Asymptotic Behavior for Random Walks in Time-Random Environment on $Z^1$

HU Xue-ping$^1$, ZHU Dong-jin$^2$

(1. Department of Mathematics, Anqing Normal College, Anhui 246011, China; 2. College of Math. & Comput. Sci., Anhui Normal University, Anhui 241000, China)

(E-mail: hxpprob@yahoo.com.cn)

Abstract In this paper, we give a general model of random walks in time-random environment in any countable space. Moreover, when the environment is independently identically distributed, a recurrence-transience criterion and the law of large numbers are derived in the nearest-neighbor case on $Z^1$. At last, under regularity conditions, we prove that the RWIRE $\{X_n\}$ on $Z^1$ satisfies a central limit theorem, which is similar to the corresponding results in the case of classical random walks.

Keywords Random walks in time-random environment; recurrence-transience criteria; strong law of large numbers; central limit theorem.

1. Introduction

The general model of random walks in space-random environment including a recurrence-transience criterion had been given in Ref. [6]. Moreover, when the environment is i.i.d., the strong law of large numbers was given in Refs. [1] and [4]. However, the limit theorem had also been investigated in Refs. [2] and [5] where the environment is stationary and ergodic. In this paper, we study some asymptotic behavior for random walks in time-random environment (RWIRE) in the nearest-neighbor case on $Z^1$ as the environment is i.i.d..

We begin with a general setup, that will be specialized later to the cases of interest to us. Now we let $N = \{0, 1, 2, \ldots\}$. For each $i \in N$, let $M_i(\chi)$ denote the collection of probability measures on $\chi$ with support $V, V \subset \chi$, where $\chi$ is countable. Formally, an element of $M_i(\chi)$, called a transition law at time $i$, is a measurable function $\omega_i : \chi \to [0, 1]$ satisfying:

(a) $\omega_i(x) \geq 0, \forall x \in V$;
(b) $\omega_i(x) = 0, \forall x \notin V$;
(c) $\sum_{x \in V} \omega_i(x) = 1$.

We equip $M_i(\chi)$ with the weak topology on probability measures which makes it become a Polish space. Furthermore, it induces a Polish structure on $\Omega = \prod_{i \in N} M_i(\chi)$. Let $F$ denote
the Borel σ-algebra on Ω. Given a probability measure P on (Ω, F). A random environment is an element ω of Ω distributed according to P. One defines naturally the shift T on Ω by

\( (T\omega)_i = \omega_{i+1}, \ i \in \mathbb{N} \).

For each ω ∈ Ω, we define the random walks in the environment ω as the space-homogeneous Markov chains \( X = \{X_n, n \geq 0\} \) taking value in \( \chi \) with transition probabilities

\[ P_\omega(x_{n+1} = y | X_n = x) = \omega_n(y - x). \]

Fix an environment \( \omega \in \Omega, X = \{X_n, n \geq 0\} \) is a time-nonhomogeneous Markov chain. We use \( P^\omega_\cdot \) to denote the law induced on \( (\chi^N, \mathcal{B}) \), where \( \mathcal{B} \) is the σ-algebra generated by cylinder functions and \( P^\omega_\cdot (X_0 = x) = 1 \).

In the sequel, we refer to \( P^\omega_\cdot (\cdot) \) as the quenched law of the random walks \( \{X_n, n \geq 0\} \). Note that for each \( x \in \chi \) and \( G \in \mathcal{B} \), the map \( \omega \mapsto P^\omega_\cdot (G) \) is \( \mathcal{F} \)-measurable.

Hence \( P^\cdot := P \otimes P^\omega_\cdot \) on \( (\Omega \times \chi^N, \mathcal{F} \times \mathcal{B}) \) is the probability measure defined by

\[ P^\cdot (F \times G) = \int_P P^\omega_\cdot (G) P(\omega), \quad F \in \mathcal{F}, G \in \mathcal{B}. \]

**Example** Let \( \chi = Z^1 \) and \( V = \{-1, 0, 1\} \). Then according to above definition the RWIRE is called the nearest-neighbor RWIRE on \( Z^1 \).

2. Recurrence-transience criteria

In this section, suppose \( \{X_n, n \geq 0\} \) is the nearest-neighbor RWIRE on \( Z^1 \), starting at 0. We let \( \omega^+_n := \omega_n(1), \omega^-_n := \omega_n(-1) \) and \( \omega^0_n := \omega_n(0) \).

