets of \( R \) (or of \( R^n \)) could be extended along two ways in the field of NSA. The 1st way is to extend the concept of “\( \sigma \)-additivity” to the case where the number of terms to be added may take any infinite integer \( \omega \in \ast N \) (the set of nonstandard positive integers). The 2nd way is that the family of covering sets consisting of open intervals of small size (used for defining outer measure via taking infimum) can be replaced by that of covering sets involving infinitesimal intervals, etc. Accordingly, a special kind of hyperstandard measure could be defined for the point-sets of \( \ast R \), and it is even possible to show that the standard real number set \( R \) is of hyperstandard measure zero in \( \ast R \) (cf. Reference [3] of [3]).

Surely, the basic idea for constructing hyperstandard measure theory in \( \ast R \) may be applied similarly to the case of [PC].

Let us notice that every monad \( m(\alpha) \) (\( \alpha \in R \)) can be matched with its right-leap \( m(\alpha)^+ \) (or left-leap \( m(\alpha)^- \)), and that the number of monads contained in the interval \( (a, b] \) (with \( a < b \)) may be counted by the semi-infinite number (integer) \((b - a)/\lambda\), namely,

\[
\sum_{a < x \leq b} 1 = \sum_{a < x + k\lambda \leq b} 1 = \frac{(b - a)/\lambda}{k=1} 1 = (b - a)/\lambda,
\]

where the leftmost summation is meaningful in [PC], since the real numbers \( x’s \) of \( (a, b] \) are ordered consecutively with \( \lambda \) as the common difference \( \lambda = x^+ - x \).

Suppose that a kind of hyperstandard linear measure concept very similar to that used in \( \ast R \) has been defined in [PC], by using the widely available covering-technics. Then we could obtain a pair of descriptive propositions as follows.

**Proposition 2** For \( a < b \) with \( a, b \in R \), let \( [a, b] \) be an interval contained in [PC]. Then the set of monads of \( \ast R \) contained in \( [a, b] \) has a hyperstandard linear measure zero. Consequently, the hyperstand linear measure of the standard point-set of \( R \) contained in \( [a, b] \) is also zero.
Proof It suffices to justify the statement for the set of monads contained in \([a, b]\). Notice that monads have no boundaries. However, we can use the usual technic with covering-relations to estimate the upper and lower bounds for any possibly existent outer and inner measures of the set of monads.

For any given positive infinitesimal \(\varepsilon\) (\(\varepsilon \in m(0)\)) and any small real number \(\delta > 0\) (\(\delta \in R\)), we can choose \(m\) so large that
\[
2(b - a)/m < \delta.
\]

Thus, starting from the obvious inclusion-relations for microintervals
\[
[x - \varepsilon, x + \varepsilon] \subset m(x) \subset [x - \lambda/m, x + \lambda/m], \quad x \in R
\]
we may easily get the estimations
\[
\sum_{x \in [a, b]} (2\varepsilon) = 2\varepsilon \cdot \left(\frac{b-a}{\lambda}\right) = 2(b - a)\left(\frac{\varepsilon}{\lambda}\right) = \text{infinitesimal} \quad \text{(cf. Proposition 1)};
\]
\[
\sum_{x \in [a, b]} (2\lambda/m) = (2\lambda/m)\left(\frac{b-a}{\lambda}\right) = 2(b - a)/m < \delta,
\]
where both the summations are taken over all the standard real numbers within \([a, b]\). This shows that the outer and inner measures of the set of monads within \([a, b]\) lie between the positive infinitesimal and the arbitrary small number \(\delta\). Hence, by taking standard part for the computation of measure, we see that the linear measure of the set \(\{m(x)|x \in (a, b), x \in R\}\) is zero.

Recall that the right-leap \(m(x)^{+}\), as a microinterval, is assumed to link \(m(x)\) \((x \in R)\) to its right-neighboring monad \(m(x)^{+}\) \((x^{+} \in R)\). Since both \(m(x)\) and \(m(x)^{+}\) have only fuzzy boundaries, the microlength of \(m(x)^{+}\), denoted by \(|m(x)^{+}|\), is a certain kind of fuzzy quantity. However, for any \(\varepsilon > 0\) with \(\text{st}(\varepsilon) = 0\) and for any given large number \(m > 0\) we have the inclusion-relations for microintervals
\[
[x + \lambda/m, x^{+} - \lambda/m] \subset m(x)^{+} \subset [x + \varepsilon, x^{+} - \varepsilon] = [x + \varepsilon, x + \lambda - \varepsilon].
\]
This implies that
\[
\lambda - \frac{2\lambda}{m} < |m(x)^{+}| < \lambda - 2\varepsilon,
\]
\[
1 - \frac{2}{m} < |m(x)^{+}|/\lambda < 1 - \frac{2\varepsilon}{\lambda}.
\]
Here \(m > 0\) is arbitrary and \(\varepsilon/\lambda\) is an infinitesimal. Hence it follows that
\[
\text{st}(|m(x)^{+}|/\lambda) = 1.
\]
This is also true when \(m(x)^{+}\) is replaced by \(m(x)^{-}\).

