An Inequality of Bohr Type on Hardy-Sobolev Classes

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Abstract Let $\beta > 0$ and $S_\beta := \{z \in \mathbb{C} : |\text{Im} z| < \beta\}$ be a strip in the complex plane. For an integer $r \geq 0$, let $H^r_{\infty, \beta}$ denote those real-valued functions $f$ on $\mathbb{R}$, which are analytic in $S_\beta$ and satisfy the restriction $|f^{(r)}(z)| \leq 1$, $z \in S_\beta$. For $\sigma > 0$, denote by $B_\sigma$ the class of functions $f$ which have spectra in $(-2\pi\sigma, 2\pi\sigma)$. And let $B_\sigma^\perp$ be the class of functions $f$ which have no spectrum in $(-2\pi\sigma, 2\pi\sigma)$. We prove an inequality of Bohr type

$$\|f\|_{\infty} \leq \frac{\pi}{\sqrt{\lambda \Lambda^{\prime}}} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^r \sinh((2k+1)2\sigma\beta)^{1/2}}, \quad f \in H^r_{\infty, \beta} \cap B_\sigma^\perp,$$

where $\lambda \in (0, 1)$, $\Lambda$ and $\Lambda^{\prime}$ are the complete elliptic integrals of the first kind for the moduli $\lambda$ and $\lambda^{\prime} = \sqrt{1-\lambda^2}$, respectively, and $\lambda$ satisfies

$$\frac{4\Lambda^{\beta}}{\pi \Lambda^{\prime}} = \frac{1}{\sigma}.$$

The constant in the above inequality is exact.

Keywords Hardy-Sobolev classes; the spectrum of a function; an inequality of Bohr type.

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1. Introduction

First we give the definition of the spectrum of a function.

Definition 1.1[1] We denote by $\Theta(\mathbb{R})$ the totality of functions $f \in C^\infty(\mathbb{R})$ such that

$$\sup_{x \in \mathbb{R}} |x^\gamma D^\alpha f(x)| < \infty$$

for every non-negative integers $\alpha$ and $\gamma$. Such functions are called rapidly decreasing (at $\infty$).

Definition 1.2[1] For any $f \in \Theta(\mathbb{R})$, define its Fourier transform $\hat{f}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} dx.$$

The smallest closed set outside which $\hat{f}(\xi)$ vanishes is called the spectrum of $f(x)$.

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For \( \sigma > 0 \), denote by \( B_{\sigma} \) the class of functions \( f \) which have spectra in \((-2\pi\sigma, 2\pi\sigma)\). And let \( B^+_{\sigma} \) be the class of functions \( f \) which have no spectrum in \((-2\pi\sigma, 2\pi\sigma)\). As usual, \( L_\infty(\mathbb{R}) \) denotes the space of real-valued functions \( f \) on \( \mathbb{R} \) with the usual norm \( \| f \|_\infty := \sup_{x \in \mathbb{R}} |f(x)| < \infty \).

We now give the classes of functions studied here. Let \( \beta > 0 \) and \( S_\beta := \{ z \in \mathbb{C} : |\text{Im} \, z| < \beta \} \) be a strip in the complex plane. For an integer \( r \geq 0 \), let \( H^r_{\infty, \beta} \) be the Hardy-Sobolev class of real-valued functions \( f \) on \( \mathbb{R} \), which are analytic in the strip \( S_\beta \) and satisfy the condition \(|f^{(r)}(z)| \leq 1, z \in S_\beta \). Denote by \( \tilde{H}^r_{\infty, \beta} \) those \( 2\pi \)-periodic functions in \( H^r_{\infty, \beta} \).

