Graded and Nongraded Properties of Partial Tilting Modules and Tilting Modules

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Abstract This paper gives the relationships among partial tilting objects (tilting objects) of categories of graded left \( A \)-modules of type \( G \), left \( A \)-modules, left \( A \)-modules and \( A^\#G \)-modules, and then proves that for graded partial tilting modules, there exist the Bongartz complements in the category of graded \( A \)-modules.

Keywords tilting module; partial tilting module; graded module; smash product.

1. Introduction and preliminaries

In 1970’s, arising from the quivers of hereditary algebras, Berstein, Gel’fand and Ponomarev introduced the notion of a reflection functor, and proved the BGP-theorem. Basing on it, they gave a new proof to the famous Gabriel theorem, turned on a new orientation of the development of the Morita theory, and brought a new research field—tilting theory\(^{[1]}\). Later, tilting theory was generalized well by Auslander, Platzek, Reiten, Bernner, Butler, Bongartz, et al. Eventually Happel and Ringel presented the notion of tilting modules with the form of axiom\(^{[2−5]}\). During the later twenty years, tilting theory developed well, which made algebraic representation theory progress quite well and promoted the developments of many research fields related with tilting theory\(^{[6,7]}\). Lately, Kleiner introduced a new grading on the preprojective algebra of a quiver\(^{[8]}\), which provided us with a tool for our research. Combining with graded ring theory and smash product theory, this paper discusses the relationships among partial tilting objects (tilting objects) of categories of graded left \( A \)-modules, left \( A \)-modules, left \( A \)-modules and \( A^\#G \)-modules, and then proves that for graded partial tilting modules, there exist the Bongartz complements in the category of graded \( A \)-modules.

Throughout this paper, let \( G \) be a multiplicative group with identity element \( e \), \( k \) be an algebraically closed field, and \( A \) be a finite dimensional associative graded \( k \)-algebra of type \( G \). The category \( A \)-gr consists of graded \( A \)-modules of type \( G \) and the morphisms are taken to be...
graded morphisms of degree $e$. We can define the restriction functor $(-)_e: A\text{-gr} \rightarrow A_e\text{-Mod}$ by putting $(M)_e = M_e$ while a morphism $f : M \rightarrow N$ in $A\text{-gr}$ restricts to $(f)_e = f_e : (M)_e \rightarrow (N)_e$. Consider $M, N \in A\text{-gr}$. Let $\text{HOM}_A(M, N)_g$ be an additive subgroup of $\text{Hom}_A(M, N)$ composed of graded morphisms of degree $g$ and $\text{HOM}_A(M, N) = \oplus_{g \in G} \text{HOM}_A(M, N)_g$. The 1st derived functors of the functors $\text{HOM}_A(-, -)_g$, $\text{HOM}_A(-, -)$ and $\text{Hom}_{A-\text{gr}}(-, -)$ are denoted by $\text{Ext}_A(-, -)_g$, $\text{Ext}_A(-, -)$ and $\text{Ext}_{A-\text{gr}}(-, -)$ respectively. It is easy to show that $\text{Ext}_{A-\text{gr}}(M, N) = \text{Ext}_A(M, N)_e$ and $\text{Ext}_A(M, N) = \oplus_{g \in G} \text{Ext}_A(M, N)_g$ for $M, N \in A\text{-gr}$. Without special explanation, all our $A$-modules will be finitely generated left $A$-modules. Hence $\text{HOM}_A(M, N) = \text{Hom}_A(M, N)$ and for every $n \geq 0$ it follows that $\text{Ext}_A^n(M, N) = \text{Ext}_A(M, N)_e$.

A $g$ with degree $g$ and the elements of $\oplus_{g \in G} M_P$ are the finite formal sums of $m_g P_g$ for $m_g \in M$ and $g \in G$. $M\sharp G$ is an $A\sharp G$-module with its addition and $A\sharp G$-modular multiplication as follows:

$$(a_P h)(b_P g) = ab h_{g^{-1}} P_g, \text{ for } g, h \in G \text{ and } a, b \in A.$$

It is clear that $A\sharp G$ is a ring.

