The Kinematic Density for Pairs of Intersecting Lines

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Abstract In this paper, we get a kinematic density formula for pairs of intersecting lines which has not yet been gotten in integral geometry by using the moving orthogonal frames method. And we obtain a kinematic formula of the intersection of the pairs of intersecting lines belonging to convex body K by using it.

Keywords pairs of intersecting lines; kinematic density; kinematic formula.

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1. Introduction

The kinematic density for pairs of intersecting lines in $\mathbb{R}^2$ is the same as the kinematic density for pairs of lines in $\mathbb{R}^2$. Let $G$ and $L$ be two lines which intersect at the point $P$, $\theta$ be the angle between $G$ and $L$, and $\alpha$ be the angle $G$ with the $x$-axis. We can get the formula\(^{[1,2]}\)
\[
dGdL = \sin \theta dPd\theta d\alpha.
\] (1)

The kinematic density for pairs of intersecting lines in $\mathbb{R}^3$ is different from the kinematic density for pairs of lines in $\mathbb{R}^3$. The kinematic density for pairs of lines in $\mathbb{R}^3$ is\(^{[3]}\)
\[
dGdL = \sin^2 \theta dP_N dQ_N dG_N dL_N dN.
\] (2)

Where $N$ is the common perpendicular of lines $G$ and $L$, $P=N \cap G$, $Q=N \cap L$, $\theta$ denotes the angle between $G$ and $L$, $dP_N$ and $dQ_N$ are the densities of $P$ and $Q$ on the common perpendicular $N$, respectively, and $dG_N$ and $dL_N$ are the densities of $G$ and $L$ for the rotations around $N$, respectively.

The kinematic density for pairs in $\mathbb{R}^n$ is\(^{[4,5]}\)
\[
dGdL = t^{n-3} \sin^2 \theta dP_N dQ_N dG_N dL_N dN,
\] (3)
where $t$ denotes the length of $PQ$, $\theta$ denotes the angle between $G$ and $L$, $dP_N$ and $dQ_N$ are the densities of $P$ and $Q$ on the common perpendicular $N$, respectively, and $dG_N$ and $dL_N$ are the densities of $G$ and $L$ for the rotations around $N$, respectively.

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Let $K$ be a convex body in $\mathbb{R}^n$, and $\sigma$ the chord intersected by a random line $G$ with $K$. Consider integrals\[^{[1]}\]

$$I_m^{(n)}(K) = \int_{G \cap K} \sigma^m dG,$$

where $m$ is a nonnegative integer and $dG$ is the density of lines. The integral $I_m^{(n)}(K)$ is called the $m$th chord-power integral of $K$.

In particular,

$$I_1^{(n)}(K) = \frac{1}{2} O_{n-1} V,$$

where $O_{n-1}$ is the surface area of $(n - 1)$-dimensional unit sphere and $V$ is the volume of $K$.

When we study geometric probability\[^{[5]}\], we may meet these problems which need using the related formulas, say, calculating the kinematic measure of the intersection of the lines $G$ and $L$ belonging to convex body $K$ in $\mathbb{R}^n$. Meanwhile, we notice that these are conditional probability problems.

2. Main results

**Lemma** The kinematic density for pairs of intersecting lines $G$ and $L$ in $\mathbb{R}^n$ is

$$d G d L = \sin^{n-1} \theta d\theta \omega_1 \omega_2 \cdots \omega_n \omega_1 \omega_2 \cdots \omega_n,$$

where $\theta$ is the angle between the lines $G$ and $L$, and $\omega_i$, $\omega_{ij}$ are 1-forms in $\mathbb{R}^n$,

$$i = 1, 2, \ldots, n; j = 1, 2, \ldots, n.$$

**Proof** Let $g_1$ be the unit vector parallel to the line $G$ and $l_1$ be the unit vector parallel to the line $L$. Let $\{P, g_1, g_2, g_3, \ldots, g_n\}$ be the moving orthogonal frames, such that the plane $\pi$ which is spanned by the lines $G$ and $L$ will be perpendicular to $\text{span}(P; g_3, g_4, \ldots, g_n)$. And the lines $l_1$ and $l_2$ are also in the plane $\pi$. Denote $g_i = l_i$, $i = 3, 4, \ldots, n$. Then