**Assumption (I)** (a) \( P \) is i.i.d.; (b) \( P\{\omega^+_n + \omega^-_n \} > 0 \} = 1 \).

**Theorem 2.1** Assume Assumption (I). Then

1. \( E_P \omega^+_0 < E_P \omega^-_0 \implies \lim_{n \to \infty} X_n = +\infty \),
2. \( E_P \omega^-_0 > E_P \omega^+_0 \implies \lim_{n \to \infty} X_n = -\infty \),
3. \( E_P \omega^+_0 = E_P \omega^-_0 \implies -\infty = \lim_{n \to \infty} \inf X_n < \lim_{n \to \infty} \sup X_n = +\infty \) hold \( P^0 \)-a.s., where \( E_P \) is the expectation operator w.r.t. \( P \).

**Proof** (1) Fix an environment \( \omega \) with \( \frac{E_P \omega^+_0}{E_P \omega^-_0} < \infty \). For each \( x \in Z^1, \ l \leq x \leq s \), define

\[ H^x_{l, s, \omega} = P^\omega_\cdot (\{X_n \} \ \text{hits} \ l \ \text{before hitting} \ s). \]

From the assumption \( \frac{E_P \omega^+_0}{E_P \omega^-_0} < \infty \), for each \( x \), it follows that \( H^x_{l, s, \omega} \) is well defined as \( P^\omega_\cdot (\{X_n \}) \) never hits \( [l, s]^{\omega} \) = 0. The Markov property of \( P^\omega_\cdot \) implies

\[
\begin{align*}
H^x_{l, s, \omega} &= \omega^+_0 H^{x+1}_{l, s, \omega} + \omega^-_0 H^x_{l, s, \omega} + \omega^+_0 H^x_{l, s, \omega}, \quad x \in (l, s); \\
H^x_{l, s, T^k \omega} &= 1, \quad k \geq 1; \\
H^x_{l, s, T^k \omega} &= 0, \quad k \geq 1. 
\end{align*}
\]

Since \( P \) is i.i.d., and \( P^\omega_\cdot \) is space-homogeneous for all \( \omega \in \Omega \), it follows by taking expectation on
There are three cases:

1. $P_{\omega}^{+} > P_{\omega}^{-}$, we have $\lim_{s \to +\infty} \lim_{s \to +\infty} P_{\omega}^{0} = 0$, but $0 \leq H_{0,s,\omega} \leq 1$, so $P(\lim_{s \to +\infty} \lim_{s \to +\infty} H_{0,s,\omega} = 0) = 1$. We also have $\lim_{s \to +\infty} P_{\omega}^{0} < 1$, hence $P(\lim_{s \to +\infty} H_{0,s,\omega} = 1) = 1$. Moreover, for any fixed $s$, $\lim_{s \to +\infty} H_{0,s,\omega} = 0$, hence $P(\lim_{s \to +\infty} H_{0,s,\omega} = 0) = 0$, so $E_{\omega}^{0} = E_{\omega}^{+} \implies -\infty = \lim_{n \to +\infty} \inf_{X_{n}} \sup_{X_{n}} +\infty$ under $P^{0}$-a.s.

2. $E_{\omega}^{-} = E_{\omega}^{+}$, for any fixed $l$, $\lim_{s \to +\infty} E_{\omega}^{0} = 1$, hence $P(\lim_{s \to +\infty} H_{0,s,\omega} = 1) = 1$. Moreover, for any fixed $s$, $\lim_{s \to +\infty} E_{\omega}^{0} = 0$, hence $P(\lim_{s \to +\infty} H_{0,s,\omega} = 0) = 0$, so $E_{\omega}^{-} = E_{\omega}^{+} \implies -\infty = \lim_{n \to +\infty} \inf_{X_{n}} \sup_{X_{n}} +\infty$ under $P^{0}$-a.s.

3. $P_{\omega}^{-} = 0, E_{\omega}^{+} = 0$, there are two cases by assumption (I)(b):

(a) When $E_{\omega}^{0} = 0, E_{\omega}^{+} > 0$, then by (2.2) and $P_{\omega}$ is space-homogeneous for all $\omega$

$E_{\omega}(H_{0,s,\omega}) = E_{\omega}(H_{0,-s,1,\omega}) = \cdots = E_{\omega}(H_{0,0,\omega}) = 0$.