**Definition 1.1** Let $A$ be a $G$-graded ring. Define the smash product $A\sharp G$ as the free $A$-module $\oplus_{g \in G} A P_g$ with multiplication as follows:

$$(a_P h)(b_P g) = ab h_{g^{-1}} P_g, \text{ for } g, h \in G \text{ and } a, b \in A.$$ 

**Definition 1.2** Let $A$ be a $G$-graded ring and $M \in A\text{-gr}$. Define $M\sharp G$ as $\oplus_{g \in G} M P_g$ where $P_g$ is just a symbol indexed by $g$ and the elements of $\oplus_{g \in G} M P_g$ are the finite formal sums of $m_g P_g$ for $m_g \in M$ and $g \in G$. $M\sharp G$ is an $A\sharp G$-module with its addition and $A\sharp G$-modular multiplication as follows:

$$\Sigma_{g \in G} m_g P_g + \Sigma_{g \in G} m'_g P_g = \Sigma_{g \in G} (m_g + m'_g) P_g, \quad (a_x P_{hg})(m_y P_g) = \begin{cases} (a_x m_y) P_g, & h = yg, \\ 0, & h \neq yg, \end{cases}$$

where $a_x \in A_x$ and $m_g \in M_g$. We call $M\sharp G$ the smash product of $M$ by $G$.

**Proposition 1.3** The functor $\sharp G: A\text{-gr} \rightarrow A\sharp G\text{-Mod}$,

$$f : M \rightarrow N \mapsto f\sharp G : M\sharp G \rightarrow N\sharp G$$

is a covariant exact functor and preserves the direct sum.

**Proposition 1.4** Let $A$ be a $G$-graded ring. Then there is an equivalence of categories $A\text{-gr} \sim A\sharp G\text{-Mod}$ by the equivalent functors:

$$(\cdot)^*: A\text{-gr} \rightarrow A\sharp G\text{-Mod}$$

$$M = \oplus_{g \in G} M_g \mapsto M^* = (M_e, M_g \cdots M_h \cdots)'$$

and the invertible functor: $(\cdot)_*: A\sharp G\text{-Mod} \rightarrow A\text{-gr}$.
Graded and nongraded properties of partial tilting modules and tilting modules

$N = (e_{(e,e)} N, e_{(h,h)} N, \ldots, e_{(g,g)} N, \ldots)' \longrightarrow N_\ast = \oplus_{g \in G} e_{(g,g)} N,$
where $e_{(g,g)} = 1, P_g$.

2. Graded partial tilting modules and the Bongartz-complement

In this section, we introduce the notion of partial tilting objects and strongly partial tilting objects in $A$-gr, discuss whether the functor $(-)_e$ and $- \sharp G$ can preserve partial tilting objects, and then prove that for graded partial tilting modules, there exist the Bongartz-complements in the category of graded $A$-modules. Note that the projective dimension of $M$ in $A$-gr will be denoted by $\text{gr.pd}_A M$.

Definition 2.1 Assume $M \in A$-gr. Then $M$ is called graded partial tilting module if it satisfies the following conditions:

1. $\text{gr.pd}_A M \leq 1$;
2. $\text{Ext}_{A-\text{gr}}(M, M) = 0$.

Obviously, if a graded module is a partial tilting module, then it is certainly a graded partial tilting module. Conversely, it is always not true.

Example 2.2 Consider the path algebra $A = k \rightarrow Q_A$, where $\rightarrow Q_A$: $1 \rightarrow 2 \rightarrow 3$. Assume $G = \mathbb{Z}_3$. Then $A$ is turned to be a graded $k$-algebra of type $G$ according to the lengths of paths (this graded structure for a path algebra is natural). It is easy to prove that $A \sharp G = k \rightarrow Q_{A \sharp G}$ where $\rightarrow Q_{A \sharp G}$ is given by:

\[
\begin{align*}
1 & \rightarrow 2 \rightarrow 3 \\
4 & \rightarrow 5 \rightarrow 6 \\
7 & \rightarrow 8 \rightarrow 9
\end{align*}
\]

