Improved Local Wellposedness of Cauchy Problem for Generalized KdV-BO Equation

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**Abstract**

In this paper we prove that the Cauchy problem associated with the generalized KdV-BO equation

\[ u_t + u_{xxx} + \lambda \mathcal{H}(u_{xx}) + u^k u_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \]

for \( \frac{1}{3} < r \leq 2, \ b > \frac{1}{r} \) and \( s \geq s(r) = \frac{1}{2} - \frac{1}{2r} \). In particular, for \( r = 2 \), we reobtain the result in [3].

**Keywords**

KdV-BO equation; Cauchy problem; local wellposedness.

1. Introduction

We investigate the initial value problem (IVP) of the generalized KdV-BO (Korteweg-de Vries-Benjamin-Ono) equation:

\[ u_t + u_{xxx} + \lambda \mathcal{H}(u_{xx}) + u^k u_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \]

\[ u(0, x) = u_0(x), \quad x \in \mathbb{R}, \]  

where \( \lambda > 0 \) and \( \mathcal{H} \) denotes the usual Hilbert transform. The integro-differential equation (1) models the unidirectional propagation of long waves in a two-fluid system, where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. It was derived by Benjamin\(^1\) to study gravity-capillary surface waves of solitary type on deep water.

Linares\(^2\) showed that there exists a local and global solution for the IVP (1)–(2) with initial data in \( L^2 \) with constant coefficient \( \lambda > 0 \) if \( k = 1 \).

Guo and Huo\(^4\) proved that IVP (1)–(2) is locally wellposed with data in \( H^s(\mathbb{R}) \) for \( s > -\frac{1}{8} \), if \( k = 1; \ s \geq \frac{1}{4}, \) if \( k = 2 \).

Notice that in equation (1), the dispersive term \( u_{xxx} \) plays the most important role and the Benjamin-Ono \( \mathcal{H}(u_{xx}) \) term can be treated as a nonlinear term. Therefore, we expect the IVP (1)–(2) has the similar wellposedness to that of the modified KdV equation:

\[ v_t + v_{xxx} + v^k v_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \]

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\[v(0, x) = v_0(x), \quad x \in R.\] (4)

Indeed, from this point of view, Huo and Guo\(^3\) proved that IVP (1)–(2) is locally wellposed with data in \(H^s(R)\) for \(s > -\frac{2}{3}\), if \(k = 1\); \(s \geq \frac{1}{4}\), if \(k = 2\); \(s > -\frac{1}{2}\), if \(k = 3\). Moreover, the solutions of IVP (1)–(2) converge to the solutions of IVP (3)–(4) if \(\lambda\) tends to zero.

Though for \(k = 1\), Guo and Huo in [4] improved the well-posedness of IVP (1)–(2) a great deal, it did not improve the wellposedness for the case of \(k = 2\). As Grünrock did in [5], by proving the wellposedness of IVP (1)–(2) in \(\hat{H}_r^3\), we manage to improve the result of the case \(k = 2\) to some extent. To this end, let us recall some definitions first.

**Definition 1**\(^5\) For \(s \in R, 1 \leq r \leq \infty\), we define the space \(\hat{H}_r^s\) by

\[\hat{H}_r^s = \{f \in S'(R) : \|f\|_{\hat{H}_r^s} < \infty\}\]

where \(\langle \xi \rangle = (1 + |\xi|)^\frac{1}{r}\) and \(\frac{1}{r} + \frac{1}{r'} = 1\).

**Definition 2**\(^5\) For \(s, b \in R, 1 \leq r \leq \infty\), we define the space \(X_{s,r}^\gamma\) to be the completion of the Schwartz function space on \(R^2\) with respect to the norm

\[\|u\|_{X_{s,r}^\gamma} = \|\langle \xi \rangle^r (\tau - \xi)^{\gamma} \hat{u}\|_{L_{\xi}^{r'}}\]

where \(\langle \xi \rangle = (1 + |\xi|)^\frac{1}{r}\), \(\frac{1}{r'} + \frac{1}{r} = 1\). \(\mathcal{F}u = \hat{u}(\tau, \xi)\) denotes the Fourier transform in variables \(t\) and \(x\) of \(u\). Denote by \(\mathcal{F}(\cdot)\) the Fourier transform in the \(\cdot\) variable. \(U(t) = \mathcal{F}_x^{-1}e^{it\xi^3}\) \(\mathcal{F}_x\) is the unitary operator associated with the Airy equation.

