Approximation by Modified Summation Integral Type Operators in the $L_p$ Spaces

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Abstract The generalized summation integral type operators with Beta basis functions are widely studied. At present, the investigations for the properties of these operators are only limited to the functions of bounded variation. Some authors studied the rate of point-wise rate of convergence, asymptotic formula of Voronovskaja type, and some direct results about these type of operators. The present paper considers the direct, inverse and equivalence theorems of modified summation integral type operators in the $L_p$ spaces.

Keywords Beta operators; $K$-functional; moduli of smoothness; Riesz-Thorin interpolation theorem; Hölder inequality.

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1. Introduction

Beta operators are important and applied widely in probability theory and approximation theory$^{[1-6]}$. The generalized summation integral type operators with Beta basis functions are widely studied$^{[6-18]}$. At present, the investigations for the properties of these operators are only limited to the functions of bounded variation$^{[6,9,17]}$. Some authors studied the rate of point-wise rate of convergence, asymptotic formula of Voronovskaja type, and some direct results about these type of operators$^{[7,8,10-18]}$.

In this paper, we will study the direct, inverse and equivalent theorems of modified summation integral type operators in the $L_p$ spaces.

For $f \in L_p[0, \infty)$, a new kind of linear positive operators are defined as$^{[6,13,15]}

$$B_n(f, x) \equiv B_n(f(t), x) = \frac{1}{n + 1} \sum_{k=1}^{\infty} b_{n,k}(x) \int_0^\infty b_{n,k}(t)f(t)dt, \quad x \in [0, \infty) \quad (1.1)$$

where

$$b_{n,k}(t) = \frac{(n+k)!}{(k-1)! n!} t^{k-1}(1+t)^{-n-k-1}, \quad k \geq 1.$$ 

Let

$$B(k, n+1) = \frac{(k-1)!}{(n+k)!} n! \quad W_n(x, t) = \frac{1}{n+1} \sum_{k=1}^{\infty} b_{n,k}(x) b_{n,k}(t).$$

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We can write 

\[ b_{n,k}(t) = \frac{t^{k-1}(1+t)^{-n-k-1}}{B(k,n+1)}, \quad B_n(f,x) = \int_0^\infty W_n(x,t)f(t)dt. \]

It is easy to verify that \( B_n(1,x) = 1 \), and the operators \( B_n(f,x) \) are linear positive ones, the rate of convergence for these operators cannot be faster than \( O(n^{-1}) \). Gupta\cite{13} obtained the direct results for these operators.

**Theorem A**\cite{13, Theorem 3.1] Let \( f \in C_r[0, \infty), \) \( \{ f \in C[0, \infty), |f(t)| \leq Ct^r, r > 0 \} \), and suppose \( f^{(r)} \) exists at a point \( x \in (0, \infty) \). Then \( \lim_{n \to \infty} B_n^{(r)}(f,x) = f^{(r)}(x) \).

**Theorem B**\cite{13, Theorem 3.3] Let \( f \in C_r[0, \infty), \) and \( 0 < a < a_1 < b_1 < b < \infty \). Then for all \( n \) sufficiently large, we have,

\[ \| B_n^{(r)}(f, \cdot) - f^{(r)} \|_{C[a,b]} \leq A_1 \omega_2(f^{(r)}), \frac{1}{\sqrt{n}}(a,b) + A_2 n^{-1} \| f \|_r, \]

where

\[ A_1 = A_1(r), \quad A_2 = A_2(r,f), \quad \| f \|_r = \sup_{0 < t < \infty} |f(t)|t^{-r}, \]

\[ \omega_k(f, \delta; a,b) = \sup \{ \Delta^k_h f(x) : |h| \leq \delta, x, x + kh \in [a,b] \}, \]

\( \Delta^k_h f(x) \) is \( k \)-th forward difference with step length \( h \).

The authors also gave the inverse result for these operators.

**Theorem C**\cite{13, Theorem 4.3] Let \( 0 < \alpha < 2, \) \( 0 < a_1 < a_2 < b_2 < b_1 < \infty \). If \( f \in C_r[0, \infty), \) then in the following statements, the implication \( 1^o \Rightarrow 2^o \) holds:

\[ 1^o \| B_n^{(r)}(f, \cdot) - f^{(r)} \|_{C[a_1,b_1]} = O(n^{-\frac{\alpha}{2}}); \quad 2^o \ f^{(r)} \in \text{Lip}^*(\alpha, a_2, b_2), \]

where \( \text{Lip}^*(\alpha, a_2, b_2) = \{ f : \forall \delta > 0, 3C > 0, \text{ such that } \omega_2(f, \delta; a,b) \leq C\delta^{\alpha} \} \).

