On \((m, n)\)-Coherent Modules and Preenvelopes

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Abstract In this paper, let \(m, n\) be two fixed positive integers and \(M\) be a right \(R\)-module, we define \((m, n)\)-\(M\)-flat modules and \((m, n)\)-coherent modules. A right \(R\)-module \(F\) is called \((m, n)\)-\(M\)-flat if every homomorphism from an \((n, m)\)-presented right \(R\)-module into \(F\) factors through a module in \(\text{add}M\). A left \(S\)-module \(M\) is called an \((m, n)\)-coherent module if \(M_{R}\) is finitely presented, and for any \((n, m)\)-presented right \(R\)-module \(K\), \(\text{Hom}(K, M)\) is a finitely generated left \(S\)-module, where \(S = \text{End}(M_{R})\). We mainly characterize \((m, n)\)-coherent modules in terms of preenvelopes (which are monomorphism or epimorphism) of modules. Some properties of \((m, n)\)-coherent rings and coherent rings are obtained as corollaries.

Keywords \((m, n)\)-\(M\)-flat module; \((m, n)\)-coherent module; \((m, n)\)-\(M\)-flat preenvelope.

1. Introduction

Throughout this paper, \(R\) denotes an associative ring with identity and all modules are unitary. For a right \(R\)-module \(A\), \(E(A)\) denotes the injective envelope of \(A\) and \(i : A \rightarrow E(A)\) denotes the inclusion map. Given a right \(R\)-module \(M\), \(M_{I}\) stands for the direct product of copies of \(M\) indexed by \(I\), and \(\text{add}M\) indicates the category consisting of all right \(R\)-modules isomorphic to direct summands of finitely direct sums of copies of \(M\). We simplify \(\text{Hom}_{R}(A, B)\) to \(\text{Hom}(A, B)\) for right \(R\)-modules \(A, B\).

Let \(\mathcal{C}\) be a class of right \(R\)-modules and \(A\) be a right \(R\)-module. A homomorphism \(\varphi : A \rightarrow C\) with \(C \in \mathcal{C}\) is called a \(\mathcal{C}\)-preenvelope of \(A\) if for any homomorphism \(f : A \rightarrow C'\) with \(C' \in \mathcal{C}\), there is a homomorphism \(g : C \rightarrow C''\) such that \(g\varphi = f^{[1]}\). Moreover, if the only such \(g\) is automorphism of \(C\) when \(C' = C\) and \(f = \varphi\), the \(\mathcal{C}\)-preenvelope \(\varphi\) is called a \(\mathcal{C}\)-envelope of \(A\). Following [2], a \(\mathcal{C}\)-envelope of \(A\) \(\varphi : A \rightarrow C\) has the unique mapping property if for any homomorphism \(f : A \rightarrow C'\) with \(C' \in \mathcal{C}\), there is a unique homomorphism \(g : C \rightarrow C'\) such that \(g\varphi = f\).

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Let $m$ and $n$ be two fixed positive integers. Recently, $(m, n)$-flat modules and $(m, n)$-coherent rings were introduced and studied in [3],[4],[5]. A right $R$-module $K$ is said to be $(m, n)$-presented if there exists an exact sequence of right $R$-modules $0 \rightarrow L \rightarrow R^m \rightarrow K \rightarrow 0$, where $L$ is $n$-generated[5]. A right $R$-module $A$ is said to be $(m, n)$-flat if $A \otimes_R I \rightarrow A \otimes_R R^m$ is a monomorphism for all $n$-generated submodule $I$ of the left $R$-module $R^m[5]$. Moreover, a ring $R$ is said to be left $(m, n)$-coherent if each $n$-generated submodule of the left $R$-module $R^m$ is finitely presented[5]. In this paper, we introduce the concepts of $(m, n)$-$M$-flat modules and $(m, n)$-coherent modules. Let $M$ be a finitely presented right $R$-module with $S = \text{End}(M_R)$ and $m, n$ fixed positive integers, it is showed that $SM$ is $(m, n)$-coherent if and only if every right $R$-module has an $(m, n)$-$M$-flat preenvelope; $SM$ is $(m, n)$-coherent and injective right $R$-modules are $(m, n)$-$M$-flat if and only if every right $R$-module has an $(m, n)$-$M$-flat preenvelope which is a monomorphism; $SM$ is $(m, n)$-coherent and every submodules of $(m, n)$-$M$-flat right $R$-modules are $(m, n)$-$M$-flat if and only if every right $R$-module has an $(m, n)$-$M$-flat preenvelope which is an epimorphism. In particular, some results of left $(m, n)$-coherent rings and left coherent rings are obtained as corollaries.

2. $(m, n)$-Coherent modules and preenvelopes

**Definition 2.1** Let $M$ be a right $R$-module. A right $R$-module $F$ is called $(m, n)$-$M$-flat if every homomorphism from a $(n, m)$-presented right $R$-module into $F$ factors through a module in $\text{add} M$, i.e., for any $(n, m)$-presented right $R$-module $K$ and any homomorphism $f : K \rightarrow F$, there exit a module $X$ in $\text{add} M$ and homomorphisms $g : K \rightarrow X$, $h : X \rightarrow F$ such that $f = hg$.