(b) When $E_{\omega}^{0} > 0, E_{\omega}^{+} = 0$, similarly, we have

$E_{\omega}(H_{0,s,\omega}) = E_{\omega}(H_{0,1,1,\omega}) = \cdots = E_{\omega}(H_{0,0,\omega}) = 1$.

So the case (a) (or (b)) of (ii) can be included in the case of (a) (or (b)) of (i). We complete the proof by (i) and (ii).

3. Strong law of large numbers

We introduce hitting times which will serve us later. Let $T_{0} = 0$ and

$T_{n} = \inf\{k : X_{k} = n\}, n \geq 1; \inf \varphi = +\infty$.

Set $\tau_{0} = 0$ and $\tau_{n} = T_{n} - T_{n-1}, n \geq 1$. Similarly, set $T_{-n} = \inf\{k : X_{k} = -n\}, n \geq 1$ and $r_{n} = T_{n} - T_{n+1}, n \geq 1$, with the convention that $r_{n} = +\infty$ if $T_{n} = +\infty$.

Theorem 3.1 Assume Assumption (I). Then under $P^{0}$-a.s.

1. $E_{\omega}^{0} > E_{\omega}^{-} \implies \lim_{n \to +\infty} T_{n}^{n} = (E_{\omega}^{0} - E_{\omega}^{-})^{-1}, \lim_{n \to +\infty} X_{n}^{n} = (E_{\omega}^{0} - E_{\omega}^{-})$;

2. $E_{\omega}^{-} < E_{\omega}^{0} \implies \lim_{n \to +\infty} T_{n}^{n} = (E_{\omega}^{0} - E_{\omega}^{-})^{-1}, \lim_{n \to +\infty} X_{n}^{n} = (E_{\omega}^{0} - E_{\omega}^{-})$;

3. $E_{\omega}^{0} = E_{\omega}^{-} \implies \lim_{n \to +\infty} T_{n}^{n} = \infty = \lim_{n \to +\infty} T_{n}^{n}, \lim_{n \to +\infty} X_{n}^{n} = 0$.

For the proof of Theorem 3.1, we need the following two lemmas.
Lemma 3.1 Assume Assumption (I). Then \( \{\tau_n, n \geq 1\} \) is i.i.d. and \( \{\tau_n, n \geq 1\} \) is i.i.d. under the law \( P^0 \)-a.s.

**Proof** When \( E_P \omega_0^+ \geq E_P \omega_0^- \), we have \( P^0(\lim_{n \to \infty} \sup X_n = +\infty) = 1 \), by the definition of \( \tau_n \), \( P^0(\tau_n < \infty) = 1 \) for all \( n \geq 1 \). To prove \( \{\tau_n\} \) is i.i.d., it suffices to show for any positive integer \( k \) and \( m \),

\[
P^0(\tau_2 = k|\tau_1 = m) = P^0(\tau_2 = k) = P^0(\tau_1 = k).
\]

For any fixed \( \omega \in \Omega \), by the Markov property and the space-homogeneity of \( P^0_\omega \), we have

\[
P^0_\omega(\tau_2 = k) = \sum_{m=1}^{\infty} P^0_\omega(\tau_2 = k|\tau_1 = m)P^0_\omega(\tau_1 = m) = \sum_{m=1}^{\infty} P^0(\tau_1 = k)P^0_\omega(\tau_1 = m).
\]

(3.1)

Since \( P \) is i.i.d., it follows by taking expectation on (3.1) w.r.t. \( P \) that

\[
P^0(\tau_2 = k) = P \otimes P^0_\omega(\tau_1 = k) \sum_{m=1}^{\infty} P \otimes P^0_\omega(\tau_1 = m) = P^0(\tau_1 = k).
\]

(3.2)

On the other hand

\[
P^0_\omega(\tau_2 = k|\tau_1 = m) = P^0_\omega(X_{m+k} = 2, X_{m+s} \neq 2, s = 1, 2, \ldots, k - 1| X_m = 1, X_n \neq 1, n = 1, 2, \ldots, m - 1) = P^0(\tau_1 = k).
\]

(3.3)

Hence by taking expectation on (3.3) w.r.t. \( P \), we also have

\[
P^0(\tau_2 = k|\tau_1 = m) = P^0(\tau_1 = k).
\]

(3.4)

So \( \{\tau_n, n \geq 1\} \) is i.i.d. under \( P^0 \)-a.s. by (3.2) and (3.4) when \( E_P \omega_0^+ \leq E_P \omega_0^- \). Similarly, we may show \( \{\tau_n, n \geq 1\} \) is i.i.d. under \( P^0 \)-a.s.