For convenience, we will introduce some of the notations. Let \( f \in L_p[0, \infty), \) \( 1 \leq p \leq \infty \). We write

\[ \| f \|_p = \begin{cases} \{ \int_0^\infty |f(x)|^pdx \}^{1/p}, & 1 \leq p < \infty, \\ \sup_{x \in [0, \infty)} |f(x)|, & p = \infty, \end{cases} \]

\[ \Delta^2_h \varphi f(x) = f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x)), \]

\[ \omega^2_\varphi(f,t)_p = \sup_{0 < h \leq t} \| \Delta^2_h \varphi f \|_p, \]

\[ D = \{ g \in L_p[0, \infty) : g' \in A.C_{loc}, \varphi'^2 g'' \in L_p \}, \]

\[ K^2_\varphi(f,t^2)_p = \inf_{g \in D} \{ \| f - g \|_p + t^2 \| \varphi'^2 g'' \|_p \}, \quad (1.2) \]

\[ \overline{D} = \{ g : g \in L_p[0, \infty), g' \in A.C_{loc}, \varphi'^2 g'' \in L_p[0, \infty) \}, \]

\[ \overline{K}^2_\varphi(f,t^2)_p = \inf_{g \in \overline{D}} \{ \| f - g \|_p + t^2 \| \varphi'^2 g'' \|_p + t^4 \| g'' \|_p \}. \]

It is well known that\cite{5}: there exist constants \( C_1, C_2 > 0, \) such that

\[ C_1^{-1} \omega^2_\varphi(f,t)_p \leq K^2_\varphi(f,t^2)_p \leq C_1 \omega^2_\varphi(f,t)_p, \]
\[ C_2^{-1}\omega^2_n(f, t) \leq \mathbf{B}_n^p(f, t) \leq C_2\omega^2_n(f, t). \]  

(1.3)

Since the first order moment of the operator \( B_n(f, x)(1.1) \) is not equal to zero, and \( \varphi^2(x) = x(1 + x) \), we need to deal with the problem by a special method. Our main results can be stated as follows:

**Theorem 1.1** (Direct result) For \( f \in L_p[0, \infty) \) \((1 \leq p \leq \infty)\), \( \varphi^2(x) = x(1 + x) \), one has

\[
\|B_n(f, x) - f(x)\|_p \leq C\left( \omega^2_n(f, \frac{1}{\sqrt{n}}) + \omega_1(f, \frac{1}{n}) + \|f\|_p \right),
\]

where

\[
\omega_1(f, \frac{1}{n}) = \sup_{0 < t \leq \frac{1}{n}} \|\Delta^1 f(x)\|_p, \quad \Delta^1 f(x) = f(x + \frac{t}{2}) - f(x - \frac{t}{2}).
\]

**Theorem 1.2** (The Bernstein-Type inequality) For \( f \in L_p[0, \infty) \) \((1 \leq p \leq \infty)\), one has

\[
\|\varphi^2 B_n f\|_p \leq C n\|f\|_p.
\]

**Theorem 1.3** (Inverse result) For \( f \in L_p[0, \infty) \) \((1 \leq p \leq \infty)\), \( 0 \leq \alpha < 2 \), one has

\[
\|B_n(f, x) - f(x)\|_p = O(n^{-\frac{\alpha}{2}}) \Rightarrow 1^\circ. \quad \omega^2_n(f, t) = O(t^\alpha); \quad 2^\circ. \quad \omega_1(f, t) = O(t^{\frac{\alpha}{2}}).
\]

Combining the direct result Theorem 1.1 and the converse result Theorem 1.3, we can obtain the equivalence theorem.