Clearly, the above result leads to the following

Proposition 3 For any interval \([a, b] \subset *R \subset [PC]\) with \(a < b\), the length of the interval is
given by the sum of all the microlengths of right-leaps \( m(x)^+ \ (x \in (a, b]) \), namely, we have

\[
\text{st}( \sum_{x \in (a, b]} |m(x)^+| ) = \text{st}( \sum_{x \in (a, b]} \lambda ) = \text{st}(\lambda \cdot \frac{b-a}{\lambda}) = b - a.
\]

**Remark 4** Clearly Proposition 2 implies that Cantor’s point-constituted continuum \( R \) and Robinson’s monad-constituted number field \( {}^*R \) are both of linear measure zero in [PC]. This is basically consistent with the intuitionists’ general view-point that any element-constituted set (totality of individuals) cannot create or represent a real continuum. Moreover, Proposition 3 has the meaning that the length \((b-a)\) for an interval \([a, b]\) in [PC] is generated by the leaps between consecutive points. In other words, Poincaré’s (IB) is the real source that yields the length of any line-segment in [PC]. However, Lebesgue’s measure theory for the point-sets of \( R \) (or \( R^n \)) is still valid, since the covering-technic involves using open intervals whose lengths are positive real numbers. As a matter of fact, both Propositions 2 and 3 simply imply that the classical measure theory is an analytic tool not for measuring the point-sets themselves, but for measuring the spaces containing the point-sets.

4. An application of \( \overline{R} \) in calculus

It may be doubtful whether the distinction between the concepts of infinitesimals and semi-infinitesimals is useful for mathematical analysis. However, the work of John Bell[1] concerning smooth analysis in which the concept of “nilsquare infinitesimal” plays an important role, could suggest that the semi-infinitesimals \( \rho := \alpha \lambda \ (\alpha \in R) \) may be used as nilsquare microquantities for smooth analysis. Generally, it may be an interesting problem of finding possible applications of the hyperstandard model [PC] in mathematical analysis. Surely, it should be a hard task to achieve a formalized theory of [PC] as well as to make some genuine applications of [PC] to analysis.

The objective of this section is to show that \( \overline{R} \) as a proper part of [PC], viz. \( \overline{R} = [PC] \setminus ({}^*R \setminus R) \), may be used straight-forwardly to construct the ordinary concepts of differentiation and integration, in which just the semi-infinitesimals play a part instead of the infinitesimals of \( {}^*R \).

For brevity, we simply call every \( \varepsilon \in m(0) \) of \( {}^*R \) and \( \alpha \lambda \ (\alpha \in R, \lambda \in \overline{R}) \) the \( {}^*R \)-infinitesimal and \( \overline{R} \)-infinitesimal, respectively. In the calculus established on \( \overline{R} \), we will make use of \( \overline{R} \)-infinitesimals throughout. Moreover, in order to get computational results in terms of visible numbers of \( \overline{R} \), the “st” (taking standard part) operation will be frequently attached to every expression (or formula) involving \( \overline{R} \)-infinitesimals \( \theta = \alpha \lambda \ (\alpha \in R) \). Here the common rule for using st-operation may be illustrated as follows.

Any two numbers \( x \) and \( y \) of \( \overline{R} \) are said to be “\( \overline{R} \)-infinitesimally close”, provided that \( x \) and \( y \) differ by a \( \overline{R} \)-infinitesimal, namely

\[
\text{st}(x - y) = ^0(x - y) = 0.
\]

This is equivalent to each of the following expression

\[
x \simeq y, \quad \text{st}(x) = \text{st}(y), \quad ^0x = ^0y,
\]
where \(0^x\) and \(0^y\) are visible elements of \(R\).

In accordance with our descriptive theory, every visible number \(x \in \overline{R}\) may be expressed as a product of two invisible numbers \(x = \nu(x) \cdot \lambda\), where \(\nu(x) = x/\lambda \in \overline{R}\). We define \(\nu(0) = 0\). Clearly, \(\nu(x)\) just represents the number of invisible numbers \(k\lambda (k \in N)\) contained in the interval \((0, x]\).