Lemma 3.2 Assume Assumption (I). Then

1) \( E_{P^0}(\tau_1) = \begin{cases} (E_P \omega_0^+ - E_P \omega_0^-)^{-1}, & E_P \omega_0^+ > E_P \omega_0^-; \\ +\infty, & E_P \omega_0^+ = E_P \omega_0^-; \end{cases} \)

2) \( E_{P^0}(\tau_{-1}) = \begin{cases} (E_P \omega_0^- - E_P \omega_0^+)^{-1}, & E_P \omega_0^- > E_P \omega_0^+; \\ +\infty, & E_P \omega_0^- = E_P \omega_0^+. \end{cases} \)

**Proof** We prove only (1) since the proof of (2) is similar. Decompose, with \( X_0 = 0 \)

\[
\tau_1 = \chi_{\{X_1=1\}} + \chi_{\{X_1=0\}}(1 + \tau'_1) + \chi_{\{X_1=-1\}}(1 + \tau''_1 + \tau'_1).
\]

(3.5)

Here \( \tau'_1 \) is the first hitting time of 1 after time 1 (possible infinite), \( 1 + \tau''_1 \) is the first hitting time of 0 after time 1, and \( 1 + \tau''_0 + \tau'_1 \) is the first hitting time of 1 after time 1 + \( \tau''_0 \).

Consider first the case \( E_{P^0}\tau_1 < \infty \). Then \( E^0_\omega\tau_1 < \infty \) under \( P \)-a.s.. Taking expectation on (3.5), one gets

\[
E^0_\omega\tau_1 = P^0_\omega(X_1 = 1) + P^0_\omega(X_1 = 0)(1 + E^0_\omega\tau_1) + \sum_{m=1}^{\infty} P^0_\omega(\tau_1 = m)E^0_\omega(\tau_{m+1} = \tau_1)
\]

\[
P^0_\omega(X_1 = -1)(1 + E^0_\omega\tau_1 + \sum_{m=1}^{\infty} P^0_\omega(\tau_1 = m)E^0_\omega(\tau_{m+1} = \tau_1))
\]
Proof of Theorem 3.1

An application of Lemmas 3.1 and 3.2 yields that in case (1)

\[
\frac{T_n}{n} = \frac{\sum_{i=1}^{n} \tau_i}{n} \longrightarrow E_{P^0} \tau_1 < \infty. \tag{3.8}
\]

Similarly, we use \(-n\) instead of \(n\), we also get that in case (2)

\[
\frac{T_{-n}}{n} = \frac{\sum_{i=1}^{n} \tau_{-i}}{n} \longrightarrow E_{P^0} \tau_{-1} < \infty. \tag{3.9}
\]

However when \(E_{P^0} \omega_0^+ = E_{P^0} \omega_0^-\), \(P^0(\omega_0 > \infty) = \lim_{n \to \infty} \inf X_n < \lim_{n \to \infty} \sup X_n = +\infty\) = 1, for \(n\) large enough, \(T_n = +\infty = T_{-n}\), so in case (3),

\[
P^0\left( \lim_{n \to \infty} \frac{T_n}{n} = +\infty = \lim_{n \to \infty} \frac{T_{-n}}{n} \right) = 1.
\]

Now we prove the second limit of the case (1). Let \(K_n\) be the unique (random) integers such that \(T_{K_n} \leq n < T_{K_{n+1}}\). Note that \(X_n \leq K_n + 1\), while \(X_n \geq K_n - (n - T_{K_n})\). Hence

\[
\frac{K_n}{n} - (1 - \frac{T_{K_n}}{n}) \leq \frac{X_n}{n} < \frac{K_n + 1}{n}. \tag{3.10}
\]