**Theorem 1.4** (Equivalence result) For \( f \in L_p[0, \infty) \) \((1 \leq p \leq \infty)\), \( 0 \leq \alpha < 2 \), the following two statements are equivalent:

1. \( \|B_n(f, x) - f(x)\|_p = O(n^{-\frac{\alpha}{2}}) \);  
2. \( \omega^2_n(f, t) = O(t^\alpha); \quad 2^\circ. \quad \omega_1(f, t) = O(t^{\frac{\alpha}{2}}) \).

**Remark 1.1** From Theorem 1.1, one can get Theorems A and B in the case \( r = 1, p = \infty \).

**Remark 1.2** From Theorem 1.3, one can get Theorem C in the case \( r = 1, p = \infty \).

**Remark 1.3** Throughout this paper, \( C \) denotes a constant independent of \( n \) and \( x \) which is not necessarily the same in different cases.

### 2. The basic properties of the mixed type Beta operators

By simple calculations, one can get the boundedness and the estimates for the moments of the operators.

**Lemma 2.1** For \( f \in L_p[0, \infty), 1 \leq p \leq \infty, \) we have \( \|B_n(f, \cdot)\|_p \leq \|f\|_p \).

**Lemma 2.2** Let \( \delta^2_n(x) = \varphi^2(x) + \frac{1}{n}, \varphi^2(x) = x(1 + x) \). We have

\[
B_n(1, x) = 1,
\]

\[
B_n(t - x, x) = \frac{2x + 1}{n},
\]

(2.1) and (2.2)
\[ B_n((t-x)^2, x) \leq \frac{C}{n} \delta^2_n(x), \quad (2.3) \]
\[ B_n((t-x)^4, x) \leq \frac{C \delta^4_n(x)}{n^2}. \quad (2.4) \]

**Lemma 2.3** For the operator \( B_n \), we have the estimation \( B_n((1+t)^{-2}, x) \leq C(1+x)^{-2} \).

**Lemma 2.4** [6] For \( \varphi^2(x) = x(1+x) \), and \( u \) between \( t \) and \( x \), we have
\[
\frac{|t-u|}{\varphi^2(u)} \leq \frac{|t-x|}{x} \left( \frac{1}{1+x} + \frac{1}{1+t} \right).
\]

**Lemma 2.5** For \( u \) between \( t \) and \( x \), \( x \in E^\infty_n = [0, n^{-1}) \), we have
\[
\frac{|t-u|}{\varphi^2(u) + \frac{u}{n}} \leq \frac{|t-x|}{\varphi^2(x) + \frac{x}{n}}. \quad (2.5)
\]

**Proof** All we have to do is to estimate \( \frac{|t-u|}{\varphi^2(u) + \frac{u}{n}} \) in the following ways.

For \( x < u < t \), we have (2.5) which in turn follows from \( \frac{t-u}{u(1+u)} \leq \frac{t-x}{x(1+x)} \) and \( t-u \leq t-x \).

For \( t < u < x < \frac{1}{n} \), (2.5) follows from \( \frac{u}{\varphi^2(u) + \frac{u}{n}} \leq \frac{x}{\varphi^2(x) + \frac{x}{n}} \) and \( \frac{u-t}{x} \leq \frac{t-x}{x} \), and the fact that \( h(x) = \frac{x}{\varphi^2(x) + \frac{1}{n}} \) increases in \([0, \frac{1}{n})\). \( \square \)

**Lemma 2.6** For \( f' \in A.C.\text{-loc}, 1 \leq p \leq \infty \), we have \( \|\varphi^2 B''_n f\|_p \leq 6\|\varphi^2 f''\|_p \).

**Proof** Using the relation[13, Lemma 2.6]
\[
B''_n(f, x) = \frac{n+2}{n(n-1)} \sum_{k=1}^\infty b_{n+2,k}(x) \int_0^\infty b_{n-2,k+2}(t) f''(t) dt,
\]
we write
\[
|\varphi^2 B''_n(f, x)| \leq \frac{1}{n} \sum_{k=1}^\infty [b_{n-1,k+1}(x) + b_{n,k+2}(x)] \int_0^\infty [b_{n-1,k+1}(t) + b_{n,k}(t)] |(t^2 + t)f''(t)| dt.
\]

For \( p = \infty \), \( |\varphi^2(x)B''_n(f, x)| \leq 6\|\varphi^2 f''\|_\infty \). For \( p = 1 \),
\[
\|\varphi^2(x)B''_n(f, x)\|_1 \leq \frac{1}{n} \int_0^\infty \sum_{k=1}^\infty [b_{n-1,k+1}(x) + b_{n,k+2}(x)] \int_0^\infty [b_{n-1,k+1}(t) + b_{n,k}(t)] |\varphi^2(t)f''(t)| dt dx
\]
\[
\leq 6\|\varphi^2 f''\|_1.
\]

By the Riesz-Thorin interpolation theorem, we get Lemma 2.6. \( \square \)

### 3. The direct theorem

We need the following lemmas to prove our direct theorem.