Let \( f \) be a real function of the real variable \( x \) with a bounded derivative. Suppose that \( f \in B^+_{\sigma} \). The inequality
\[
\| f \|_\infty \leq (4\sigma)^{-1} \| f' \|_\infty
\]
was given by Bohr\footnote{Bohr, 1914} for almost periodic \( f(x) \) with a proof based on the theory of analytic functions. The constant \( (4\sigma)^{-1} \) in (1.1) is the best possible. Iteration of (1.1) gives the inequality
\[
\| f \|_\infty \leq \sigma^{-n} t_n \| f^{(n)} \|_\infty
\]
with \( t_n = 4^{-n} \). With methods from the theory of real functions, Favard\footnote{Favard, 1924} found that the best possible value of \( t_n \) is
\[
t_n = (2\pi)^{-n} \cdot \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(n+1)}}{(2k+1)^n + 1}.
\]
Hörmander\footnote{Hörmander, 1963} obtained the following generalization of the inequality of Bohr type: if \( f(x) \) is real and
\[
-M_1 \leq f^{(n)}(x) \leq M_2,
\]
then
\[
-\sigma^{-n}\mu_1^{(n)}(M_1, M_2) \leq f(x) \leq \sigma^{-n}\mu_2^{(n)}(M_1, M_2),
\]
where \( \mu_1^{(n)} \) and \( \mu_2^{(n)} \) denote the best possible constants and are defined by
\[
-\mu_1^{(n)}(M_1, M_2) = \min_x h_n(x; M_1, M_2), \quad \mu_2^{(n)}(M_1, M_2) = \max_x h_n(x; M_1, M_2),
\]
where
\[
h_n(x; M_1, M_2) = \frac{M_1 + M_2}{(n + 1)!} \{ \overline{B}_{n+1}(x + \frac{M_2}{2(M_1 + M_2)}) - \overline{B}_{n+1}(x - \frac{M_2}{2(M_1 + M_2)}) \},
\]
and the functions \( \overline{B}_n(x) \) have the period 1 and coincide with the Bernoulli polynomials \( B_n(x) \) in the interval \((0, 1)\).

In this paper, we get an inequality of Bohr type for the class of functions \( H^r_{\infty, \beta} \).

We introduce the function \( \Phi_{\lambda, r, \beta} \) which will be proved to be the extremal function of the inequality of Bohr type for some \( \lambda \in (0, 1) \), and give the explicit presentation of its uniform norm \( \| \Phi_{\lambda, r, \beta} \|_\infty \).

Let \( \Lambda \) and \( \Lambda' \) be the complete elliptic integrals of the first kind for the moduli \( \lambda \in (0, 1) \) and
\( \lambda' = \sqrt{1 - \lambda^2} \), respectively. Put

\[
\Phi_{\lambda,0,\beta}(z) := \sqrt{\frac{\lambda}{\Lambda}} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\frac{\pi\Lambda'}{2\lambda})}{(2k+1) \sinh((2k+1)\frac{\pi\Lambda'}{2\lambda})},
\]

\[
\Phi_{\lambda,2j-1,\beta}(z) := \int_{\frac{2\lambda}{\Lambda'}}^z \Phi_{\lambda,2j-2,\beta}(\mu)d\mu, \quad j = 1, 2, \ldots.
\]

\[
\Phi_{\lambda,2j,\beta}(z) := \int_0^z \Phi_{\lambda,2j-1,\beta}(\mu)d\mu,
\]

Then from [6], we have

\[
\Phi_{\lambda,r,\beta}(z) = \sqrt{\frac{\lambda}{\Lambda}} \sum_{k=0}^{\infty} \frac{(\frac{\pi\Lambda'}{2\lambda})^r}{(2k+1)^r \sinh((2k+1)\frac{\pi\Lambda'}{2\lambda})}, \quad r = 0, 1, \ldots.
\]

When \( \lambda \) satisfies \( 4\Lambda\beta/(\pi\Lambda') = 1/n \) for some fixed \( n \in \mathbb{N} \), we know that \( \Phi_{\lambda,r,\beta}(z) = \Phi_{\beta,n,r}(z) \) [7].

We are now ready to state the main result.

**Theorem 1.3** Let \( \sigma > 0 \) and \( r = 0, 1, 2, \ldots \) Then

\[
\|f\|_{\infty} \leq \|\Phi_{\lambda,r,\beta}\|_{\infty} = \sqrt{\frac{\lambda}{\Lambda}} \sum_{k=0}^{\infty} \frac{(\frac{\pi\Lambda'}{2\lambda})^r}{(2k+1)^r \sinh((2k+1)2\sigma\beta)}, \quad f \in H_{\infty,\beta} \cap B_\sigma^+,
\]

where \( \lambda \in (0, 1) \) satisfying

\[
4\Lambda\beta/(\pi\Lambda') = 1/\sigma.
\]

