The Crossing Number of the Cartesian Products of $W_m$ with $P_n$

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Abstract Most results on crossing numbers of graphs focus on some special graphs, such as the Cartesian products of small graphs with path, star and cycle. In this paper, we obtain the crossing number formula of Cartesian products of wheel $W_m$ with path $P_n$ for arbitrary $m \geq 3$ and $n \geq 1$.

Keywords drawing; crossing number; wheel; path; Cartesian product.

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1. Introduction

Let $G$ be a simple graph with vertex set $V$ and edge set $E$. The crossing number $\text{cr}(G)$ of a graph $G$ is the minimum number of pairs of intersected edges in a drawing of $G$ in the plane. It is well known that the crossing number of a graph is attained only in good drawings of the graph, which are those drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point except their common vertex. Let $D$ be a good drawing of the graph $G$, denote by $\text{cr}_D(G)$ the number of crossings in $D$. If $D$ is a good drawing of $G$ satisfying $\text{cr}_D(G) = \text{cr}(G)$, then $D$ is called an optimal drawing of $G$.

The suspension of a graph $G_1$ from a graph $G_2$ is obtained by adjoining every vertex of $G_1$ to every vertex of $G_2$, and is denoted by $G_1 + G_2^{[1]}$. The wheel $W_m$ is obtained by the suspension of $K_1$ from cycle $C_m$ of length $m$.

The Cartesian product $G_1 \times G_2$ of graphs $G_1$ and $G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \times G_2) = \{(u_i, v_j), (u_h, v_k) : u_i = u_h \text{ and } \{v_j, v_k\} \in E(G_2) \text{ or } v_j = v_k \text{ and } \{u_i, u_h\} \in E(G_1)\}$$

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Let $G_1$ be a graph homeomorphic to $G_2$, (for the definition\cite{2}), it is readily seen that $\text{cr}(G_1) = \text{cr}(G_2)$. And if $G_1$ is a subgraph of $G_2$, then $\text{cr}(G_1) \leq \text{cr}(G_2)$.

The crossing number of a graph is a tantalizingly open problem. Let $P_n$ be the path of length $n$, $S_n$ be the complete bipartite graph $K_{1,n}$. So far, most researches on crossing number focused on the estimation of the crossing number of special graphs, such as the Cartesian products of small graphs with path $P_n$, star $S_n$, and cycle $C_n$\cite{3-8}. Klešč determined that the crossing number of $W_3 \times P_n$ is $2n$ in [3], and that the crossing number of $W_4 \times P_n$ is $3n - 1$ in [7]. Yu and Huang\cite{9} gave the upper bounds of the crossing number of $W_m \times P_n$ for arbitrary $m$ and $n$, and proved that $\text{cr}(W_m \times P_n) = (n - 1)(\lceil \frac{m - 1}{2} \rceil + (n + 1)$ for arbitrary $m$ when $n = 1, 2, 3$. Recently, Bokal\cite{10} determined the crossing number of the Cartesian products of $S_m$ with $P_n$ for an arbitrary star $S_m$ and path $P_n$ using a newly introduced operation. Stimulated by these results, we consider the crossing number of the Cartesian products of wheel $W_m$ with path $P_n$ for arbitrary $m \geq 3$ and $n \geq 1$, and get the main result of this paper:

\textbf{Theorem 1} For $m \geq 3$ and $n \geq 1$, we have $\text{cr}(W_m \times P_n) = (n - 1)(\lceil \frac{m}{2} \rceil \lceil \frac{m - 1}{2} \rceil + 1) + 2$.