$$l_1 = \cos \theta g_1 + \sin \theta g_2, \quad l_2 = -\sin \theta g_1 + \cos \theta g_2,$$

$$dl_1 = \cos \theta dg_1 + \sin \theta dg_2 + (\cos \theta g_2 - \sin \theta g_1)d\theta,$$

$$dl_2 = -\sin \theta dg_1 + \cos \theta dg_2 - (\cos \theta g_1 + \sin \theta g_2)d\theta.$$

Because the arbitrary line $G$ always intersects the line $L$, we have

$$d L = dP l_2 \wedge dl_1 l_2 \wedge dl_1 g_3 \wedge \cdots \wedge dl_1 g_n$$

$$= dP(- \sin \theta g_1 + \cos \theta g_2)$$

$$\wedge (\cos \theta dg_1 + \sin \theta dg_2 + (\cos \theta g_2 - \sin \theta g_1)d\theta)(- \sin \theta g_1 + \cos \theta g_2)$$

$$\wedge (\cos \theta dg_1 + \sin \theta dg_2 + (\cos \theta g_2 - \sin \theta g_1)d\theta)g_3$$

$$\wedge \cdots$$

$$\wedge (\cos \theta dg_1 + \sin \theta dg_2 + (\cos \theta g_2 - \sin \theta g_1)d\theta)g_n$$

$$= (- \sin \theta dP g_1 + \cos \theta dP g_2) \wedge (d\theta + dg_1 g_2) \wedge (\cos \theta dg_1 g_3 + \sin \theta dg_2 g_3)$$

$$\wedge \cdots$$
The kinematic density for pairs of intersecting lines

\[ \lVert \cos \theta d g_1 g_n + \sin \theta d g_2 g_n \rVert. \]

Since \(^2\) \(dG = dP g_2 \wedge \cdots \wedge dP g_n \wedge d g_1 g_2 \wedge \cdots \wedge d g_1 g_n\), we have

\[ dGdL = dP g_2 \wedge \cdots \wedge dP g_n \wedge d g_1 g_2 \wedge \cdots \wedge d g_1 g_n \]
\[ \wedge (- \sin \theta d P g_1 + \cos \theta d P g_2) \wedge (d \theta + d g_1 g_2) \wedge (\cos \theta d g_1 g_3 + \sin \theta d g_2 g_3) \]
\[ \wedge \cdots \wedge \]
\[ \wedge (\cos \theta d g_1 g_n + \sin \theta d g_2 g_n) \]
\[ = \sin^{n-1} \theta d \theta d P g_1 \wedge dP g_2 \wedge \cdots \wedge dP g_n \wedge d g_1 g_2 \wedge \cdots \wedge d g_1 g_n \wedge d g_2 g_3 \]
\[ \wedge \cdots \wedge d g_2 g_n \]
\[ = \sin^{n-1} \theta d \theta \omega_1 \omega_2 \cdots \omega_n \theta_1 \theta_2 \cdots \omega_1 \omega_2 \cdot \cdots \omega_{n-2} \omega_{n-2} \cdots \omega_{2n}. \]

**Theorem 1** The kinematic density for pairs of intersecting lines \(G\) and \(L\) in \(\mathbb{R}^n\) is

\[ dGdL = \sin^{n-1} \theta d \theta d P u_{n-1} d u_{n-2}, \]

where \(P\) is the intersection of the lines \(G\) and \(L\), \(du_{n-1}\) denotes \((n-1)\)-dimensional volume element of unit sphere \(U_{n-1}\), and \(du_{n-2}\) denotes \((n-2)\)-dimensional volume element of unit sphere \(U_{n-2}\) in \(\mathbb{R}^n\).