Hence $Ae_3 = k\{e_3\}$, $Ae_2 = k\{e_2, \beta\}$ and $Ae_1 = k\{e_1, \alpha, \beta \alpha\}$ are $A$-graded modules respectively with graded structures as follows:

\[
(Ae_3)_\mathbb{0} = \begin{cases} k\{e_3\}, & \mathbb{i} = \mathbb{0} \\ 0, & \mathbb{i} \neq \mathbb{0} \end{cases}, \quad (Ae_2)_\mathbb{0} = \begin{cases} k\{e_2\}, & \mathbb{i} = \mathbb{0} \\ k\{\beta\}, & \mathbb{i} = \mathbb{1} \\ 0, & \mathbb{i} = \mathbb{2} \end{cases},
\]

\[
(Ae_1)_\mathbb{0} = \begin{cases} k\{e_1\}, & \mathbb{i} = \mathbb{0} \\ k\{\alpha\}, & \mathbb{i} = \mathbb{1} \\ k\{\beta \alpha\}, & \mathbb{i} = \mathbb{2} \end{cases}
\]

Let $M \triangleq \tau^{-1} Ae_3$. Obviously, $M$ is graded with $(\tau^{-1} Ae_3)_\mathbb{0} = \begin{cases} k\{\beta^*\}, & \mathbb{i} = \mathbb{0} \\ 0, & \mathbb{i} \neq \mathbb{0} \end{cases}$. In fact, owing to [8], if we consider the case when $a(\gamma) = b(\gamma) = 1$ for $\forall \gamma \in Q_1$, then the $(a, b)$-preprojective algebra of the quiver $\rightarrow Q_A$ is turned out to be $A' = (kQ')/I$ where $kQ'$ is given by:
and the two-sided ideal \( I = \langle \{ \alpha^* \alpha, \beta^* \beta - \alpha \alpha^*, -\beta \beta^* \} \rangle \). It follows that \( M \triangleq \tau^{-1} A e_3 = V_1^e = k\{ \beta^* \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \} \), which can be a \( G \)-graded \( A \)-module with \( M_i = \{ k\{ \beta^* \} \}, i = 0, i \neq 0 \).

We claim that graded module \( M \oplus A e_3 \) is a graded partial tilting module but not partial tilting module. Actually, \( A \# G \)-module \( M P_0 \) and \( A e_3 P_0 \) can be represented by their dimension vectors as follows:

\[
\begin{array}{cccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 1 \\
0 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

And it is easy to check that \( B \triangleq M P_0 \oplus A e_3 P_0 \) is a partial tilting \( A \# G \)-module. Applying the Proposition 1.4, we prove that \( B^* = M \oplus A e_3 \) is a graded partial tilting module where \( M \) and \( A e_3 \) are graded modules. However, since \( \text{Ext}_A(M, A e_3) \neq 0 \), we have \( \text{Ext}_A(B_*, B_*) \neq 0 \). Hence \( A \)-module \( B_* \) is not a partial tilting module.

From [9], we know that for \( \forall g \in G \) the translational functor \( T_g : A-\text{gr} \rightarrow A-\text{gr} \) is an equivalent functor and if the ring \( A \) is a strongly graded ring of type \( G \), then \( (\cdot)_e : A-\text{gr} \Rightarrow A_e-\text{Mod} : A \otimes_{A_e} - \) are mutually equivalent functors. We get

**Proposition 2.3** Let \( A \) be a strongly graded ring of type \( G \) and \( M = \oplus_{g \in G} M_g \) be a graded \( A \)-module. Then the followings are equivalent:

1. \( M \) is a graded partial tilting \( A \)-module;
2. \( M_e \) is a partial tilting \( A_e \)-module;
3. There is \( g \in G \) such that \( M_g \) is a partial tilting \( A_e \)-module;
4. \( M_g \) is a partial tilting \( A_e \)-module for \( \forall g \in G \).

Because of the difference between the derived functor \( \text{EXT}_A(M, N) \) and \( \text{Ext}_{A-\text{gr}}(M, N) \), we introduce the notion of graded strongly partial tilting module, that is,

**Definition 2.4** \( M \in A-\text{gr} \) is called graded strongly partial tilting module if it satisfies the following conditions

1. \( \text{gr.pd}_A M \leq 1 \);
2. \( \text{EXT}_A(M, M) = 0 \).

Clearly, graded module \( M \) is a graded strongly partial tilting module if and only if \( M \) is a partial tilting module. In addition, graded strongly partial tilting modules are graded partial tilting modules. Conversely, it is not always true, as shown in Example 2.2.

**Proposition 2.5** Assume the multiplicative group \( G \) is finite. Then \( M \) is a graded strongly partial tilting module if and only if \( M \# G \) is a partial tilting \( A \# G \)-module.