\textbf{2. Some lemmas}  

In [10], Bokal defined that $\hat{G}$ is the graph obtained from $G$ by adding two vertices $v_1$ and $v_2$ and the edges $v_iv$ for $i = 1, 2$ and each $v \in V(G)$. If a vertex $v \in V(G)$ is adjacent to all other vertices in $G$, then $v$ is called a dominating vertex. Let $0, 1, \ldots, n$ be the vertices of the path $P_n$ with 0 and $n$ be its origin and terminus, respectively. With $G \times P_n$ we denote the capped Cartesian products of $G$ and $P_n$, i.e. the graph, obtained from $G \times P_n$ by adding two new vertices $v_0$ and $v_n$ and connecting $v_0$ with all the vertices of $G \times \{0\}$ and $v_n$ with all the vertices of $G \times \{n\}$. Bokal gave a relationship between the crossing numbers of $G \times P_n$ and $\hat{G}$:

\textbf{Lemma 2}\cite{10} \hspace{1cm} Let $G$ be a graph with a dominating vertex. Then for $n \geq 0$, $\text{cr}(G \times P_n) = (n + 1)\text{cr}(\hat{G})$. \hfill \Box

$W_m \times P_n$ contains a subgraph homeomorphic to $W_m \hat{\times} P_{n-2}$. To obtain the crossing number of $W_m \hat{\times} P_{n-2}$ by Lemma 2, we need to consider the crossing number of $W_m$, which is isomorphic to $P_2 + C_m$, see Figure 1.

![Figure 1 A good drawing of $P_2 + C_m$](image)

\textbf{Lemma 3} For $m \geq 3$, we have $\text{cr}(P_2 + C_m) = \lceil \frac{m}{2} \rceil \lceil \frac{m - 1}{2} \rceil + 1$. 

Proof Figure 1 shows that $\text{cr}(P_2 + C_m) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 1$. Now we prove the reverse inequality by assuming that there exists a good drawing $D$ of $P_2 + C_m$ with $\text{cr}_D(P_2 + C_m) < \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 1$. Since $P_2 + C_m$ contains a subgraph isomorphic to $K_{3,m}$ and $\text{cr}_D(P_2 + C_m) \geq \text{cr}(K_{3,m}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$, so $\text{cr}_D(P_2 + C_m) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$.

We partition the edges of $P_2 + C_m$ into three parts: these edges in $P_2$, these edges in $C_m$, and these edges in $K_{3,m}$. Obviously, there isn’t any crossing on the edges of $P_2$ or $C_m$, or else deleting the crossed edges of $P_2$ or $C_m$ from $D$, we get a graph containing a subgraph isomorphic to $K_{3,m}$ and a drawing with crossing number less than $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$, a contradiction! Let $x_1, x_2, x_3$ be the vertices of $P_2$, and let $v_1, v_2, \ldots, v_m$ be the vertices of $C_m$, see Figure 1. Then $x_1, x_2, x_3$ must lie in the same region of $C_m$, without loss of generality, we may assume that they lie in the finite region. The edges $x_1v_i$ ($1 \leq i \leq m$) divide the finite region into $m$ subregions, in one of which $x_2$ lies. Thus the $m$ edges $x_2v_i$ ($1 \leq i \leq m$) must cross those edges adjacent to $x_1$ at least $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$ times. Similarly, the edges adjacent to $x_3$ also contribute at least $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$ crossings. None of these crossings are counted more than once, so there are at least $2 \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$ crossings in $D$, a contradiction to our assumption.

According to Lemmas 2 and 3, it is easy to get that $\text{cr}(W_m \times \hat{P}_{n-2}) = (n-1)\text{cr}(\hat{W}_m) = (n-1)(\left\lfloor \frac{m}{2} \right\rfloor |m-1|) + 1$.

![Figure 2 A good drawing of $W_m \times P_n$](image)

3. The proof of Theorem 1
We always assume that $m \geq 4$ and $n \geq 4$ in the proof since the results in [3,7,9] are consistent with Theorem 1. It will be convenient to consider the graph $W_m \times P_n$ in the following way: it has $(m+1)(n+1)$ vertices and edges in the $n+1$ copies $W_m \times \{i\}, i = 0, 1, \ldots, n$, and in the $m+1$ paths of length $n$ (see Figure 2). Furthermore, we color the former edges red and the latter ones blue.

**Proof** Figure 2 shows that $cr(W_m \times P_n) \leq (n-1)(\left\lceil \frac{m}{2} \right\rceil + \frac{m-1}{2}) + 1$. Now we move to the proof of the reverse inequality.