The constant in the inequality (1.7) is best possible, which means that

\[
\sup_{f \in H_{\infty,\beta} \cap B_\sigma^+} \|f\|_{\infty} = \|\Phi_{\lambda,r,\beta}\|_{\infty} = \sqrt{\frac{\lambda}{\Lambda}} \sum_{k=0}^{\infty} \frac{(\frac{\pi\Lambda'}{2\lambda})^r}{(2k+1)^r \sinh((2k+1)2\sigma\beta)}.
\]

**Remark 1.4** From [7, 8], we know that

\[
\sup_{f \in H_{n,\beta} \cap \mathcal{T}_n^+} \|f\|_{\infty} = \|\Phi_{\beta,n,r}\|_{\infty} = \sqrt{\frac{\lambda}{\Lambda}} \sum_{k=0}^{\infty} \frac{(\frac{\pi\Lambda'}{2\lambda})^r}{(2k+1)^r \sinh((2k+1)n\beta)},
\]

where \( \mathcal{T}_n := \text{span}\{1, \cos t, \sin t, \ldots, \cos((n - 1)t), \sin((n - 1)t)\} \) is the space of trigonometric polynomials with order \( n - 1 \), and \( f \in \mathcal{T}_n^+ \) means that

\[
\int_0^{2\pi} f(t) \sin(kt)dt = 0, \quad k = 0, 1, \ldots, n - 1,
\]

\[
\int_0^{2\pi} f(t) \cos(kt)dt = 0,
\]

Thus, Theorem 1.3 is the generalization of this result.

**2. Proof of main result**

First we give some auxiliary results.
Lemma 2.1  For any \( \sigma > 0 \), there exists a \( \lambda_\sigma \in (0, 1) \) such that
\[
4\Lambda_\sigma \beta/(\pi \Lambda'_\sigma) = 1/\sigma, \tag{2.1}
\]
where \( \Lambda_\sigma \) and \( \Lambda'_\sigma \) are the complete elliptic integrals of the first kind for the moduli \( \lambda_\sigma \in (0, 1) \) and \( \lambda'_\sigma = \sqrt{1 - \lambda_\sigma^2} \), respectively.

**Proof** Since \( \Lambda \to \frac{\pi}{2} \) and \( \Lambda' \to +\infty \) as \( \lambda \to 0^+ \), \( \frac{4\Lambda \beta}{\pi \Lambda'} \to 0^+ \) as \( \lambda \to 0^+ \). On the other hand, when \( \lambda \to 1^- \), \( \Lambda \to +\infty \) and \( \Lambda' \to \frac{\pi}{2} \), i.e., when \( \lambda \to 1^- \), \( \frac{4\Lambda \beta}{\pi \Lambda'} \to +\infty \). So from the fact that \( 4\Lambda \beta/(\pi \Lambda') \) continuously depends on \( \lambda \), it follows that for any \( \sigma > 0 \), there exists a \( \lambda_\sigma \in (0, 1) \) such that (2.1) holds. Lemma 2.1 is proved. \( \square \)

We now consider continuous functions \( \varphi \) on \( \mathbb{R} \) with the properties
\[
\varphi(x) \geq 0, \quad \sum_{-\infty}^{+\infty} \varphi(x + n) \leq 1, \quad \varphi(0) = 1. \tag{2.2}
\]
An example of such a function is \( \varphi(x) = (\pi x)^{-2} \sin^2(\pi x) \). So we can take a fixed function \( \varphi \) on \( \mathbb{R} \) having the properties (2.2). If \( g \) is a bounded function on \( \mathbb{R} \), we set
\[
g_h(x) = \sum_{-\infty}^{+\infty} \varphi(hx + n)g(x + nh^{-1}), \tag{2.3}
\]
where \( h > 0 \). It is evident that the series converges on \( \mathbb{R} \) and that \( g_h(x) \) has the period \( h^{-1} \).

Lemma 2.2\(^5\) If \(-1 \leq g(x) \leq 1 \) for all \( x \in \mathbb{R} \), then \(-1 \leq g_h(x) \leq 1 \) and \( g_h(x) \) tends to \( g(x) \) as \( h \to 0^+ \), uniformly on every bounded set of \( \mathbb{R} \).