**Remark**

(1) By [5], a right $R$-module $F$ is $(m, n)$-flat if and only if for every homomorphism from $(n, m)$-presented right $R$-module into $F$ factors through a free module. Hence $(m, n)$-$R$-flat modules are just $(m, n)$-flat right $R$-modules.

(2) It is easy to see that $F$ is an $(m, n)$-$M$-flat right $R$-module if and only if for any $(n, m)$-presented right $R$-module $K$, any homomorphism $f : K \rightarrow F$, there exist a positive integer $s$ and homomorphisms $g : K \rightarrow M^s$, $h : M^s \rightarrow F$ such that $f = hg$.

(3) By definition, the class of $(m, n)$-$M$-flat right $R$-modules is closed under direct summands, finitely direct sums.

(4) If $X$ is in $\text{add} M$, then $X$ is $(m, n)$-$M$-flat; if $X$ is $(n, m)$-presented, and $X$ is $(m, n)$-$M$-flat, then $X$ is in $\text{add} M$.

(5) If $M$ is a projective right $R$-module, $F$ is an $(m, n)$-$M$-flat right $R$-module, then $F$ is $(m, n)$-flat.

**Definition 2.2** For a right $R$-module $M$, $S$ denotes the ring $\text{End}(M_R)$. $SM$ is called an $(m, n)$-coherent module if $M_R$ is finitely presented, and for any $(n, m)$-presented right $R$-module $K$, $\text{Hom}(K, M)$ is a finitely generated left $S$-module.

Let $M$ be a right $R$-module, $S = \text{End}(M_R)$ and $A, B$ be right $R$-modules. We use $\sigma_{A,B}$ denote the homomorphism $\text{Hom}(M, A) \otimes_S \text{Hom}(B, M) \rightarrow \text{Hom}(B, A)$ given by $\sigma_{A,B}(f \otimes g) = fg$.
where \( f \in \text{Hom}(M, A) \), \( g \in \text{Hom}(B, M) \). It is easy to see that if \( C \) is a direct summand of \( B \), and \( \sigma_{A,B} \) is an isomorphism, then \( \sigma_{A,C} \) is also an isomorphism. Hence if \( B \in \text{add}M \), then \( \sigma_{A,B} \) is an isomorphism. Moreover, it is clear that if \( A \in \text{add}M \), then \( \sigma_{A,B} \) is also an isomorphism.

**Proposition 2.1** Let \( M \) be a right \( R \)-module. The following statements are equivalent:

1. \( F \) is an \((m,n)\)-\( M \)-flat right \( R \)-module;
2. For any \((n,m)\)-presented right \( R \)-module \( K \), \( \sigma_{F,K} \) is an epimorphism.

**Proof** (1) \( \Rightarrow \) (2). Let \( K \) be an \((n,m)\)-presented right \( R \)-module. For any \( f \in \text{Hom}(K, F) \), by the definition of \((m,n)\)-\( M \)-flatness, there exist a positive integer \( k \) and homomorphisms \( g : K \to M^k \) and \( h : M^k \to F \) such that \( f = hg \). Let \( p_i : M^k \to M \) and \( \lambda_i : M \to M^k \) denote the \( i \)-th canonical projection and canonical injection respectively. Put \( g_i = p_i g \), \( h_i = h \lambda_i \), then \( \sum_{i=1}^{k} (h_i \otimes g_i) \in \text{Hom}(M, F) \otimes_S \text{Hom}(K, M) \) and \( \sigma_{F,K}(\sum_{i=1}^{k} (h_i \otimes g_i)) = \sum_{i=1}^{k} h_i g_i = \sum_{i=1}^{k} h_i \lambda_i p_i g = f \). Hence \( \sigma_{F,K} \) is an epimorphism.

(2) \( \Rightarrow \) (1). Let \( K \) be an \((n,m)\)-presented right \( R \)-module, and \( f \in \text{Hom}(K, F) \). By hypothesis, \( f = \sigma_{F,K}(\sum_{i=1}^{k} (h_i \otimes g_i)) \) for some \( h_i \in \text{Hom}(M, F) \), \( g_i \in \text{Hom}(K, M) \). Put \( X = M^k \). We define \( g : K \to M^k \) by \( g(x) = (g_1(x), \ldots, g_k(x)) \) for every \( x \in K \) and \( h : M^k \to F \) by \( h(m_1, \ldots, m_k) = \sum_{i=1}^{k} h_i (m_i) \) for every \( (m_1, \ldots, m_k) \in M^k \). Then \( f = hg \) and so \( F \) is \((m,n)\)-\( M \)-flat.

**Proposition 2.2** Let \( M \) be a pure projective right \( R \)-module. Then every pure submodule of \((m,n)\)-\( M \)-flat right \( R \)-module is \((m,n)\)-\( M \)-flat.