It is easily seen that \(x = \nu(x)\lambda\) and \(y = \nu(y)\lambda\) imply the following relations

\[
\nu(x) \pm \nu(y) = \nu(x \pm y),
\nu(x)\nu(y) = \nu(1)\nu(xy),
\nu(x)/\nu(y) = \nu(x/y)\lambda = x/y, \quad y \neq 0.
\]

Evidently we can construct the summations of \(\overline{R}\)-infinitesimals of the form

\[
\sum_{a \leq x \leq b} f(x)(x^+ - x) = \sum_{0 \leq k \leq \nu(b-a)} f(a + k\lambda)\lambda.
\]

In fact, this type of summation is the source for Leibniz’s invention of integration process. According to Leibniz, the summation on the left-hand side (LHS) may be written as \(\int_a^b f(x)dx\).

In parallel to the classical calculus, a special kind of nonstandard analysis may easily be established on \(\overline{R}\). Let us introduce a few basic definitions as follows.

1\(^{0}\). Suppose that \(f : \overline{R} \rightarrow \overline{R}\) is a single-valued mapping such that for \(x \in \overline{R}\) there holds \(f(x + \theta) \simeq f(x)\) whenever \(st(\theta) = 0\) (i.e., \(\theta\) is a \(\overline{R}\)-infinitesimal). Then \(f(x)\) is called a continuous function. Of course, the above definition can also be stated for the mapping \(f\) defined between two intervals of \(\overline{R}\).

2\(^{0}\). Given a continuous function \(f(x)\). Suppose that for every \(\theta > 0\) with \(st(\theta) = 0\) we have definite values

\[
0((f(x + \theta) - f(x))/\theta) = f'_+(x), \quad 0((f(x - \theta) - f(x))/(-\theta)) = f'_-(x).
\]

Then \(f'_+\) and \(f'_-\) are called the right-derivative and left-derivative of \(f(x)\), respectively. In particular, if \(f'_+(x) = f'_-(x)\), then \(f(x)\) is said to be a differentiable function, and \(f'(x) = f'_+(x)\) is called the derivative of \(f(x)\).

3\(^{0}\). Suppose that \(\phi(x)\) is a continuous function defined on \([a, b]\) of \(\overline{R}\), where \(a\) and \(b\) are visible real numbers. Then

\[
\int_a^b \phi(x)dx := 0(\sum_{a \leq x < b} \phi(x)(x^+ - x)) = 0(\sum_{1 \leq k \leq \nu(b-a)} \phi(a + (k-1)\lambda) \cdot \lambda)
\]

is called the Newton-Leibniz-type integration, where the range of summation \(a \leq x < b\) may also be written as \(a \leq x \leq b^-\).

Also, we shall need an analogue of Duhamel’s principle of the form: Let \(\theta > 0\) with \(st(\theta) = 0\) and let \(\nu = \alpha/\theta, \alpha > 0, \alpha \in R\). Suppose that there are given two sequences of numbers \(\{a_k\}\) and \(\{b_k\}\) \((1 \leq k \leq \nu)\) in \(\overline{R}\) such that \(a_k \simeq b_k\) for every \(k\). Then we have

\[
0((\sum_{1 \leq k \leq \nu} a_k\theta)) = 0((\sum_{1 \leq k \leq \nu} b_k\theta)).
\]
This may be checked easily. Notice that the condition \( a_k \simeq b_k \) for all \( k(1 \leq k \leq \nu) \) implies that \( \max_k |a_k - b_k| \) is a \( \overline{R} \)-infinitesimal. Thus the conclusion of the principle follows from the fact as shown below:

\[
\begin{align*}
\left| \sum_{1 \leq k \leq \nu} a_k \theta - \sum_{1 \leq k \leq \nu} b_k \theta \right| &= \sum_{1 \leq k \leq \nu} |a_k - b_k| \theta \\
&\leq \max_k |a_k - b_k| \cdot \sum_{1 \leq k \leq \nu} \theta \\
&= (\nu \theta) \max_k |a_k - b_k| = \overline{R} \text{-infinitesimal}.
\end{align*}
\]

It is now easy to state and prove 3 basic theorems for the integral calculus as follows.

**Theorem 1** (Newton-Leibniz 1st formula) Let \( f(x) \) have a continuous derivative \( f'(x) \) on \([a, b] \subset \overline{R} \). Then there holds the fundamental formula

\[
\int_a^b f'(x)dx = f(b) - f(a).
\]

**Proof** Notice that the definition of derivative implies that

\[
\{f(a + k\lambda) - f(a + (k - 1)\lambda)\}/\lambda \simeq f'_\nu(a + (k - 1)\lambda) = f'(a + (k - 1)\lambda).
\]

Thus, using the analogue of Duhamel’s principle with \( \theta = \lambda \), we obtain

\[
\int_a^b f'(x)dx = 0\left(\sum_{1 \leq k \leq \nu(b-a)} f'(a + (k - 1)\lambda) \cdot \lambda \right) \\
= 0\left(\sum_{1 \leq k \leq \nu(b-a)} \frac{f(a + k\lambda) - f(a + (k - 1)\lambda)}{\lambda} \cdot \lambda \right) \\
= 0(f(a + \nu(b-a)\lambda) - f(a)) = f(b) - f(a). \quad \Box
\]