Let $D$ be an optimal drawing of $W_m \times P_n$. We denote the $m$–cycles of $W_m \times \{0\}$ and $W_m \times \{n\}$ by $C^0_m$ and $C^m_m$, respectively. The following two cases are discussed:

**Case 1** Suppose that there is not any crossing on the edges of $C^0_m$ or $C^m_m$ in $D$. We may assume, without loss of generality, that there is not any crossing on the edges of $C^0_m$. We denote the dominating vertex of $W_m \times \{0\}$ by $x_0$, and the vertices of $C^0_m$ by $u_1, u_2, \ldots, u_m$.

Since there is not any crossing on the edges of $C^0_m$, $x_0$ and all of the other vertices of $W_m \times \{1\}$, $W_m \times \{2\}, \ldots, W_m \times \{n\}$ must lie in the same region of $C^0_m$. Without loss of generality, we may assume that they lie in the interior region of $C^0_m$. Let $|A_i|$ denote the number of crossings on the edge $x_0u_i$ ($i = 1, 2, \ldots, m$). Without loss of generality, we assume that $|A_i| = \min_{1 \leq i \leq m} \{A_i\}$.

Now we move to obtain a good drawing $D'$ of a new graph which is homeomorphic to $W_m \times P_{n-2}$: firstly, deleting the edges $x_0u_i$ ($i = 2, 3, \ldots, m$) from $D$; secondly, drawing curves connecting $u_1u_i$ along the exterior region of $C^0_m$ ($i = 3, 4, \ldots, m-1$) such that there is not any crossing on the curves $u_1u_i$ ($i = 3, 4, \ldots, m-1$); thirdly, deleting $u_iu_{i+1}$ ($i = 2, 3, \ldots, m-1$) from $D$ (see Figure 3); and lastly, deleting the edges of $C^m_m$ from $W_m \times \{n\}$. We distinguish two subcases:

**Subcase 1.1** Suppose that $|A_1| = 0$. Let $x_1$ denote the dominating vertex of $W_m \times \{1\}$. If all of the vertices of $W_m \times \{1\}$ except $x_1$ lie in the same region, without loss of generality, we may assume that they lie in the region $x_0u_2u_3$, then there will be at least $m - 2$ crossings on the edges of $x_0u_2$ and $x_0u_3$ made by the blue edges adjacent to $u_i$ ($i \neq 2, 3$). If all of the vertices of $W_m \times \{1\}$ except $x_1$ lie in different regions of $C^0_m$, then there will be at least 2 crossings on the edges $x_0u_i$ ($i \neq 1$) made by the $m$–cycle of $W_m \times \{1\}$. Thus the first step decreases at least 2 crossings.
Subcase 1.2 Suppose that $|A_1| \neq 0$, that is $|A_1| \geq 1$. It is easy to see that the first step decreases at least $(m - 1)|A_1|$ crossings.

Therefore, the first step decreases at least 2 crossings no matter $|A_1| = 0$ or not. The latter three steps don’t increase the number of crossings, so we have

$$\text{cr}_D(W_m \times P_n) \geq \text{cr}'_D(W_m \hat{\times} P_{n-2}) + 2 \geq (n - 1)(\left\lceil \frac{m}{2} \right\rceil |\frac{m - 1}{2}| + 1) + 2.$$

Case 2 Suppose that both $C^0_m$ and $C^n_m$ are crossed in $D$. If the crossings are made between the edges of $C^0_m$ and $C^n_m$, then they must cross at least twice; if not, the edges of $C^0_m$ and $C^n_m$ are crossed at least once respectively. Deleting the edges of $C^0_m$ and $C^n_m$ from $D$, we obtain a graph homeomorphic to $W_m \hat{\times} P_{n-2}$ with at least $(n - 1)(\left\lceil \frac{m}{2} \right\rceil |\frac{m - 1}{2}| + 1)$ crossings. So we have

$$\text{cr}_D(W_m \times P_n) \geq (n - 1)(\left\lceil \frac{m}{2} \right\rceil |\frac{m - 1}{2}| + 1) + 2.$$

In all, for an optimal drawing $D$ of $W_m \times P_n$, we have $\text{cr}_D(W_m \times P_n) \geq (n - 1)(\left\lceil \frac{m}{2} \right\rceil |\frac{m - 1}{2}| + 1) + 2$, which completes the proof of Theorem 1.

References