Deduced directly from the definition, the embedding relation

\[\|u\|_{X_{s_1, b_1}^r} \leq \|u\|_{X_{s_2, b_2}^r}\]

holds if \(s_1 \leq s_2, b_1 \leq b_2\); for \(r = 2\), we reobtain Bourgain space \(X_{s,b}^\gamma\).

For \(b > \frac{1}{r}\), we have

\[X_{s,b}^r \subset C(R, \hat{H}_r^s)\]

Let \(\psi \in C_0^\infty(R)\) with \(\psi \equiv 1\) on \([-\frac{1}{2}, \frac{1}{2}]\) and \(\text{supp}\psi \subseteq (-1, 1)\). Denote \(\psi_\delta(t) = \psi(\frac{t}{\delta})\). Then the time restricted space is defined by

\[X_{s,b}^r(\delta) := \{f = \hat{f}|_{[-\delta, \delta] \times R} = \psi_\delta(t)\hat{f} : \hat{f} \in X_{s,b}^r\},\]

endowed with the norm

\[\|f\|_{X_{s,b}^r(\delta)} := \inf\\{\|\hat{f}\|_{X_{s,b}^r} : \hat{f}|_{[-\delta, \delta] \times R} = f\}\]

Our main result reads as follows (for \(k = 2\)):

**Theorem** Let \(\frac{1}{3} < r \leq 2\). Then for any \(b > \frac{1}{r}\) and \(s \geq s(r) = \frac{1}{2} - \frac{1}{2r}\), there exist \(T = \)

\[\|u\|_{X_{s,b}^r} = \|\langle \xi \rangle^r (\tau - \xi)^3 \hat{u}(\tau, \xi)\|_{L_{\xi}^{r'}}\].

\(^1\)We call \(X_{s,b}^\gamma\) Bourgain space because it was first introduced by Bourgain in [7]. The norm of \(X_{s,b}^\gamma\) is given by
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T(\|u_0\|_{H^r_x}) > 0 and a unique solution \( u \in C([0,T],\dot{H}^r_x) \cap X^{r,b}_{s,t} \) of IVP (1)–(2) with \( u_0(x) \in \dot{H}^r_x \). Moreover, for any given \( t \in (0,T) \), the mapping \( u_0(x) \to u(t,x) \) is Lipschitz continuous from \( \dot{H}^r_x \) to \( C([0,T],\dot{H}^r_x) \).

In the sequel, \( C \) will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters.

2. Estimate of the Benjamin-Ono term

In this section, we concentrate on the estimate of Benjamin-Ono term \( \mathcal{H}(u_{xx}) \).

**Lemma 1** Let \( s \in R, b' + 1 > b > 0 > b' > -\frac{1}{r}, \) and \( b - b' > \frac{2}{r}, 1 < r < \infty. \) We have

\[
\| \mathcal{H}(u_{xx}) \|_{X^{r,b}_{s,t}} \leq C \| u \|_{X^{r,b}_{s,t}}.
\] (5)

**Proof** We have

\[
\| \mathcal{P}f \|_{X^{r,b}_{s,t}} = \left( \int d\xi d\tau <\xi>^{r'} <\tau>^{b'} |\mathcal{F}[U(\cdots)\mathcal{P}f](\tau,\xi)|^{r'} \right)^{\frac{1}{r'}}
\]

\[
= \left( \int d\xi d\tau \frac{\langle \xi \rangle^{2r'}}{\langle \tau \rangle^{b-b'}} \langle \xi \rangle^{s-2r'} <\tau>^{b'} |\mathcal{F}[U(\cdots)\mathcal{P}f](\tau,\xi)|^{r'} \right)^{\frac{1}{r'}}
\]

\[
\leq \left( \int d\xi d\tau \frac{\langle \xi \rangle^{2r'} \chi_{|\tau|<|\xi|^\beta}}{\langle \tau \rangle^{b-b'}} \langle \xi \rangle^{s-2r'} <\tau>^{b'} |\mathcal{F}[U(\cdots)\mathcal{P}f](\tau,\xi)|^{r'} \right)^{\frac{1}{r'}}
\]

\[
\leq \left( \int d\xi d\tau \langle \xi \rangle^{(s-2)'} <\tau>^{b'} |\mathcal{F}[U(\cdots)\mathcal{P}f](\tau,\xi)|^{r'} \right)^{\frac{1}{r'}}
\]

\[
= C \| f \|_{X^{r-2,b}_{s,t}}.
\]

This completes the proof. \( \square \)