**Proof** Let \( A \) be a pure submodule of \((m,n)\)-\( M \)-flat module \( B \), and \( K \) be an \((n,m)\)-presented right \( R \)-module. Let \( j : A \to B \) denote the inclusion map, and \( \pi : B \to B/A \) denote a canonical epimorphism. For any \( f \in \text{Hom}(K, A) \), then there exist a module \( X \) in \( \text{add}M \) and homomorphisms \( g : K \to X \), \( h : X \to B \) such that \( jf = hg \). Note that there is a pure projective right \( R \)-module \( Y \) such that \( \alpha : Y \to B \) is a pure epimorphism by [6]. Since \( M \) is pure projective, so \( X \) is pure projective, and there is a homomorphism \( \beta : X \to Y \) such that \( \alpha \beta = h \). Put a pullback of \( \alpha : Y \to B \) and \( j : A \to B \), we have the following commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{g} & X \\
\downarrow{\beta} & & \downarrow{\pi}
\end{array}
\]

\[
\begin{array}{cccc}
0 & \xrightarrow{\lambda} & U & \xrightarrow{\pi} & B/A & \xrightarrow{g} & 0 \\
\downarrow{\delta} & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\pi}
\end{array}
\]

\[
\begin{array}{cccc}
0 & \xrightarrow{j} & A & \xrightarrow{\pi} & B/A & \xrightarrow{g} & 0 \\
\end{array}
\]

Note that \( \pi \alpha \beta g = \pi hg = \pi jf = 0 \), it follows that \( \beta g(K) \subseteq \text{Ker}(\pi \alpha) = \text{Im} \lambda \). But \( \lambda \) is a monomorphism, hence there is a submodule \( V \) of \( U \) such that for any \( y \in K \), there is unique \( x \in V \) satisfying \( \beta g(y) = \lambda(x) \). It is easy to see that \( \lambda \) is a pure monomorphism since \( j \) is a pure monomorphism and \( \alpha \) is a pure epimorphism. Thus since \( K \) is finitely generated, \( V \) is finitely generated. By [7], there exists a homomorphism \( \gamma : Y \to U \) such that \( \gamma \lambda(x) = x \) for any \( x \in V \).
Let \( k = \delta \gamma \beta : X \rightarrow A \). Then for any \( y \in K \),

\[
kg(y) = \delta \gamma \beta g(y) = \delta \gamma \lambda(x) = \delta = \alpha \lambda(x) = hg(y) = f(y).
\]

Therefore \( A \) is \((m,n)\)-M-flat.

**Lemma 2.3**\(^8\) Let \( M, A \) be right \( R \)-modules and \( S = \text{End}(M_R) \). Then \( A \) has an add\(M\)-preenvelope if and only if \( \text{Hom}(A,M) \) is a finitely generated left \( S \)-module.

**Proposition 2.4** Let \( M \) be a right \( R \)-module and \( S = \text{End}(M_R) \). The following statements are equivalent:

1. \( sM \) is an \((m,n)\)-coherent module;
2. \( M_R \) is finitely presented and every \((n,m)\)-presented right \( R \)-module has an add\(M\)-preenvelope;
3. \( M_R \) is finitely presented and every \((n,m)\)-presented right \( R \)-module has an \((m,n)\)-\(M\)-flat preenvelope.

**Proof** By Lemma 2.3, (1) \( \Leftrightarrow \) (2) is clear.

(2) \( \Rightarrow \) (3). For any \((n,m)\)-presented right \( R \)-module \( K \), it has an add\(M\)-preenvelope \( f : K \rightarrow X \) with \( X \in \text{add}M \). For any \((m,n)\)-flat right \( R \)-module \( F \) and any homomorphism \( g : K \rightarrow F \), there exist \( X_1 \in \text{add}M \) and homomorphisms \( g_1 : K \rightarrow X_1, g_2 : X_1 \rightarrow F \) such that \( g = g_2 g_1 \).

Note that \( f \) is an add\(M\)-preenvelope, it follows that there is a homomorphism \( h : X \rightarrow X_1 \) such that \( hf = g_1 \). Hence \( g_2 h f = g_2 g_1 = g \), that is, \( f \) is an \((m,n)\)-\(M\)-flat preenvelope.

(3) \( \Rightarrow \) (2). For any \((n,m)\)-presented right \( R \)-module \( K \), it has an \((m,n)\)-\(M\)-flat preenvelope \( f : K \rightarrow F \). Hence there exist a right \( R \)-module \( X \in \text{add}M \) and homomorphisms \( g : K \rightarrow X, h : X \rightarrow F \) such that \( f = h g \). It is easy to see that \( g : K \rightarrow X \) is an add\(M\)-preenvelope of \( K \).