From [5], we know that a function \( f \) has no spectrum in \((-2\pi \sigma, 2\pi \sigma)\) means explicitly that \( \int_{-\infty}^{+\infty} f(x) \psi(x) \, dx = 0 \) if \( \psi(x) \in \Psi \) and \( \dot{\psi}(\xi) \) vanishes outside a compact set in \((-2\pi \sigma, 2\pi \sigma)\), where \( \Psi \) is the class of all infinitely differentiable functions on \( \mathbb{R} \) which vanish at infinity together with all their derivatives more rapidly than any inverse power of \( x \).

It also follows from [5] that there exists a function \( \varphi \) on \( \mathbb{R} \) having the properties (2.2) and the properties that \( \varphi \in \Psi \) and \( \dot{\varphi} (\xi) \) shall vanish outside a bounded set of \( \mathbb{R} \). Denote by \( M \) a number such that \( \dot{\varphi}(\xi) = 0 \) for \( |\xi| \geq M \). The Fourier transform of \( h\varphi(hx)e^{-ikhx} \) is \( \hat{\varphi}(\xi + kh)/h = \dot{\varphi}(\xi h^{-1} + k) \). It vanishes outside an interval contained in \((-2\pi \sigma, 2\pi \sigma)\) if \( |kh| < 2\pi \sigma - Mh \).

Lemma 2.3\(^5\) \( g_h(x) \) has no spectrum in \((-2\pi \sigma - Mh, 2\pi \sigma - Mh)\) if \( g(x) \) has no spectrum in \((-2\pi \sigma, 2\pi \sigma)\).

Lemma 2.4 Let \( \sigma > 0 \) and \( r = 0, 1, 2, \ldots \). Then for any \( f \in H^r_{\infty, \beta} \cap B^1_{\sigma} \), there exists a \( F \in H^{r+1}_{\infty, \beta} \) such that
\[
F'(z) = f(z), \quad z \in S_{\beta},
\]
and
\[
\|F\|_{\infty} \leq \|\Phi_{\lambda_\sigma, r + 1, \beta}\|_{\infty}, \tag{2.4}
\]
where \( \lambda_\sigma \in (0, 1) \) satisfying equality (2.1).

**Proof** Let \( \sigma > 0 \) and \( r = 0, 1, 2, \ldots \). For any \( f \in H^r_{\infty, \beta} \cap B^1_{\sigma} \), define the periodic approximating
function (2.3) of \( f^{(r)} \) by \( f^{(r)}_h \) on \( \mathbb{R} \). Since \( f \in H^r_{\infty, \beta} \), by Lemma 2.2, we have \(-1 \leq f^{(r)}_h(x) \leq 1\) for all \( x \in \mathbb{R} \) and \( f^{(r)}_h \) tends to \( f^{(r)} \) as \( h \to 0^+ \), uniformly on every bounded set of \( \mathbb{R} \). Since \( f \in B^+_\sigma \), \( f^{(r)} \) has no spectrum in \((-2\pi \sigma, 2\pi \sigma)\). So from Lemma 2.3, it follows that \( f^{(r)}_h \) has no spectrum in \((-2\pi \sigma - Mh, 2\pi \sigma - Mh)\), where the function \( \varphi \) on \( \mathbb{R} \) has the properties (2.2) and the properties that \( \varphi \in \Psi \) and \( \varphi(\xi) = 0 \) for \( |\xi| \geq M \), and \( h \) is so small that \( 2\pi \sigma - Mh > 0 \). Then there exists a periodic function \( f_h \) on \( \mathbb{R} \) with zero mean value satisfying the \( r \)-th derivative of \( f_h(x) \) is \( f^{(r)}_h(x) \) and \( f_h \) tends to \( f \) as \( h \to 0^+ \), uniformly on every bounded set of \( \mathbb{R} \). So there exists a periodic integral of \( f_h \) on \( \mathbb{R} \), denoted by \( F_h(x) := \int_0^x f_h(t) \, dt + c_0 \), such that \( F'_h(x) = f_h(x), \quad x \in \mathbb{R}, \) and

\[
\|F_h\|_\infty \leq \|\Phi_{\lambda, r+1, \beta}\|_\infty,
\]

where \( c_0 \in \mathbb{R} \) and \( \lambda \in (0, 1) \) satisfying (2.1). Denote by \( F(x) := \int_0^x f(t) \, dt + c_0 \) the integral of \( f(x) \) on \( \mathbb{R} \). Then \( F'(x) = f(x), \ x \in \mathbb{R} \) and \( F_h(x) \) tends to \( F(x) \) as \( h \to 0^+ \), uniformly on every bounded set of \( \mathbb{R} \). So from the fact that \( \|F\|_\infty \leq \|F - F_h\|_\infty + \|F_h\|_\infty \), we know that \( \|F\|_\infty \leq \|\Phi_{\lambda, r+1, \beta}\|_\infty \), as \( h \to 0^+ \). Now it follows from the uniqueness theorem of analytic functions that there exists a \( F \in H^r_{\infty, \beta} \) such that \( F'(z) = f(z), \ z \in S_{\beta} \) and (2.4) holds. Lemma 2.4 is proved.