**Lemma 3.1** Let
\[
H_n(f, x) = \frac{1}{n+1} \sum_{k=1}^\infty b_{n-1,k+1}(x) \int_0^\infty b_{n+1,k}(t) f(t) dt.
\]
Then \( H_n((x-t)^2, x) = \frac{2x}{n+1} + \frac{2x}{(2n+1)^2} - x^2 \frac{n}{n+1}(1+x)^{-n-1} \).
Lemma 3.2 For $g \in \mathcal{D}_p$, $\delta_n^2(x) = \varphi^2(x) + \frac{1}{n}$, $1 \leq p \leq \infty$, we have

$$\| B_n(\int_x^t (t-u)g''(u)du, x) \|_p \leq \frac{C}{n} \| \delta_n^2 g'' \|_p.$$  

**Proof** We separate the proof for $p = \infty$ and $p = 1$ and prove the result for $p = \infty$ first.

When $x \in E_n = [\frac{1}{n}, \infty)$, we have $\delta_n^2(x) \sim \varphi^2(x)$. Using Lemmas 2.2, 2.4 and Hölder inequality, we can deduce

$$| B_n(\int_x^t (t-u)g''(u)du, x) | \leq \frac{C}{n} \| \varphi^2 g'' \|_\infty.$$  

When $x \in E_n^\circ = [0, \frac{1}{n})$, by Lemma 2.5(2.5) and Lemma 2.2, we have

$$| B_n(\int_x^t (t-u)g''(u)du, x) | \leq \| \delta_n^2 g'' \|_\infty \left| B_n(\int_x^t \frac{|t-u|}{\varphi^2(u) + \frac{1}{n}} du, x) \right| \leq C \| \delta_n^2 g'' \|_\infty \frac{\varphi^2(x)}{n-1} \cdot \frac{1}{\varphi^2(x) + \frac{1}{n}} \leq \frac{C}{n} \| \delta_n^2 g'' \|_\infty.$$  

Therefore, Lemma 3.2 is valid for the case $p = \infty$.

To prove the rest of the result for $p = 1$, we observe that

$$\| B_n(\int_x^t (t-u)g''(u)du, x) \|_1 = \int_0^\infty \frac{1}{n+1} \left| \sum_{k=1}^\infty b_{n,k}(x) \left( \int_0^x + \int_x^\infty \right) b_{n,k}(t) \int_x^t (t-u)g''(u)du, x \right| dx$$

$$\leq \frac{1}{n+1} \int_0^\infty |g''(u)| \left\{ \int_u^\infty \int_u^\infty - \int_0^u \int_0^\infty \right\} (u-t) \sum_{k=1}^\infty b_{n,k}(x)b_{n,k}(t)dt\ dx.$$  

We need to prove that

$$J := \left\{ \int_u^\infty \int_0^u - \int_0^u \int_u^\infty \right\} (u-t) \sum_{k=1}^\infty b_{n,k}(x)b_{n,k}(t)dt\ dx$$

$$\leq \frac{1}{2} \sum_{k=1}^\infty (\int_u^\infty \int_0^u - \int_0^u \int_u^\infty ) b_{n,k}(t) d(u-t)^2 b_{n,k}(x) dx$$

$$:= \frac{1}{2} (J_1 + J_2) \leq C \delta_n^2(u) = C (\varphi^2(u) + \frac{1}{n}).$$  

We will estimate $J_1$. Using the integration by parts and recalling the fact

$$b'_{n,k}(t) = (n+1)(b_{n+1,k-1}(t) - b_{n+1,k}(t)), \ k \geq 1,$$
Thus, Using the Taylor formula with integral remainder, we have

\[ J_1 \leq (n + 1)u^2(1 + u)^{-n - 1} + \frac{n + 1}{n} \sum_{k=1}^{\infty} \int_0^u (t - u)^2 b_{n+1,k}(t)b_{n-1,k}(u)dt. \]