**Theorem 2.5** Let \( M \) be a finitely presented right \( R \)-module and \( S = \text{End}(M_R) \). The following statements are equivalent:

1. \( sM \) is an \((m,n)\)-coherent module;
2. Every right \( R \)-module has an \((m,n)\)-\(M\)-flat preenvelope;
3. Every \((n,m)\)-presented right \( R \)-module has an add\(M\)-preenvelope;
4. Every \((n,m)\)-presented right \( R \)-module has an \((m,n)\)-\(M\)-flat preenvelope;
5. \( \prod_{i \in I} M \) is an \((m,n)\)-\(M\)-flat right \( R \)-module for any index set \( I \);
6. The direct products of \((m,n)\)-\(M\)-flat right \( R \)-modules is \((m,n)\)-\(M\)-flat;
7. \( \text{Hom}_S(P,M) \) is an \((m,n)\)-\(M\)-flat right \( R \)-module for any projective left \( S \)-module \( P \).

**Proof** (2) \( \Rightarrow \) (1), (6) \( \Rightarrow \) (5) are trivial and by Proposition 2.4, (1) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4).

(1) \( \Rightarrow \) (6). Let \( \{F_i\}_{i \in I} \) be a family of \((m,n)\)-\(M\)-flat right \( R \)-modules and \( K \) be an \((n,m)\)-presented right \( R \)-module. We denote by \( p_i : \prod_{i \in I} F_i \rightarrow F_i \) the ith canonical projection. For any \( f \in \text{Hom}(K, \prod_{i \in I} F_i) \), \( p_i f \) factors through \( X_i \) with \( X_i \in \text{add}M \), that is, there are homomorphisms \( g_i : K \rightarrow X_i \) and \( h_i : X_i \rightarrow F_i \) such that \( p_i f = h_i g_i \). Note that \( sM \) is \((m,n)\)-coherent, it follows that \( K \) has an add\(M\)-preenvelope \( g : K \rightarrow X \). Hence there is a homomorphism
k_i : X → X_i such that k_ig = g_i. By the universal property of direct products, there is a homomorphism h : X → \prod_{i \in I} F_i such that p_ih = h_i k_i, then p_ihg = h_ik_ig = h_ig_i = p_i f. Therefore hg = f.

(6) ⇒ (2). Let A be a right R-module with CardA ≤ \aleph. For any \((m,n)\)-M-flat module F and any homomorphism f : A → F, Cardf(A) ≤ \aleph. Let \(\aleph_{\alpha} = \max\{\text{Card}R, \aleph\} \). Then by [9], there is a pure submodule S of F such that f(A) ⊆ S and CardS ≤ \(\aleph_{\alpha}\). By Proposition 2.2, S is \((m,n)\)-M-flat. Let \((\varphi_i)_{i \in I}\) give all such homomorphisms \(\varphi_i : A → S_i\) with CardS_i ≤ \(\aleph_{\alpha}\). So any homomorphism A → F has a factorization A → S_j → F for some j ∈ I. Hence A → \(\prod_{i \in I} S_i\) is an \((m,n)\)-M-flat preenvelope since \(\prod_{i \in I} S_i\) is \((m,n)\)-M-flat by (6).

(5) ⇒ (1). We need to prove for any \((n,m)\)-presented right R-module A, Hom(A,M) is a finitely generated left S-module. Let \(p_i : M^I → M\) denote the ith canonical projection. By Lemma 3.2.21 in [9], this is equivalent to prove that for any index set I, \(\tau : \text{Hom}(M,M^I) \otimes S \text{Hom}(A,M) → \text{Hom}(A,M)^I\) given by \(\tau(g \otimes f) = (p_igf)\) is an epimorphism, where f ∈ Hom(A,M), \(g \in \text{Hom}(M,M^I)\). Note that \(\sigma_{M^I,A} : \text{Hom}(M,M^I) \otimes \text{Hom}(A,M) → \text{Hom}(A,M^I)\) is an epimorphism since \(M^I\) is \((m,n)\)-M-flat. \(\theta : \text{Hom}(M,M^I) → \text{Hom}(A,M^I)\) given by \(\theta(f) = (pf)_{i \in I}\) is an isomorphism and \(\tau = \theta \sigma_{M^I,A}\), it follows that \(\tau\) is an epimorphism. Therefore \(sM\) is \((m,n)\)-coherent.

(5) ⇒ (7). Since P is a projective left S-module, Hom(P,M) is isomorphic to a direct summand of \(M^I\) for some index set I. By hypothesis, \(M^I\) is \((m,n)\)-M-flat, therefore Hom(P,M) is \((m,n)\)-M-flat.

(7) ⇒ (5). Put \(sP = S^{(I)}\) for any index set I.

In Ref. [5], it was proved that R is a left \((m,n)\)-coherent ring if and only if \(R^I\) is an \((m,n)\)-flat right R-module for any index set I. Hence by Theorem 2.5, when \(M_R = R_R\), it is easy to see that \(R\) is \((m,n)\)-coherent if and only if \(R\) is left \((m,n)\)-coherent ring. So the equivalence (1)–(3) and (6)–(9) of Theorem 3.1 in Ref. [3] is just Corollary 2.6.