**Proof of Lemma 1**

**Case 1** If the Fourier transform \( \mathcal{F}\mathcal{H}(\partial_{xx}u) \) is supported in \( \{ (\tau,\xi) : |\xi| \leq 1 \} \), we obtain

\[
\| \mathcal{H}(u_{xx}) \|_{X^{r,b}_{s,t}} = \left\| \frac{\langle \xi \rangle^s}{\langle \tau - \xi^3 \rangle^{b-b'}} \mathcal{F}\mathcal{H}(u_{xx})(\tau,\xi) \right\|_{L^r_{t,\xi}}
\]

\[
= \left\| \frac{\xi}{\langle \tau - \xi^3 \rangle^{b-b'}} \langle \xi \rangle^s (\tau - \xi^3)^b \mathcal{H}(u_{xx})(\tau,\xi) \right\|_{L^r_{t,\xi}}
\]

\[
\leq C \| u \|_{X^{r,b}_{s,t}}.
\]

**Case 2** If the Fourier transform \( \mathcal{F}\mathcal{H}(u_{xx}) \) is supported in \( \{ (\tau,\xi) : |\xi| \geq 1 \} \), we rewrite \( \mathcal{H}(u_{xx}) \) as \( \mathcal{H}(u_{xx}) = \mathcal{P}\mathcal{H}(u_{xx}) + (1 - \mathcal{P})\mathcal{H}(u_{xx}) \) and estimate the two terms respectively.
For $\mathcal{P}\mathcal{H}(u_{xx})$, by Lemma 2, we have
\[
\|\mathcal{P}\mathcal{H}(u_{xx})\|_{X_{s,b}^{r'}} \leq C\|\mathcal{H}(u_{xx})\|_{X_{s-2,b}^r} \leq C\|u\|_{X_{s,b}^r}.
\]

For $(1 - \mathcal{P})\mathcal{H}(u_{xx})$, it is clear that $\mathcal{F}(1 - \mathcal{P})\mathcal{H}(u_{xx})$ is supported in
\[
\{(\tau, \xi) : |\tau| \ll |\xi|^3 \text{ or } |\tau| \gg |\xi|^3\},
\]
correspondingly, we have either $|\tau - \xi|^3 \sim |\xi|^3$ or $|\tau - \xi|^3 \gg |\xi|^3$. Then we have
\[
\|(1 - \mathcal{P})\mathcal{H}(u_{xx})\|_{X_{s,b}^{r'}}
\]
\[
= \left\| \frac{\xi|\xi|^2}{(\tau - |\xi|^3)^b} \left(\xi^3(\tau - |\xi|^3)\chi_{|\tau|\sim |\xi|^3} \text{ or } |\tau|, |\xi| \gg |\xi|^3\right) \hat{u}(\tau, \xi) \right\|_{L_{r'}^r}
\]
\[
\leq C\|u\|_{X_{s,b}^r}.
\]
Thus, collecting all the estimates together, we obtain
\[
\|\mathcal{H}(u_{xx})\|_{X_{s,b}^{r'}} \leq \|\mathcal{P}\mathcal{H}(u_{xx})\|_{X_{s,b}^{r'}} + \|(1 - \mathcal{P})\mathcal{H}(u_{xx})\|_{X_{s,b}^{r'}} \leq C\|u\|_{X_{s,b}^r}.
\]

3. Proof of the main results

In order to smooth the proof of the main result, let us recall some facts as lemmas first.

**Lemma 3** [5] 1) For $\phi \in \hat{H}_s^r$, we have
\[
\|\psi(t)S(t)\phi\|_{X_{s,b}^r} \leq C\psi\|\hat{\phi}\|_{\hat{H}_s^r}.
\]

2) Assume $1 < r < \infty$ and $b' + 1 \geq b \geq 0 \geq b' > -\frac{1}{r}$. Then
\[
\left\| \psi(t) \int_0^t S(t - t')f(t')dt' \right\|_{X_{s,b}^r} \leq C\delta^{1+b'-b}\|f\|_{X_{s,b}^{r'}}.
\]

**Lemma 4** [5] Let $\frac{4}{3} < r \leq 2$ and $s \geq s(r) = \frac{1}{2} - \frac{1}{2r}$. Then for all $b' < \frac{1}{2r} - \frac{5}{8}$ and $b > \frac{1}{r}$ the estimate
\[
\|(\Pi_{i=1}^3 u_i)\|_{X_{s,b}^r} \leq C\Pi_{i=1}^3 \|u_i\|_{X_{s,b}^r}
\]
holds true.