Similarly, \( J_2 \leq 2\sum_{k=1}^{\infty} \int_0^\infty (t - u)^2 b_{n+1,k}(t)b_{n-1,k}(u)dt. \)

Using Lemma 3.1,

\[ \frac{1}{2}(J_1 + J_2) \leq (n + 1)u^2(1 + u)^{-n - 1} + \sum_{k=1}^{\infty} \int_0^\infty (t - u)^2 b_{n+1,k}(t)b_{n-1,k}(u)dt \]

\[ \leq u^2 + \varphi^2(u) \leq C \delta_{n}^2(u). \]

Thus,

\[ J \leq \frac{1}{2}(J_1 + J_2) \leq C \delta_{n}^2(u). \]

Lemma 3.2 is true for \( p = 1 \). Again using the Riesz-Thorin interpolation theorem completes the proof of Lemma 3.2 for \( 1 \leq p \leq \infty \).

Now we can prove our direct theorem.

**Proof of Theorem 1.1** We construct a function \( A_n(f, x) \) such that

\[ A_n(f, x) = B_n(f, x) + L_n(f, x). \]

Then \( A_n(1, x) = 1 \), \( A_n(t - x, x) = \frac{2x}{n} \), \( A_n((t - x)^2, x) \leq \frac{CG(x)}{n} \), \( \|A_n\| \leq 3 \),

\[ \|L_n(f, x)\|_p = \left\{ \int_0^\infty |f(x) - f(x + 1/n)|^p dx \right\}^{2/p} \leq \omega(f, 1/n)_p. \]

By the definition of \( K^2_\varphi(f, t^2)_p \), we now choose \( g \in \overline{D} \) satisfying

\[ \|f - g\|_p + \frac{1}{n} \|\varphi^2 g''\|_p + \frac{1}{n^2} \|g''\|_p \leq 2K^2_\varphi(f, 1/n)_p. \]

Using the Taylor formula with integral remainder, we have

\[ \|A_n(g, x) - g(x)\|_p \leq \|g'(x)A_n(t - x, x)\|_p + \|A_n(\int_x^t (t - u)g''(u)du, x)\|_p \]

\[ \leq \frac{2}{n} \|x g'(x)\|_p + \|B_n(\int_x^t (t - u)g''(u)du, x)\|_p + \| \int_x^{x+1/n} (x + 1/n - u)g''(u)du\|_p. \]

Recalling

\[ \| \int_x^{x+1/n} (x + 1/n - u)g''(u)du\|_p \leq \frac{1}{n^2} \|g''\|_p, \]

by the relation\[20], \( \|x f'\|_p \leq C(\|x^2 f''\|_p + \|f\|_p) \), and Lemma 3.2 and the choice of \( g \) (3.2), we write

\[ \|A_n(g, x) - g(x)\|_p \leq C K^2_\varphi(f, 1/n)_p + \frac{2}{n} \|f\|_p. \]

Combining (3.1)–(3.3) and the equivalence relation (1.3), we get

\[ \|B_n(f, x) - f(x)\|_p \leq \|A_n(f, x) - f(x)\|_p + \|L_n(f, x)\|_p \]

\[ \leq 4\|g\|_p + \|A_n g - g\|_p + \omega_1(f, 1/n)_p \]
Using the integration by parts, we write

\[
\leq C(\omega^2_p(f, \frac{1}{\sqrt{n}}) + \omega_1(f, \frac{1}{n}) + \frac{\|f\|_p}{n}).
\]

\[\square\]

4. The inverse theorem

We also need the following lemmas to prove Theorems 1.2 and 1.3.