**Proof** Since \( (\cdot)^* : A-\text{gr} \rightarrow A \# G-\text{Mod} \) is an equivalent functor and \( M \# G \cong (\oplus_{g \in G}(M(g)))^* \), we
have
\[
\text{pd}_{A^\circ G} M^\sharp G = \text{pd}_{A^\circ G} (\oplus_{g \in G} (M(g))^\ast) = \text{gr.pd}_A (\oplus_{g \in G} (M(g))) \\
= \text{pd}_A (\oplus_{g \in G} (M(g))) = \sup \{ \text{pd}_A M(g) \} = \text{pd}_A M = \text{gr.pd}_A M.
\]
So gr.pd_A M \leq 1 \iff \text{pd}_{A^\circ G} M^\sharp G \leq 1.

Moreover, it is easy to show that
\[
\text{Ext}_{A^\circ G} (M^\sharp G, M^\sharp G) = \text{Ext}_{A^\circ G} ((\oplus_{g \in G} (M(g)))^\ast, (\oplus_{g \in G} (M(g)))^\ast) \\
= \text{Ext}_{A\text{-gr}} (\oplus_{g \in G} (M(g)), \oplus_{g \in G} (M(g))) \\
= \oplus_{g \in G} \oplus_{h \in G} \text{Ext}_{A\text{-gr}} (M(g), M(h)) = \oplus_{g \in G} \oplus_{h \in G} \text{Ext}_{A\text{-gr}} (M, M(g^{-1}h)) \\
= \oplus_{g \in G} \oplus_{h \in G} (\text{Ext}_A (M, M))_{g^{-1}h} = \oplus_{g \in G} \text{EXT}_A (M, M).
\]
Thus Ext_{A^\circ G} (M^\sharp G, M^\sharp G) = 0 \iff \text{EXT}_A (M, M) = 0.

Next let us discuss the existence of the Bongartz-complement in A-gr. For convenience, we introduce the notion of the graded tilting module firstly.

**Definition 2.6** M \in A-gr is called graded tilting module if it satisfies the following conditions:

(T_1) gr.pd_A M \leq 1;

(T_2) Ext_{A\text{-gr}} (M, M) = 0;

(T_3) There exists a short exact sequence 0 \to A \to M' \to M'' \to 0 with M' and M'' direct sums of direct summands of M in A-gr, that is, M' and M'' \in \text{add}_{A\text{-gr}} (M).

**Remark 2.7** It is well known that one of conditions in the definition of the tilting module, similar to (T_3) above, can be replaced by (T_3)': the number of mutually non-isomorphic indecomposable summands of M equals to rankK_0(A). But this case is not true for graded tilting modules.

**Example 2.8** For Example 2.2, it is clear that A = \oplus_{i=1}^3 A e_i is a graded tilting module and A e_1, A e_2 and A e_3 are mutually non-isomorphic and indecomposable in A-gr. But the number of mutually non-isomorphic indecomposable summands of A 3 does not equal to rankK_0 (f.g.A-gr \mathfrak{M}).

In fact, rankK_0 (f.g.A-gr \mathfrak{M}) = rankK_0 (A^\sharp G) = |G| rankK_0 (A) = 9 \neq 3. \qed

Now we show the existence of Bongartz-complements of graded partial tilting modules in categories of graded modules.

**Theorem 2.9** Let T be a graded partial tilting A-module. Then there exists a graded A-module X such that T \oplus X is a graded tilting A-module.

**Proof** It is not hard to prove that for \forall A', B \in A-gr it follows that e_{A\text{-gr}} (B, A') \cong Ext_{A\text{-gr}} (B, A') where e_{A\text{-gr}} (B, A') = \{ short exact sequences 0 \to A' \to C \to B \to 0 in A-gr | A', B, C \in A-gr \}.