**Remark 1** The formula (6) of kinematic density for pairs of intersecting lines is different from the formula (3) about kinematic density for pairs of intersecting lines.

**Remark 2** For \(n = 3\), we have

\[ dGdL = \sin^2 \theta d \theta d P u_1 d u_2, \]

which is different from the formula (2).

**Remark 3** But, for \(n = 2\), we have

\[ dGdL = \sin \theta d \theta d P u_1, \]

which is the same as the formula (1).

**Proof** Since \(^2\)

\[ dP = \omega_1 \omega_2 \cdots \omega_n, \quad du_{n-1} = \omega_1 \cdots \omega_{n-1}, \quad du_{n-2} = \omega_{n-1} \cdots \omega_{n-2}, \]

by formula (5), we get

\[ dGdL = \sin^{n-1} \theta d \theta \omega_1 \omega_2 \cdots \omega_n \omega_1 \cdots \omega_{n-1} \omega_{n-2} \cdots \omega_{2n} \]
\[ = \sin^{n-1} \theta d \theta d P u_{n-1} d u_{n-2}. \]

**Theorem 2** The kinematic density for pairs of intersecting lines \(G\) and \(L\) in \(\mathbb{R}^n\) is

\[ dGdL = \sin^{n-1} \theta d \theta d \Sigma_G d P_G d G, \]

where \(\Sigma\) is the plane spanned by the lines \(G\) and \(L\), \(d \Sigma_G\) is the kinematic density of \(\Sigma\) for the rotations around \(G\), and \(d P_G\) is the kinematic density of \(P\) on \(G\).
Proof Since \( dP_G = \omega_1, \ dG = \omega_2 \cdot \cdots \cdot \omega_n \cdot \omega_1, \ d\Sigma_G = \omega_3 \cdot \cdots \cdot \omega_2, \)

it follows from formula (5) that
\[
\begin{align*}
dGdL &= \sin^{n-1} \theta d\theta \omega_1 \cdot \cdots \cdot \omega_n \omega_2 \\
&= \sin^{n-1} \theta d\theta d\Sigma_G dP_G dG.
\end{align*}
\]

Theorem 3 The kinematic formula of the intersection of the lines \( G \) and \( L \) belonging to convex body \( K \) in \( \mathbb{R}^n \) is
\[
\int_{G \cap L \in K} dGdL = J_n O_{n-1} O_{n-2} V,
\]
where \( J_n = \int_0^{\pi} \sin^{n-1} \theta d\theta \), \( O_{n-1} \) is the surface area of the \((n-1)\)-dimensional unit sphere, \( O_{n-2} \) is the surface area of the \((n-2)\)-dimensional unit sphere, and \( V \) is the volume of \( K \).

Proof By formula (6), we obtain
\[
\begin{align*}
\int_{G \cap L \in K} dGdL &= \int_{P \in K} \sin^{n-1} \theta d\theta dP d\mu_{n-1} d\mu_{n-2} \\
&= O_{n-1} O_{n-2} \int_0^{\pi} \sin^{n-1} \theta d\theta \int_{P \in K} dP \\
&= J_n O_{n-1} O_{n-2} V.
\end{align*}
\]
Also by formula (4) and (7), we obtain
\[
\begin{align*}
\int_{G \cap L \in K} dGdL &= \int_{G \cap L \in K} \sin^{n-1} \theta d\theta dP_G dGd\Sigma_G \\
&= \int_0^{\pi} \sin^{n-1} \theta d\theta \int_{G \cap L \in K} dP_G \int_{G \cap K \neq \emptyset} dG \\
&= \int_0^{\pi} \sin^{n-1} \theta d\theta \int_{G \cap K \neq \emptyset} \text{vol}(G \cap K) dG \\
&= 2J_n O_{n-2} \int_{G \cap K \neq \emptyset} \sigma dG \\
&= 2J_n O_{n-2} I_1^{(n)}(K) = J_n O_{n-1} O_{n-2} V.
\end{align*}
\]

References
\[\begin{align*}
\end{align*}\]