Lemma 4.1 For \( x \in E_n = [\frac{1}{n}, \infty) \), we have

\[
|\varphi^2(x) b''_{n,k}(x)| \leq C \sum_{i=0}^{2} n^{1+\frac{1}{2}} b_{n,k}(x) \frac{k - 1 - 2x}{n} - x|i\varphi^{-i}(x).
\]

Proof Since

\[
b''_{n,k}(x) = \frac{b_{n,k}(x)}{x^2(1 + x)^2} \cdot \left[ -(n + 2)x(1 + x) + (k - 1 - (n + 2)x)^2 - (1 + 2x)(k - 1 - (n + 2)x) \right]
\]

\[= \frac{1}{x^2(1 + x)^2} \sum_{i=0}^{2} k - 1 - \frac{(n + 2)x}{n} i \cdot Q_i(x)n^i b_{n,k}(x),
\]

where

\[
Q_0(x) = -x(1 + x)(n + 2), \quad Q_1(x) = -\frac{nx(1 + x)}{n + 2}, \quad Q_2(x) = \frac{n^2}{(n + 2)^2},
\]

on the other hand, for \( i = 0, 1, 2 \), one has

\[|x^{-2}(1 + x)^{-2} Q_i(x)n^i| \leq C\left(\frac{n}{x(1 + x)}\right)^{1+i},\]

which completes the proof of Lemma 4.1. \[\square\]

Lemma 4.2 For \( x \in E_n = [\frac{1}{n}, \infty) \), \( \varphi^2(x) = x(1 + x) \), one has

\[
\int_{E_n} b_{n,k}(x) \varphi^{-2m}(x) \left( \frac{k}{n} - x \right)^{2m} dx \leq \frac{C}{n^m}, \quad m = 0, 1, 2, \ldots .
\]

Proof We recall that \( \int_{E_n} b_{n,k}(x) dx \leq 1 \), which is Lemma 4.2 for the case \( m = 0 \).

We assume, using induction, that

\[
\int_{E_n} b_{n,k}(x) \varphi^{-2l}(x) \left( \frac{k}{n} - x \right)^{2l} dx \leq Cn^{-l}.
\]

Now we shall show for the case \( m = l + 1 \) that

\[
\int_{E_n} b_{n,k}(x) \varphi^{-2(l+1)}(x) \left( \frac{k}{n} - x \right)^{2(l+1)} dx \leq Cn^{-(l+1)}.
\] (4.1)

Using the integration by parts, we write

\[
\int_{E_n} \frac{b_{n,k}(x)}{x^{l+1}(1 + x)^{l+1}} \left( \frac{k}{n} - x \right)^{2l+1} dx = \frac{b_{n,k}(\frac{1}{n})}{(2l + 1)(\frac{1}{n})^{l}(1 + \frac{1}{n})^{l}} \int_{E_n} \frac{k}{n} - x \left( \frac{k}{n} - x \right)^{2l+1} dx.
\]

\[
= \frac{2l + n + 2}{2l + 1} \int_{E_n} \frac{b_{n,k}(x)}{x^{l+1}(1 + x)^{l+1}} \left( \frac{k}{n} - x \right)^{2l+1} \left( \frac{k - l - 1}{2l + n + 2} - x \right) dx.
\]
By the assumption, one has
\[
\left| \int_{E_n} \frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}} \left( \frac{k}{n} - x \right)^{2l+1} \left( \frac{k - l - 1}{2l + n + 2} - x \right) dx \right| \\
\leq C \left( n^{-l-1} + \frac{b_{n,k}(\frac{k}{n})}{n^{l+1}((1 + \frac{k}{n})^{l+1})} \left( \frac{k}{n} - \frac{1}{n} \right)^{2l+1} \right).
\]

Since \( \frac{b_{n,k}(\frac{k}{n})}{n^{l+1}((1 + \frac{k}{n})^{l+1})} (\frac{k}{n} - \frac{1}{n})^{2l+1} \leq C n^{-l} \), we get
\[
\left| \int_{E_n} \frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}} \left( \frac{k}{n} - x \right)^{2l+1} \left( \frac{k - l - 1}{2l + n + 2} - x \right) dx \right| = O(n^{-l-1}).
\]

Let \( G_{n,k} = \{ x : nk - 2(l+1)(n+k) < x \leq \frac{k+1}{n} \} \subset E_n = [\frac{1}{n}, \infty) \).