Let e_1, e_2, \ldots, e_d be a basis of the k-vector space Ext_{A\text{-gr}} (T, A), and consider the exact sequence
\[
0 \to A \xrightarrow{g_1} X \to T^{(d)} \to 0 \quad (*)
\]
defined as the push out in $A$-$\text{gr}$ along the graded morphism $f : A^{(d)} \rightarrow A(a_1, a_2, \ldots, a_d) \hookrightarrow \Sigma_{i=1}^d a_i$ of the exact sequence $\oplus_{i=1}^d e_i$, that is,

$$
\begin{array}{c}
0 \longrightarrow A^{(d)} \xrightarrow{f_1} \oplus_{i=1}^d X_i \longrightarrow T^{(d)} \longrightarrow 0 \\
\downarrow f \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
0 \longrightarrow A \xrightarrow{g_1} X \longrightarrow T^{(d)} \longrightarrow 0
\end{array}
$$

Since $\text{pd}_A T = \text{gr} \cdot \text{pd}_A T \leq 1$, $\text{pd}_A A = 0$ and $(\ast)$ is exact in the $A$-modules category, we have $\text{pd}_A X < 1$. Hence $\text{gr} \cdot \text{pd}_A (T \oplus X) = \text{pd}_A (T \oplus X) \leq 1$.

Applying the functor $\text{Hom}_{A\text{-}gr}(T, -)$ to $(\ast)$ gives a long exact sequence:

$$
\text{Hom}_{A\text{-}gr}(T, T^{(d)}) \xrightarrow{\varphi} \text{Ext}_{A\text{-}gr}(T, A) \longrightarrow \text{Ext}_{A\text{-}gr}(T, X) \longrightarrow \text{Ext}_{A\text{-}gr}(T, T^{(d)}) = 0.
$$

By construction, $\varphi$ is surjective. Hence $\text{Ext}_{A\text{-}gr}(T, X) = 0$.

Moreover, applying $\text{Hom}_{A\text{-}gr}(-, T)$ to $(\ast)$ yields:

$$
0 = \text{Ext}_{A\text{-}gr}(T^{(d)}, T) \longrightarrow \text{Ext}_{A\text{-}gr}(X, T) \longrightarrow \text{Ext}_{A\text{-}gr}(A, T) = (\text{EXT}_A(A, T))_e = 0.
$$

Thus $\text{Ext}_{A\text{-}gr}(X, T) = 0$.

And applying $\text{Hom}_{A\text{-}gr}(-, X)$ to $(\ast)$ yields:

$$
0 = \text{Ext}_{A\text{-}gr}(T^{(d)}, X) \longrightarrow \text{Ext}_{A\text{-}gr}(X, X) \longrightarrow \text{Ext}_{A\text{-}gr}(A, X) = (\text{EXT}_A(A, X))_e = 0.
$$

It follows that $\text{Ext}_{A\text{-}gr}(X, X) = 0$.

Hence $\text{Ext}_{A\text{-}gr}(T \oplus X, T \oplus X) = 0$. Since $(\ast)$ is the sequence of (T3), $T \oplus X$ is indeed a graded tilting module. \hfill $\Box$

### 3. Tilting modules $M^*_G$

Assume $G$ is a finite multiplicative group. And in this section we mainly talk about how the functor $\sharp G$ can preserve tilting modules. Note that according to the $A\sharp G$-modular multiplication of $M^*_G$, $MP_g$ is an $A\sharp G$-submodule of $M^*_G$ for all $g \in G$. So we have

**Theorem 3.1** If the partial tilting graded $A$-module $M$ in $A$-$\text{gr}$ has an indecomposable decomposition $M = \bigoplus_{i=1}^n T_i$, where $n = \text{rank} K_0(A)$, $T_i$ are mutually non-isomorphic and $T_i \not\cong T_j (g)$ ($\forall g \in G$, $g \neq e; i, j = 1, \ldots, n$) in $A$-$\text{gr}$, then the $A\sharp G$-module $M^*_G$ is the tilting module.

**Proof** Since $(\ast) : A$-$\text{gr} \rightarrow A\sharp G$-$\text{Mod}$ is an equivalent functor and $M^*_G \cong (\bigoplus_{g \in G} (M(g)))^*$, we have

$$
\text{pd}_{A\sharp G} M^*_G = \text{pd}_{A\sharp G} (\bigoplus_{g \in G} (M(g)))^* = \text{gr} \cdot \text{pd}_A (\bigoplus_{g \in G} (M(g)))
$$

$$
= \text{pd}_A (\bigoplus_{g \in G} (M(g))) = \sup \{ \text{pd}_A M(g) \} = \text{pd}_A M \leq 1
$$

and

$$
\text{Ext}_{A\sharp G}(M^*_G, M^*_G) = \text{Ext}_{A\sharp G}((\bigoplus_{g \in G} (M(g)))^*, (\bigoplus_{g \in G} (M(g)))^*)
$$

$$
= \text{Ext}_{A\text{-}gr}((\bigoplus_{g \in G} (M(g))), (\bigoplus_{g \in G} (M(g)))^*)
$$
And we claim that

\[ \text{Ext} A\#G T \]

isomorphic. If \( A\#G T \)

Example 3.3
tilting module.