**Corollary 2.6** The following statements are equivalent:

1. R is a left \((m,n)\)-coherent ring;
2. Every right R-module has an \((m,n)\)-flat preenvelope;
3. Every \((n,m)\)-presented right R-module has a finitely generated projective preenvelope;
4. Every \((n,m)\)-presented right R-module has an \((m,n)\)-flat preenvelope;
5. \(\prod_{i \in I} R\) is \((m,n)\)-flat right R-module for any index set I;
6. The direct products of \((m,n)\)-flat right R-modules is \((m,n)\)-flat;
7. Hom(P,R) is an \((m,n)\)-flat right R-module for any projective left R-module.

Let M be a right R-module \(S = \text{End}(M_R)\). We denote Hom(A,M) by \(A^\ast\), where \(A^\ast\) is a left S-module. Put \(A^{**} = \text{Hom}_S(A^\ast, M)\). Define \(\delta_A : A → A^{**}\) by \(\delta_A(a)(f) = f(a)\) for any a ∈ A and f ∈ Hom(A,M). It is clear that if A ∈ addM, then \(\delta_A\) is an isomorphism.

**Proposition 2.7** Let \(sM\) be \((m,n)\)-coherent and K be an \((n,m)\)-presented right R-module. Then K has an addM-envelope if and only if \(K^\ast\) has a projective cover.
Proof Let $f : K \to X$ be an addM-envelope of $K$. Then $f^* : X^* \to K^*$ is an epimorphism and $X^*$ is a finitely generated projective left $S$-module. For any $h \in \text{Hom}(X^*, X^*)$ such that $f^*h = f^*$, since $f^{**}\delta_K = \delta_Xf$, it follows that $\delta_X^{-1}h^*\delta_Xf = \delta_X^{-1}f^{**}\delta_K = \delta_X^{-1}(f^*h)^*\delta_K = \delta_X^{-1}f^{**}\delta_K = f$. Note that $f$ is an addM-envelope of $K$, hence $\delta_X^{-1}h^*\delta_X$ is an isomorphism, and so $h = \delta_X^{-1}h^{**}\delta_X$. It is also an isomorphism. This proves that $f^*$ is a projective cover of $K^*$. Conversely, let $h : P \to K^*$ be a projective cover of $K^*$. Note that $S M$ is $(m, n)$-coherent, $K$ is a finitely generated left $S$-module and so $P$ is finitely generated. It is clear that $P^* \in \text{addM}$. Let $f = h^*\delta_K : K \to P^*$. For any $\alpha : K \to X$ with $X \in \text{addM}$, $X^*$ is a projective left $S$-module, hence there is a homomorphism $\beta : X^* \to P$ such that $h\beta = \alpha^*$. Put $\gamma = \delta_X^{-1}\beta^* : P^* \to X$, then we have $\gamma f = \delta_X^{-1}\beta^*f = \delta_X^{-1}\beta^*h^*\delta_K = \delta_X^{-1}(h\beta)^*\delta_K = \delta_X^{-1}\alpha^*\delta_K = \alpha$. Therefore $f$ is an addM-preenvelope of $K$. If $\varphi : P^* \to P^*$ satisfies $\varphi f = f$. Note that $h^{**}\delta_P = \delta_Kh$ and $\delta_K^*\delta_K = 1_{K^*}$, it follows that $h = \delta_K^*h^{**}\delta_P$. Hence we have $h(\delta_P^{-1}\varphi^*\delta_P) = \delta_K^*h^{**}\varphi^*\delta_P = (\varphi h^*\delta_K)^*\delta_P = (\varphi f)^*\delta_P = f^*\delta_P = \delta_K^*h^{**}\delta_P = h$. Since $h$ is a projective cover of $K^*$, $\delta_P^{-1}\varphi^*\delta_P$ is an isomorphism. And so $\varphi = \delta_P^{-1}\varphi^*\delta_P$ is also an isomorphism. This proves that $f$ is an addM-envelope.

Corollary 2.8 Let $R$ be a left $(m, n)$-coherent ring and $K$ be an $(n, m)$-presented right $R$-module. Then $K$ has a projective envelope if and only if $\text{Hom}(K, R)$ has a projective cover.

3. Preenvelopes which are monomorphism

In this section, we mainly investigate when every right $R$-module has an $(m, n)$-$M$-flat preenvelope which is a monomorphism.

Theorem 3.1 Let $M$ be a finitely presented right $R$-module, $S = \text{End}(M_R)$.

1. $S M$ is $(m, n)$-coherent, every injective right $R$-module is $(m, n)$-$M$-flat.
2. $S M$ is $(m, n)$-coherent, injective envelope of every $(n, m)$-presented right $R$-module is $(m, n)$-$M$-flat.
3. $S M$ is $(m, n)$-coherent, injective envelope of every simple right $R$-module is $(m, n)$-$M$-flat.
4. $S M$ is $(m, n)$-coherent, every $(n, m)$-presented right $R$-module is cogenerated by $M_R$.
5. $S M$ is $(m, n)$-coherent, every $(n, m)$-presented right $R$-module may be embedded in a module in $\text{addM}$.
6. Every right $R$-module has an $(m, n)$-$M$-flat preenvelope which is a monomorphism.
7. Every $(n, m)$-presented right $R$-module has an $(n, m)$-$M$-flat preenvelope which is a monomorphism.
8. Every $(n, m)$-presented right $R$-module has an add$M$-preenvelope which is a monomorphism.