Since \(P^0(\lim_{n \to \infty} X_n = +\infty) = 1\), from the definition of \(K_n\), \(P^0\left( \lim_{n \to \infty} \frac{K_n}{n} = \lim_{n \to \infty} \frac{\tau_n}{n} \right) = 1\). But the definition \(K_n\) also implies

\[
P^0\left( \lim_{n \to \infty} \frac{K_n}{n} = \lim_{n \to \infty} \frac{n}{T_n} \right) = 1.
\]

Thus it follows from (3.8) and (3.10) that

\[
P^0\left( \lim_{n \to \infty} \frac{X_n}{n} = \lim_{n \to \infty} \frac{n}{T_n} = E_{P^0} \omega_0^+ - E_{P^0} \omega_0^- \right) = 1.
\]

Similarly, we may prove the second limit of (2) and (3).

4. The central limit Theorem

In this section, we study the limiting distribution of the RWIRW \(\{X_n\}\), we use following notations:

\[
\mu = (E_{P^0} \tau_1)^{-1}, \quad \sigma^2 = \frac{E_{P^0} (\tau_1)^2 - (E_{P^0} \tau_1)^2}{(E_{P^0} \tau_1)^3}, \quad D_{P^0} \tau_1 = E_{P^0} (\tau_1)^2 - (E_{P^0} \tau_1)^2;
\]

\[
\mu_{-1} = (E_{P^0} \tau_{-1})^{-1}, \quad \sigma^{-1}_2 = \frac{E_{P^0} (\tau_{-1})^2 - (E_{P^0} \tau_{-1})^2}{(E_{P^0} \tau_{-1})^3}, \quad D_{P^0} \tau_{-1} = E_{P^0} (\tau_{-1})^2 - (E_{P^0} \tau_{-1})^2.
\]
Theorem 4.1 Assume Assumption (I), under $P^0$-a.s.,

1. If $E_P(\omega_0^+ - \omega_0^-) > 0$ and $D_P \tau_1 < \infty$, then

$$\frac{X_n - n\mu}{\sigma \sqrt{n}} \rightarrow N(0,1); \quad \frac{T_n - n\mu^{-1}}{\sqrt{n}D_P \tau_1} \rightarrow N(0,1), \quad n \rightarrow \infty.$$ 

2. If $E_P\omega_0^+ = E_P\omega_0^-$, then

$$\frac{X_n}{\sqrt{2nE_P\omega_0^+}} \rightarrow N(0,1), \quad n \rightarrow \infty.$$ 

3. If $E_P\omega_0^+ < E_P\omega_0^-$ and $D_P \tau_{-1} < \infty$, then

$$\frac{X_n - n\mu^{-1}}{\sigma^{-1} \sqrt{n}} \rightarrow N(0,1); \quad \frac{T_n - n\mu^{-1}}{\sqrt{n}D_P \tau_{-1}} \rightarrow N(0,1), \quad n \rightarrow \infty.$$ 

Proof (1) For any positive integers $n$, $L$ and $M$, by the definition of $T_n$, we have

$$\{T_L \geq n\} \subset \{X_n \leq L\} \subset (\{T_{L+M} \geq n\} \cup (\inf_{s \geq T_{L+M}} X_s \leq L) \cap (T_{L+M} < n))$$

$$\subset (\{T_{L+M} \geq n\} \cup \{\inf_{s \geq T_{L+M}} X_s - (L + M) \leq -M\}). \quad (4.1)$$

Since $P$ is i.i.d., it follows by the Markov property and the space-homogeneity of $P_0^0$ that

$$P^0(\inf_{s \geq T_{L+M}} X_s - (L + M) \leq -M) = P^0(\inf_{s \geq 0} X_s \leq -M). \quad (4.2)$$

By Theorem 2.1 we have

$$\lim_{M \rightarrow \infty} P^0(\inf_{s \geq 0} X_s \leq -M) = 0. \quad (4.3)$$

Hence for any positive integers $l$ and for any $\delta > 0$, there exists $M$ large enough such that

$$P^0(T_L \geq n) \leq P^0(X_n \leq L) \leq P^0(T_{L+M} \geq n) + \delta. \quad (4.4)$$

Now for any given real $x$, take

$$L = L(n, x) = n\mu + x\sigma \sqrt{n} + o(\sqrt{n}), \quad (4.5)$$

we have $L(n, x) \rightarrow +\infty$ as $n \rightarrow \infty$ and

$$P^0(X_n \leq L(n, x)) \approx P^0(\frac{X_n - n\mu}{\sigma \sqrt{n}} \leq x). \quad (4.6)$$