It is easy to show that

\[ \text{Ext} A\#G T \]

\[ \text{Ext} A\#G M(g), \oplus_{g \in G} M(g) \]

Thus \( \text{Ext} A\#G M(g), \oplus_{g \in G} M(g) \)

By conditions, there exists an \( A\#G \)-modular decomposition

\[ M\#G = (\oplus_{i=1}^{n} T_{i})\#G = \oplus_{i=1}^{n} (T_{i}\#G) = \oplus_{i=1}^{n} \oplus_{g \in G} T_{i}P_{g}. \]

And we claim that \( T_{i}P_{g} \) are indecomposable \( A\#G \)-modules and mutually non-isomorphic. In fact, assume \( T_{i}P_{g} \) are decomposable, i.e., \( T_{i}P_{g} = D \oplus B. \) Since \( D \oplus B = T_{i}P_{g} \cong (T_{i}(g^{-1}))^{*} \) and \( (\cdot)_{*}: A\#G \text{-Mod} \to A\text{-gr} \) is equivalent, we have \( T_{i}(g^{-1}) \cong D_{*} \oplus B_{*}, \) that is, \( T_{i} \) are decomposable, a contradiction.

Next let us divide into three steps to prove that \( T_{i}P_{g}(\forall g \in G, i = 1, \ldots, n) \) are mutually non-isomorphic. If \( T_{i}P_{g} \cong T_{j}P_{h} \) where \( i \neq j, \) then \( (T_{i}(g^{-1}))^{*} \cong (T_{j}(g^{-1}))^{*}. \) Thus \( T_{i}(g^{-1}) \cong T_{j}(g^{-1}). \)

Hence \( T_{i} \cong T_{j}, \) a contradiction; if \( T_{i}P_{g} \cong T_{i}P_{h} \) where \( g \neq h, \) then it is similar to prove that \( T_{i}(g^{-1}) \cong T_{i}(h^{-1}). \) And since the translational functors are equivalent functors, \( T_{i} \cong T_{i}(gh^{-1}), \) a contradiction; if \( T_{i}P_{g} \cong T_{j}P_{h} \) with \( g \neq h \) and \( i \neq j, \) then from \( (T_{i}(g^{-1}))^{*} \cong T_{i}P_{g} \cong T_{j}P_{h} \cong (T_{j}(h^{-1}))^{*} \) it follows that there exists a graded isomorphism \( T_{i} \cong T_{j}(gh^{-1}), \) a contradiction.

To sum up the above-mentioned, the theorem is proved.

\[ \square \]

**Corollary 3.2** If graded \( A\text{-module} M \) is a tilting module and in \( A\text{-gr} \) has an indecomposable decomposition \( M = \oplus_{i=1}^{n} T_{i} \) where \( T_{i} \cong T_{i}(g) (\forall g \in G \text{ and } g \neq e), \) then \( A\#G \)-module \( M\#G \) is a tilting module.

**Example 3.3** For Example 2.2, it is obvious that \( BB(APR) \)-tilting module \( T = Ae_{1} \oplus Ae_{2} \oplus \tau^{-1} Ae_{3} \) satisfies the conditions of the Theorem 3.1. Hence \( T\#G \) is a tilting \( A\#G \)-module.

In fact, for \( \forall \theta \in Z_{3}, \) it follows that \( Ae_{1}I_{\theta} = (A\#G)\epsilon_{1}P_{\theta}, Ae_{2}I_{\theta} = (A\#G)\epsilon_{2}P_{\theta}, \) Clearly, \( A\#G \)-module \( MP_{0}, MP_{1} \) and \( MP_{2} \) can be represented respectively by the dimension vectors as follows:

\[
\begin{array}{cccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
0 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

Then according to the definition of the tilting module, it is proved directly that \( T\#G \) is an \( A\#G \)-tilting module. On the other hand, although \( T \) is a \( BB(APR) \)-tilting module, \( T\#G \) is not so. Thus \( -\#G \) does not preserve the \( BB(APR) \)-tilting module.