Proof (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (6), (6) $\Rightarrow$ (7), (8) $\Rightarrow$ (5) are trivial.

(2) $\Rightarrow$ (1). Let $E$ be an injective right $R$-module and $K$ be an $(n, m)$-presented right $R$-module. For any homomorphism $f : K \to E$, by hypothesis of (2), $E(K)$ is an $(m, n)$-$M$-flat. Hence there are a right $R$-module $X$ in $\text{addM}$ and homomorphisms $g : K \to X$, $h : X \to E(K)$.
such that \( i = hg \). Note that \( E \) is injective, there exists a homomorphism \( k : E(K) \to E \) such that \( f = ki = khg \). Therefore \( E \) is \((m,n)\)-\(M\)-flat.

(3) \( \Rightarrow \) (4). Let \( K \) be an \((n,m)\)-presented right \(R\)-module. By [10], we need to prove that for any nonzero \( x \in K \), there is a homomorphism \( \varphi : K \to M \) such that \( \varphi(x) \neq 0 \). In fact, there exists a maximal submodule \( L \) of \( xR \), then \( E(xR/L) \) is \((m,n)\)-\(M\)-flat. We let \( \lambda : xR \to K \) denote an inclusion map and \( \pi : xR \to xR/L \) be the canonical epimorphism. By the injectivity of \( E(xR/L) \), there exists a homomorphism \( f : K \to E(xR/L) \) such that \( f\lambda = i\pi \). Note that \( E(xR/L) \) is \((m,n)\) \(-\)flat, it follows that there are a positive integer \( n \) and homomorphisms \( g : K \to M^n, h : M^n \to E(xR/L) \) such that \( hg = f \). Since \( x \neq 0 \), we have \( i\pi(x) \neq 0 \), thus \( g(x) \neq 0 \). Suppose \( g(x) = (m_1, \ldots, m_i, \ldots, m_n) \). Then there is a nonzero element \( m_i \in M \) for some \( 1 \leq i \leq n \). Hence \( p_i g(x) \neq 0 \). Therefore (4) holds.

(4) \( \Rightarrow \) (5). Let \( K \) be an \((n,m)\)-presented right \(R\)-module. By hypothesis, there is a monomorphism \( \lambda : K \to M^I \) for some index set \( I \). Note that \( sM \) is \((n,m)\)-coherent, hence \( M^I \) is \((m,n)\)-\(M\)-flat. It follows that there are a right \(R\)-module \( X \) in \( \text{add} M \) and homomorphisms \( g : K \to X, h : X \to M^I \) such that \( \lambda = hg \). Since \( \lambda \) is monomorphism, \( g \) is a monomorphism, i.e., (5) holds.

(5) \( \Rightarrow \) (1). Let \( E \) be an injective right \(R\)-module, \( K \) an \((n,m)\)-presented right \(R\)-module and \( f : K \to E \) a homomorphism. By hypothesis, there is a monomorphism \( \lambda : K \to X \) with \( X \in \text{add} M \). Note that \( E \) is injective, there is a homomorphism \( g : X \to E \) such that \( f = g\lambda \). Hence \( E \) is \((m,n)\)-\(M\)-flat.

(7) \( \Rightarrow \) (8). For any \((n,m)\)-presented right \(R\)-module \( K \), by hypothesis, there is an \((m,n)\)-\(M\)-flat preenvelope \( f : K \to F \) which is a monomorphism. Then there exist a right \(R\)-module \( X \) in \( \text{add} M \) and homomorphisms \( g : K \to X, h : X \to F \) such that \( f = hg \). It is easy to see that \( g \) is an \( \text{add} M \)-preenvelope which is a monomorphism. Therefore (8) holds.

**Corollary 3.2** The following statements are equivalent:

1. \( R \) is a left \((m,n)\)-coherent ring, every injective right \(R\)-module is \((m,n)\)-flat;
2. \( R \) is a left \((m,n)\)-coherent ring, injective envelope of every \((n,m)\)-presented right \(R\)-module is \((m,n)\)-flat;
3. \( R \) is a left \((m,n)\)-coherent ring, injective envelope of every simple right \(R\)-module is \((m,n)\)-flat;
4. \( R \) is a left \((m,n)\)-coherent ring, every \((n,m)\)-presented right \(R\)-module is torsionless;
5. \( R \) is a left \((m,n)\)-coherent ring, every \((n,m)\)-presented right \(R\)-module may be embedded in a finitely generated projective right \(R\)-module;
6. Every right \(R\)-module has an \((m,n)\)-flat preenvelope which is monomorphism;
7. Every \((n,m)\)-presented right \(R\)-module has an \((m,n)\)-flat preenvelope which is monomorphism;
8. Every \((n,m)\)-presented right \(R\)-module has a finitely generated projective preenvelope which is monomorphism.