**Lemma 2.5** Let \( r = 0, 1, 2, \ldots \). If \( f \in H^r_{\infty, \beta} \) and the inequality \( \|f\|_\infty \leq \|\Phi_{\lambda, r, \beta}\|_\infty \) holds for some \( \lambda \in (0, 1) \), then for all \( 1 \leq l \leq r + 1 \),

\[
\|f^{(l)}\|_\infty \leq \|\Phi^{(l)}_{\lambda, r, \beta}\|_\infty = \frac{\pi}{\sqrt{\lambda \Lambda}} \left( \frac{4\lambda \beta}{\pi \Lambda'} \right)^{r-l} \sum_{k=0}^{\infty} \frac{(-1)^{k+(r-l+1)}}{(2k+1)^{r-l} \sinh((2k+1)\frac{\pi \Lambda}{\Lambda})}. \tag{2.5}
\]

**Lemma 2.6** Let \( \sigma > 0 \). Then \( \Phi_{\lambda, r, \beta} \in H^r_{\infty, \beta} \cap B^+_{\sigma} \), where \( \lambda \) satisfies (2.1).

**Proof** Let \( \sigma > 0 \) and \( \lambda \) satisfy (2.1). It is obvious that \( \Phi_{\lambda, r, \beta} \in H^r_{\infty, \beta} \). We only need to prove that \( \Phi_{\lambda, r, \beta} \in B^+_{\sigma} \). In fact, it follows from the condition: \( \lambda \) satisfies (2.1) that

\[
\int_{-\infty}^{+\infty} \Phi_{\lambda, r, \beta}(x) \psi(x) \, dx = 0
\]

if \( \psi(x) \in \Psi \) and \( \hat{\psi}(\xi) \) vanishes outside a compact set in \((-2\pi \sigma, 2\pi \sigma)\). So \( \Phi_{\lambda, r, \beta} \in B^+_{\sigma} \). □

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3** Let \( \sigma > 0 \) and \( r = 0, 1, 2, \ldots \). Then from Lemma 2.4, it follows that for any \( f \in H^r_{\infty, \beta} \cap B^+_{\sigma} \), there exists an \( F \in H^r_{\infty, \beta} \) such that

\[
F'(z) = f(z), \quad z \in S_{\beta},
\]

and

\[
\|F\|_\infty \leq \|\Phi_{\lambda, r+1, \beta}\|_\infty,
\]
which together with Lemma 2.5 gives
\[
\sup_{f \in H_{\infty, \beta}^r \cap B_{\sigma}^\perp} \|f\|_{\infty} \leq \sup_{F \in H_{\infty, \beta}^{r+1}} \|F\|_{\infty} = \|\Phi_{\lambda, r+1, \beta}\|_{\infty},
\]
i.e.
\[
\sup_{f \in H_{\infty, \beta}^r \cap B_{\sigma}^\perp} \|f\|_{\infty} \leq \|\Phi_{\lambda, r, \beta}\|_{\infty},
\]
where \(\lambda_{\sigma} \in (0, 1)\) satisfying (2.1). And from Lemma 2.6, we know that \(\Phi_{\lambda, r, \beta} \in H_{\infty, \beta}^r \cap B_{\sigma}^\perp\) for \(\lambda_{\sigma}\) satisfying (2.1). So
\[
\sup_{f \in H_{\infty, \beta}^r \cap B_{\sigma}^\perp} \|f\|_{\infty} = \|\Phi_{\lambda, r, \beta}\|_{\infty} = \frac{\pi}{\sqrt{\lambda_{\sigma}} \Lambda_{\lambda_{\sigma}} \sigma^r} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^r \sinh((2k+1)^2 \sigma \beta)}.
\]

Theorem 1.3 is proved.

References