**Lemma 5** Let $\frac{4}{3} < r \leq 2$ and $s \geq s(r) = \frac{1}{2} - \frac{1}{2r}$. Then for all $b' < \frac{1}{2r} - \frac{5}{8}$ and $b > \frac{1}{r}$ the estimates
\[
\|u^2u_x\|_{X_{s,b}^{r'}} \leq C\|u\|_{X_{s,b}^r}^3
\]
and
\[
\|u^2u_x - v^2v_x\|_{X_{s,b}^{r'}} \leq C\|u\|_{X_{s,b}^r}^2 + \|v\|_{X_{s,b}^r}^2\|u - v\|_{X_{s,b}^r}
\]
hold true.

**Proof** Inequality (6) is a direct result of Lemma 4. Inequality (7) follows from Lemma 4 and
\[
3(u^2u_x - v^2v_x) = (u^3 - v^3)_x = [(u - v)(u^2 + uv + v^2)]_x
\]
\[
= [(u - v)u^2]_x + [(u - v)uv]_x + [(u - v)v^2]_x.
\]
Sketch of proof of the Theorem Take $b = \frac{1}{r} + \varepsilon$, $b' = -\frac{1}{r} + 2\varepsilon$, for $0 < \varepsilon < \frac{1}{3r}$. Then $b + 1 > b > 0 > b' > -\frac{1}{r}$, and $b - b' = \frac{2}{3}, b > \frac{1}{r}$.

For $u \in X_{r,b}^s(\delta)$ with extension $\tilde{u} \in X_{r,b}^s$, define operator

$$T(u) = \psi(t)U(t)u_0 - i\psi_\delta(t) \int_0^t U(t - \tau)F(u(\tau))d\tau,$$

then extension of $T(u)$ is given by

$$\tilde{T}(u) = \psi(t)U(t)u_0 - i\psi_\delta(t) \int_0^t U(t - \tau)F(\tilde{u}(\tau))d\tau,$$

where $F(\tilde{u}(\tau)) = (\tilde{u}^3)_x + \lambda H(\tilde{u}_{xx})$.

By Lemma 3 and the first inequality of Lemma 5, we obtain

$$\|T(u)\|_{X_{r,b}^s(\delta)} \leq \|\psi(t)U(t)u_0\|_{X_{r,b}^s} + \|\psi_\delta(t)\| \int_0^t \|U(t - \tau)F(\tilde{u}(\tau))\|d\tau \|X_{r,b}^s\|

\leq C\|u_0\|_{\tilde{H}_r^s} + C\delta^{1-b+b'}\|\tilde{u}\|_{X_{r,b}^s} + C\delta^{1-b+b'}\|\tilde{u}\|_{X_{r,b}^s}.

This holds for all extension $\tilde{u} \in X_{r,b}^s$ of $u \in X_{r,b}^s(\delta)$. Hence

$$\|T(u)\|_{X_{r,b}^s(\delta)} \leq C\|u_0\|_{\tilde{H}_r^s} + C\delta^{1-b+b'}\|u\|_{X_{r,b}^s(\delta)} + C\delta^{1-b+b'}\|u\|_{X_{r,b}^s(\delta)}.

(8)$$

Similarly, by Lemma 3 and the second inequality of Lemma 5, for given $u \in X_{r,b}^s(\delta)$, $v \in X_{r,b}^s(\delta)$, we obtain

$$\|T(u) - T(v)\|_{X_{r,b}^s(\delta)} \leq C\delta^{1-b+b'}\|u\|^3_{X_{r,b}^s(\delta)} + \|v\|^3_{X_{r,b}^s(\delta)} \|u - v\|_{X_{r,b}^s(\delta)} + C\delta^{1-b+b'}\|u - v\|_{X_{r,b}^s(\delta)}.

(9)$$

Inequalities (8) and (9) show that for $R = 2C\|u_0\|_{\tilde{H}_r^s}$ and $\delta^{1-b+b'} < \min\{\frac{1}{2(4r)}, \frac{1}{2}\}$, the mapping $T$ is a contract mapping of the closed ball of radius $R$ in $X_{r,b}^s(\delta)$ into itself. The Banach fixed point theorem now guarantees the existence of a solution of $T(u) = u$. Because of $b > \frac{1}{r}$, any solution $u \in X_{r,b}^s(\delta)$ also belongs to $C([0, \delta], \tilde{H}_r^s)$. Finally, the statement about continuous dependence can be shown in a straightforward manner using the same estimates as the above.

References