**Proposition 3.3** Let \( M \) be an injective right \(R\)-module. The following statements are equivalent:
(1) Every \((n, m)\)-presented right \(R\)-module has an \((m, n)\)-\(M\)-flat envelope which is a monomorphism;

(2) Injective envelope of every \((n, m)\)-presented right \(R\)-module is in \(\text{add}\, M\);

(3) Every injective right \(R\)-module is \((m, n)\)-\(M\)-flat;

(4) For any \((n, m)\)-presented right \(R\)-module \(K\), \((m, n)\)-\(M\)-flat envelope of \(K\) exists, and coincides with its injective envelope.

**Proof** (1) \(\Rightarrow\) (2). For any \((n, m)\)-presented right \(R\)-module \(K\), by hypothesis, \(K\) has an \((m, n)\)-\(M\)-flat envelope \(f : K \to F\) which is a monomorphism. Hence there are a right \(R\)-module \(X\) in \(\text{add}\, M\) and homomorphisms \(g : K \to X\), \(h : X \to F\) such that \(f = hg\). Note that \(g\) is a monomorphism since \(f\) is a monomorphism and \(X\) is injective since \(M\) is injective, it follows that there are \(\alpha : X \to E(K)\) and \(\beta : E(K) \to X\) such that \(\alpha g = i, \beta i = g\). Then \(\alpha \beta = 1\) since \(E(K)\) is an injective envelope, i.e., \(E(K)\) is isomorphic to a direct summand of \(X\). Therefore \(E(K)\) is in \(\text{add}\, M\).

(2) \(\Rightarrow\) (3). Let \(E\) be an injective right \(R\)-module and \(K\) be an \((n, m)\)-presented right \(R\)-module. By hypothesis, \(E(K)\) is in \(\text{add}\, M\). For any homomorphism \(f : K \to E\), there is a homomorphism \(g : E(K) \to E\) such that \(f = gi\). This shows that \(f\) factors through a right \(R\)-module in \(\text{add}\, M\) and hence \(E\) is \((m, n)\)-\(M\)-flat.

(3) \(\Rightarrow\) (4). Let \(K\) be an \((n, m)\)-presented right \(R\)-module. By hypothesis, \(E(K)\) is \((m, n)\)-\(M\)-flat. For any \((m, n)\)-\(M\)-flat right \(R\)-module \(F\), and any homomorphism \(f : K \to F\), there are a right \(R\)-module \(X\) in \(\text{add}\, M\) and homomorphisms \(g : K \to X\) and \(h : X \to F\) such that \(f = hg\). Note that \(X\) is injective since \(M\) is injective, it follows that there is a homomorphism \(k : E(K) \to X\) with \(ki = g\). Hence \(hki = hg = f\). Therefore \(i\) is an \((m, n)\)-\(M\)-flat preenvelope. But \(E(k)\) is an injective envelope of \(K\), hence \(i\) is an \((m, n)\)-\(M\)-flat envelope.

(4) \(\Rightarrow\) (1) is trivial.

4. Preenvelopes which are epimorphisms

In this section, we consider when every right \(R\)-module has an \((m, n)\)-\(M\)-flat preenvelope which is an epimorphism.

**Theorem 4.1** Let \(M\) be a finitely presented right \(R\)-module and \(S = \text{End}(M_R)\). The following statements are equivalent:

(1) \(S\) is \((m, n)\)-coherent, and a submodule of \((m, n)\)-\(M\)-flat right \(R\)-module is \((m, n)\)-\(M\)-flat;

(2) Every \((n, m)\)-presented right \(R\)-module has an \((m, n)\)-\(M\)-flat preenvelope which is an epimorphism;

(3) Every right \(R\)-module has an \((m, n)\)-\(M\)-flat preenvelope which is an epimorphism;

(4) Every right \(R\)-module has an \((m, n)\)-\(M\)-flat envelope which is an epimorphism;

(5) Every \((n, m)\)-presented right \(R\)-module has an \(\text{add}\, M\)-preenvelope which is an epimorphism.

**Proof** (1) \(\Rightarrow\) (3). By Theorem 2.5, every right \(R\)-module \(A\) has an \((m, n)\)-\(M\)-flat preenvelope
$f : A \to F$. Let $F_1 = \Im f$. Then $F_1$ is $(m, n) - M$-flat by (1). Hence $f : M \to F_1$ is an $(m, n) - M$-flat preenvelope which is an epimorphism.

(3) $\implies$ (2) is trivial.

(2) $\implies$ (5). Let $K$ be an $(n, m)$-presented right $R$-module. Then $K$ has an $(m, n) - M$-flat preenvelope $f : K \to F$ which is an epimorphism. Since $F$ is $(m, n) - M$-flat, there exist a right $R$-module $X$ in $\text{add} M$ and homomorphisms $g : K \to X$ and $h : X \to F$ such that $f = hg$. But $X$ is $(m, n) - M$-flat, hence there exists $k : F \to X$ with $kf = g$. Thus $f = kfk$, and so $hk = 1_F$ since $f$ is epimorphism. Therefore $F \in \text{add} M$ and (5) holds.