\[ \square \]

**Definition 3.4** \( M \in A\text{-gr} \) is called graded strongly tilting module if it satisfies the following conditions:

\[(1)\, \text{gr.} pd_{A} M \leq 1;\]
(2) $\text{EXT}_A(M, M) = 0$;
(3) There exists a short exact sequence $0 \to A \to M' \to M'' \to 0$ with $M'$ and $M''$ direct sums of direct summands of $M$ in $A\text{-gr}$, that is, $M' \text{ and } M'' \in \text{add}_{A\text{-gr}}(M)$.

Obviously, graded strongly tilting modules are graded tilting modules. Conversely, it is not always true.

**Example 3.5** For example 2.2, it follows that $Ae_1P_2 = (A^gG)e_1P_2, Ae_2P_2 = (A^gG)e_2P_2$ and $Ae_3P_2 = (A^gG)e_3P_2$ are all indecomposable $A^gG$-projective modules for $\forall i \in \mathbb{Z}_3$. It is easy to prove that $A^gG$-module

$$C \triangleq MP_2 \oplus Ae_2P_2 \oplus Ae_1P_2 \oplus Ae_1P_2 \oplus Ae_2P_2 \oplus Ae_1P_2 \oplus Ae_2P_2 \oplus Ae_3P_2$$

is an $A^gG$-tilting module. Applying the Proposition 1.4 yields that $C_\ast$ is a graded tilting module. But since $\text{Ext}_A(M, Ae_3) \neq 0$, $\text{EXT}_A(C_\ast, C_\ast) = \text{Ext}_A(C_\ast, C_\ast) \neq 0$. Hence $C_\ast$ is a graded tilting module but not graded strongly tilting module. \hspace{1cm} $\square$

Applying the Proposition 2.5 and the exactness of the functor $-\sharp G$, we have

**Proposition 3.6** If $M$ is a graded strongly tilting $A$-module, then $M^gG$ is an $A^gG$-tilting module.

### 4. $A_e$-tilting modules

In this section, we discuss the relationships among tilting objects (partial tilting objects) of categories $A\text{-gr}$, $A\text{-Mod}$ and $A_e\text{-Mod}$.

From [9], we know that for $\forall g \in G$ the translational functor $T_g : A\text{-gr} \to A\text{-gr}$ is an equivalent functor and if the ring $A$ is a strongly graded ring of type $G$, then $(\cdot)_e : A\text{-gr} = A_e\text{-Mod} : A \otimes A_e$-are mutually equivalent functors. We have

**Proposition 4.1** Let $A$ be a strongly graded ring of type $G$ and $M = \oplus_{g \in G} M_g$ be a graded $A$-module. Then the followings are equivalent:

1. $M$ is a graded tilting $A$-module;
2. $M_e$ is a tilting $A_e$-module;
3. There exists $g \in G$ such that $M_g$ is a tilting $A_e$-module;
4. $M_g$ is a tilting $A_e$-module for $\forall g \in G$.

**Theorem 4.2** Let $A$ be a strongly graded ring of type $G$. If the partial tilting graded $A$-module $M$ in $A\text{-gr}$ has an indecomposable decomposition $M = \oplus_{i=1}^n N_i^{k_i}$, where $n = \text{rank} K_0(A_e)$, $k_i \in \mathbb{Z}^+$ and $N_i$ are mutually non-isomorphic, then $M_g$ is a tilting $A_e$-module for $\forall g \in G$ ($M_g$ is $g$-component of $M$).

**Proof** Since $A$ is a strongly graded ring of type $G$, then $(\cdot)_e : A\text{-gr} = A_e\text{-Mod} : A \otimes A_e$-are mutually equivalent functors. Thus $\text{pd} M_g \leq 1$. Observe that $\text{Ext}_A(M_g, M_g) \cong \text{Ext}_A((M(g))_e, (M(g))_e) \cong \text{Ext}_{A\text{-gr}}(M(g), M(g)) = (\text{EXT}_A(M(g), M(g)))_e = 0$.