**Theorem 2** (Summation formula of \( \overline{R} \)-infinitesimals) Let \( f(x) \) be given as in Theorem 1. Then for every \( \theta > 0 \) with \( \text{st}(\theta) = 0 \) we have the formula

\[
\sum_{0 \leq k < (b-a)/\theta} f'(a + k\theta)\theta \simeq \int_a^b f'(x)dx = f(b) - f(a),
\]

where the summation-range \( 0 \leq k < (b-a)/\theta \) may be written as \( 0 \leq k \leq [(b-a)/\theta] \), where \( [x] \) denotes the integer part of \( x \) (\( x \in \overline{R} \)) with \( x - 1 < [x] \leq x \).

The proof of Theorem 2 is entirely similar to that of Theorem 1, and may be omitted.

Recall that in the definition of integration, the lower and upper limits \( a \) and \( b \) are assumed to be visible numbers of \( R \subset \overline{R} \). Generally, we may also assumed that the limits of integration may take invisible real numbers such as \( m\lambda \) (\( m \in N \)) and \( \nu\lambda \) (\( \nu \) being \( \overline{R} \)-infinite integer). Consequently \( x \pm m\lambda \) (\( x \in R, m \in N \)) are allowed to be limits of integration, and an integration of the form \( \int_a^{\lambda/2} f(x)dx \) has no meaning.

**Theorem 3** (Newton-Leibniz 2nd formula) Let \( F(x) = \int_a^x f(t)dt \), where \( a < x \) and \( f(t) \) is a continuous function. Also, we define \( F'_+(x) \) and \( F'_-(x) \) to be the right and left derivatives of
F(x) by the following micro-difference-quotients

\[ F'(x) = \frac{F(x + \theta) - F(x)}{\theta} = F'_+(x), \quad F'(x - \theta) = F'_-(x), \]

where \( \theta = m\lambda \) \((m \in \mathbb{N})\) so that \( \text{st}(\theta) = 0 \). Then we have the formula

\[ F'_+(x) = F'_-(x) = f'(x). \]

**Proof** It suffices to show that \( F'_+(x) = f(x) \). Observe that the definition of integration implies the simple properties \( \int_a^b dx = b - a, \int_a^c + \int_c^b = \int_a^b \) with \( a < b < c \). Thus we have

\[ F'_+(x) = 0 \left\{ \frac{1}{\theta} \left( \int_x^{x+\theta} f(t) dt - \int_x^x f(t) dt \right) \right\} = 0 \left\{ \frac{1}{\theta} \int_x^{x+\theta} f(t) dt \right\}. \]

Denote \( \underline{f} = \min \{ f(t) | x \leq t \leq x + \theta \} \) and \( \overline{f} = \max \{ f(t) | x \leq t \leq x + \theta \} \). The continuity of \( f(x) \) implies that \( \underline{f} \simeq \overline{f} \simeq f(x) \). Consequently we obtain

\[ 0 \left\{ \frac{1}{\theta} \int_x^{x+\theta} f(t) dt \right\} = 0 \left\{ \frac{1}{\theta} \int_x^{x+\theta} f(x) dt \right\} = 0 \left\{ \frac{1}{\theta} f(x) \cdot \theta \right\} = f(x). \]

**Remark 5** It may be worth mentioning that the quasi-Duhamel Principle\(^2\) used in NSA can be transferred to our non-standard analysis on \( \mathbb{R} \): Let \( \{ \theta_j \} \) and \( \{ \theta'_j \} \) \((1 \leq j \leq \nu)\) be two sequences of position \( \mathbb{R} \)-infinitesimals such that \( \theta_j/\theta'_j \simeq 1 \) and \( \sum_{j=1}^{\nu} \theta_j = s \) (finite number), where \( \nu = \alpha/\lambda \) with \( 0 < \alpha \in \mathbb{R} \) is a \( \mathbb{R} \)-infinite integer. Suppose that \( \{ r_j \} \) and \( \{ r'_j \} \) are sequences of finite numbers in \( \mathbb{R} \) such that \( r_j \simeq r'_j \) \((1 \leq j \leq \nu)\). Then we have

\[ \sum_{j=1}^{\nu} r_j \theta_j \simeq \sum_{j=1}^{\nu} r'_j \theta'_j, \quad \text{i.e.,} \quad 0 \left\{ \sum_{j=1}^{\nu} r_j \theta_j \right\} = 0 \left\{ \sum_{j=1}^{\nu} r'_j \theta'_j \right\}. \]

The proof is entirely similar to that for the quasi-Duhamel Principle, and the result may be used for treating the Riemann-Stieltjes integration in the non-standard analysis in \( \mathbb{R} \).

**References**