(5) $\implies$ (1). By Proposition 4.2, $sM$ is $(m, n)$-coherent. Let $N$ be $(m, n) - M$-flat and $N_1$ a submodule of $N$. For any $(n, m)$-presented right $R$-module $K$, and any homomorphism $f : K \to N_1$, let $\lambda : N_1 \to N$ be the inclusion map. Then there exist a right $R$-module $X$ in $\text{add} M$ and homomorphisms $g : K \to X$ and $h : X \to N$ such that $\lambda f = hg$. Note that $K$ has an $M$-preenvelope $k : K \to Y$ which is an epimorphism, then there is a homomorphism $\alpha : Y \to X$ with $ak = g$. Thus $\lambda f = h\alpha k$, whence $\text{Ker} k \subseteq \text{Ker} f$. Define $\beta : Y \to N_1$ by $\beta(k(x)) = f(x)$ for any $x \in K$. It is clear that $\beta$ is well-defined and $f = \beta k$. Therefore $N_1$ is $(m, n) - M$-flat and (1) holds.

(1) $\iff$ (4). By Remark in Section 2, the class of $(m, n) - M$-flat right $R$-modules is closed under direct summands. Then from Theorem 2 in Ref. [11], it follows that every right $R$-module has an $(m, n) - M$-flat envelope if and only if the class of $(m, n) - M$-flat right $R$-modules is closed under direct products and submodules. Note that $M_R$ is a a finitely presented right $R$-module, hence the class of $(m, n) - M$-flat right $R$-modules is closed under direct products if and only if $sM$ is an $(m, n)$-coherent module by Theorem 2.5. Therefore (1) $\iff$ (4).

**Corollary 4.2** The following statements are equivalent:

1. $R$ is a left $(m, n)$-coherent ring, and a submodule of $(m, n)$-flat right $R$-module is $(m, n)$-flat;
2. Every $(n, m)$-presented right $R$-module has an $(m, n)$-flat preenvelope which is an epimorphism;
3. Every right $R$-module has an $(m, n)$-flat preenvelope which is an epimorphism;
4. Every right $R$-module has an $(m, n)$-flat envelope which is an epimorphism;
5. Every $(n, m)$-presented right $R$-module has a finitely generated projective preenvelope which is an epimorphism.

**Proposition 4.3** Let $M_R$ be a finitely presented right $R$-module with $S = \text{End}(M_R)$. The following statements are equivalent:

1. $sM$ is $(m, n)$-coherent, for any $(n, m)$-presented right $R$-module $K$, $K^*$ is a projective left $S$-module and $\delta_R$ is an epimorphism;
2. Every $(n, m)$-presented right $R$-module has an $\text{add} M$-envelope which is an epimorphism;
3. Every $(n, m)$-presented right $R$-module has an $\text{add} M$-envelope with the unique mapping property and $\delta_R$ is an epimorphism.
Proof (1) ⇒ (2). By (1), $K^*$ is a finitely generated projective left $S$-module, hence $K^{**} \in \text{add}M$. Next we prove that $\delta_K : K \rightarrow K^{**}$ is an add$M$-envelope of $K$. For any $X \in \text{add}M$, and any homomorphism $f : K \rightarrow X$, then $\delta_X$ is an isomorphism and $\delta_X f = f^{**} \delta_K$. Hence $f = \delta_X^{-1} f^{**} \delta_K$. This proves that $\delta_K$ is an add$M$-preenvelope. If $\alpha \in \text{Hom}(K^{**}, K^{**})$ with $\alpha \delta_K = \delta_K$, then $\delta_K^* \alpha^* = \delta_K^*$. Note that $K^*$ is finitely generated projective, $\delta_K^*$ is an isomorphism, and so $\delta_K^*$ is an isomorphism since $\delta_K^* \delta_K = 1_{K^*}$. It follows that $\delta_K^* \alpha^* = 1_{K^*}$, whence $\alpha = \delta_K^{-1} \alpha^* \delta_K^{**}$ is an isomorphism. Therefore $K$ has an add$M$-envelope $\delta_K$ which is an epimorphism and (2) holds.

(2) ⇒ (3). For any $(n, m)$-presented right $R$-module $K$, there is an add$M$-envelope $f : K \rightarrow X$ which is an epimorphism. Hence we have the following exact sequence $0 \rightarrow \text{Hom}(X, L) \rightarrow \text{Hom}(K, L) \rightarrow 0$ for any $L \in \text{add}M$. Therefore $f$ has the unique mapping property. Put $L = M$. Then $f^*$ is an isomorphism, and so $f^{**}$ is also isomorphism. Since $f^{**} \delta_K = \delta_X f$, it is clear that $\delta_K$ is an epimorphism.

(3) ⇒ (1). For any $(n, m)$-presented right $R$-module $K$, by (3), there is an add$M$-preenvelope $f : K \rightarrow X$ and $X^* \cong K^*$. Hence $K^*$ is a finitely generated projective left $S$-module. Therefore $S M$ is $(m, n)$-coherent, $K^*$ is a projective left $S$-module, and (1) holds.

References