Next, we claim that $N_i$ is a strongly graded module. Since $A$ is a strongly graded ring
of type $G$, we have $(N_i)_g \neq 0$. In addition, applying the functor $(T_g(-))_e$ to $M = \oplus_{i=1}^{n} N_i^k$, yields $(M(g))_e = \oplus_{i=1}^{n} (N_i(g))^k_e$, that is, $M_g = \oplus_{i=1}^{n} (N_i)_g^k$. Since $N_i$ is indecomposable in $A$-gr, $(N_i)_g$ is also indecomposable in $A$-gr. (If not, then there is a decomposition of $A$-modules $(N_i)_g = B \oplus D$, which leads to that $N_i(g) \cong A \otimes_{A_n} (N_i)_g = A \otimes_{A_n} B \oplus A \otimes_{A_n} D$, that is, $N_i$ is decomposable in $A$-gr. It yields the contradiction). Then since $N_i \neq N_j (i \neq j)$, it is easy to prove that $(N_i)_g \not\cong (N_j)_g$. So the number of mutually non-isomorphic indecomposable summands of $M_g$ equals to rank$K_0(A_e)$.

Remark 4.3 The condition that $A$ is a strongly graded ring of type $G$ in Theorem 4.2 cannot be lost but is not necessary.

Example 4.4 For example 2.2, $A$ is not a strongly graded ring of type $G$ and $M \triangleq \tau^{-1} A e_3 = V_{i=1}^{n} = k\{\beta^*\}$ is a graded $A$-module with

$$(\tau^{-1} A e_3)_T = \begin{cases} k\{\beta^*\}, & \overline{t} = \overline{T} \\
0, & \overline{t} \neq \overline{T} \end{cases}.$$  

Then it is easy to show that $T = A e_1 \oplus A e_2 \oplus M$ is a graded $A$-module and tilting module and $T_e = k\{e_1\} \oplus k\{e_2\} \oplus 0$. But the number of mutually non-isomorphic indecomposable summands of $T_e$ does not equal to rank$K_0(A_e) = 3$. Hence it is not a tilting $A_e$-module.

Example 4.5 Consider the path algebra $A = k\overrightarrow{Q_A}$, where $\overrightarrow{Q_A} = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. It is seen from [8] that if we consider the case when $a(\gamma) = b(\gamma) = 1$ for $\forall \gamma \in Q_1$, then the $(a, b)$-preprojective algebra of the quiver $\overrightarrow{Q_A}$ is turned out to be $A' = (k\overrightarrow{Q')}/I$, where $\overrightarrow{Q'}$ is given by:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and the two-sided ideal $I = \langle \{\alpha^* \alpha, \beta^* \beta - \alpha \alpha^*, -\beta \beta^*\} \rangle$. Then the preprojective algebra is turned into a graded $k$-algebra of type $Z_4$ by the degrees of pathes. Obviously, it is not strongly graded.

We can prove that $A' e_1 = k\{e_1, \alpha, \beta \alpha\}$, $A' e_2 = k\{e_2, \beta, \beta^* \beta, \alpha^*\}$ and $A' e_3 = k\{e_3, \beta^*, \alpha^* \beta^*\}$ are graded $A'$-modules respectively with their graded structures as follows:

$$(A' e_1)_T = \begin{cases} k\{e_1, \alpha, \beta \alpha\}, & \overline{t} = \overline{0} \\
0, & \overline{t} \neq \overline{0} \end{cases}, \quad (A' e_2)_T = \begin{cases} k\{e_2, \beta\}, & \overline{t} = \overline{0} \\
k\{\beta^* \beta, \alpha^*\}, & \overline{t} = \overline{T} \\
0, & \overline{t} \neq \overline{T} \end{cases},$$

$$(A' e_3)_T = \begin{cases} k\{e_3\}, & \overline{t} = \overline{0} \\
k\{\beta^*\}, & \overline{t} = \overline{T} \\
k\{\alpha^* \beta^*\}, & \overline{t} = \overline{\overline{T}} \end{cases}.$$  

It is clear that $A' = (k\overrightarrow{Q'_A})/I = A' e_1 \oplus A' e_2 \oplus A' e_3$ is a tilting $A'$-module and $(A')_T = (A' e_1)_T \oplus (A' e_2)_T \oplus (A' e_3)_T = k\{e_1, \alpha, \beta \alpha\} \oplus k\{e_2, \beta\} \oplus k\{e_3\} = A e_1 \oplus A e_2 \oplus A e_3$ is also a tilting $A'_T = A$-module.
References