On the other hand

$$P^0(T_L \geq n) = P^0(\frac{T_{L(n, x)} - L(n, x)\mu^{-1}}{\sqrt{L(n, x)D_P \tau_1}} \geq \frac{n - L(n, x)\mu^{-1}}{\sqrt{L(n, x)D_P \tau_1}}).$$

By the definition of $L(n, x)$, we have

$$\lim_{n \rightarrow \infty} \frac{n - L(n, x)\mu^{-1}}{\sqrt{L(n, x)D_P \tau_1}} = -x.$$ 

By the Central Limit Theorem of i.i.d. random variable sequence and the given condition we obtain

$$\lim_{n \rightarrow \infty} P^0(T_L \geq n) = \lim_{n \rightarrow \infty} P^0(T_{L(n, x) + M} \geq n) = 1 - \Phi(-x) = \Phi(x),$$
Suppose that the characteristic function of $X$ can prove the case (3).

(2) When $E_pω_0^+ = E_pω_0^-$, suppose $X_n = \sum_{i=1}^n Y_i, Y_i \in \{-1, 0, 1\}$, where

$E_ω(Y_i = 1) = ω_i^+, E_ω(Y_i = -1) = ω_i^-, E_ω(Y_i = 0) = ω_i^0$.

Since $P$ is i.i.d., by the given condition we have

$E_pX_n = E_p[\sum_{i=1}^n(ω_i^+ - ω_i^-)] = \sum_{i=1}^n(E_pω_i^+ - E_pω_i^-) = 0$,

$E_pX_n^2 = E_p(E_ω(\sum_{i=1}^n Y_i Y_j)) = E_p(E_ω(\sum_{i=1}^n Y_i^2 + 2 \sum_{1≤i<j≤n} Y_i Y_j))$

$= E_p(\sum_{i=1}^n(ω_i^+ + ω_i^-) + 2 \sum_{1≤i<j≤n} (ω_i^+ - ω_i^-)(ω_j^+ - ω_j^-)) = 2\sum_{i=1}^n E_pω_i^+ = 2nE_pω_0^+$.

Suppose that the characteristic function of $\frac{X_n}{\sqrt{2nE_pω_0^+}}$ is $φ_n(t)$. Then

$φ_n(t) = E_p(\exp(it\frac{X_n}{\sqrt{2nE_pω_0^+}})) = E_p(E_ω(\prod_{j=1}^n(\exp(it\frac{Y_j}{\sqrt{2nE_pω_0^+}}))))$

$= E_p(\prod_{j=1}^n(\exp(it\frac{ω_j}{\sqrt{2nE_pω_0^+}})ω_j^+ + \exp(-it\frac{ω_j}{\sqrt{2nE_pω_0^+}})ω_j^- + 0))$

$= \prod_{j=1}^n(\exp(it\frac{ω_j}{\sqrt{2nE_pω_0^+}})E_pω_j^+ + \exp(-it\frac{ω_j}{\sqrt{2nE_pω_0^+}})E_pω_j^- + E_pω_j^0)$

$= [E_pω_0^+(\exp(it\frac{ω_0}{\sqrt{2nE_pω_0^+}}) + \exp(-it\frac{ω_0}{\sqrt{2nE_pω_0^+}})] + E_pω_0^0)^n$

$= [E_pω_0^+(1 + \frac{it}{\sqrt{2nE_pω_0^+}} - \frac{1}{2} \frac{t^2}{2nE_pω_0^+} + o(\frac{1}{n}) + 1 - \frac{it}{\sqrt{2nE_pω_0^+}} - \frac{1}{2} \frac{t^2}{2nE_pω_0^+} + o(\frac{1}{n})) + E_pω_0^0)^n$

$= [E_pω_0^+(2 - \frac{t^2}{2nE_pω_0^+} + o(\frac{1}{n})) + E_pω_0^0)^n$

$= [1 - \frac{t^2}{2n} + o(\frac{1}{n})]^n → \exp(-\frac{t^2}{2}), n → ∞$. (4.7)

It follows by the Continuous Theorem and (4.7) that $\frac{X_n}{\sqrt{2nE_pω_0^+}} → N(0, 1)$ as $n → ∞$.

References