On \( E_n \setminus G_{n,k} \), we have the following relations
\[
\frac{k}{n} - x \left( \frac{k - l - 1}{2l + n + 2} - x \right) > 0, \quad (4.2)
\]
\[
\left| \frac{k}{n} - x \right| \leq C \left| \frac{k - l - 1}{2l + n + 2} - x \right|. \quad (4.3)
\]

Next, we will prove (4.1) for the case \( m = 0, 1, 2, \ldots \). Write
\[
\int_{E_n} b_{n,k}(x) \varphi^{2l+1}(x) \left( \frac{k}{n} - x \right)^{2l+1} dx = \int_{E_n \setminus G_{n,k}} + \int_{G_{n,k}} := G_1 + G_2.
\]

From the relations (4.2) and (4.3), we write
\[
G_1 \leq C \int_{E_n \setminus G_{n,k}} \frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}} \left( \frac{k}{n} - x \right)^{2l+1} \left( \frac{k - l - 1}{2l + n + 2} - x \right) dx \\
\leq C \left( \int_{E_n} + \int_{G_{n,k}} \right) = O(n^{-l-1}) + G_3.
\]

On \( G_{n,k} \), for \( x \in G_{n,k} \), we have
\[
\left| \frac{k}{n} - x \right| = O\left( \frac{1}{n} \right), \quad (4.4)
\]
\[
\left| \frac{k - l - 1}{2l + n + 2} - x \right| = O\left( \frac{1}{n} \right), \quad (4.5)
\]
\[
\varphi^{2}(x) \geq C \frac{k}{n} \left( 1 + \frac{k}{n} \right). \quad (4.6)
\]

Since the methods to estimate \( G_2, G_3 \) are similar, we deal here only with \( G_2 \). By (4.4), (4.6),
\[
G_2 = \int_{G_{n,k}} \frac{b_{n,k}(x)}{x^{l+1}(1+x)^{l+1}} \left( \frac{k}{n} - x \right)^{2l+2} dx \\
\leq C \int_{0}^{\infty} b_{n,k}(x) \left( \frac{k}{n} \left( 1 + \frac{k}{n} \right)^{-l-1} \right) \left( \frac{1}{n} \left( 1 + \frac{k}{n} \right) \right)^{2l+1} dx \leq C n^{-l-1}.
\]

**Remark** In the estimate of \( G_3 \), we use (4.5). From the estimates of \( G_1, G_2, G_3 \), (4.1) is valid for the case \( m = l + 1 \). By induction, Lemma 4.2 is true for the case \( m = 0, 1, 2, \ldots \).
Lemma 4.3 For $x \in E_n = \left[\frac{1}{n}, \infty\right)$, $m = 0, 1, 2$, we have
\[ \sum_{k=1}^{\infty} b_{n,k}(x) \left(\frac{k}{n} - x\right)^{2m} \leq C \varphi^{2m}(x). \]

Proof From the following equalities
\[ \sum_{k=1}^{\infty} b_{n,k}(x) \frac{k}{n} = \frac{(n+1)(n+2)}{n} x + \frac{n+1}{n}, \]
\[ \sum_{k=1}^{\infty} b_{n,k}(x) \frac{k^2}{n^2} = \frac{(n+1)(n+2)(n+3)}{n^2} x^2 + \frac{3(n+1)(n+2)}{n^2} x + \frac{n+1}{n^2}, \]
\[ \sum_{k=1}^{\infty} b_{n,k}(x) \frac{k^3}{n^3} = \frac{(n+1)(n+2)(n+3)(n+4)}{n^3} x^3 + \frac{6(n+1)(n+2)(n+3)}{n^3} x^2 + \frac{7(n+1)(n+2)}{n^3} x + \frac{n+1}{n^3}, \]
\[ \sum_{k=1}^{\infty} b_{n,k}(x) \frac{k^4}{n^4} = \frac{(n+1)\cdots(n+5)}{n^4} x^4 + \frac{10(n+1)\cdots(n+4)}{n^4} x^3 + \frac{25(n+1)(n+2)(n+3)}{n^4} x^2 + \frac{15(n+1)(n+2)}{n^4} x + \frac{n+1}{n^4}, \]
we can obtain Lemma 4.3. \hfill \Box

Lemma 4.4 For $f \in L_p[0, \infty)$, $1 \leq p \leq \infty$, we have $\|B'_n(f, x)\|_p \leq C_n \|f\|_p$.

Proof Noticing that $b'_{n,k}(t) = (n+1)(b_{n+1,k-1}(t)) - b_{n+1,k}(t)$,
\[ B'_n(f, x) = \sum_{k=1}^{\infty} b_{n+1,k}(x) \int_0^{\infty} [b_{n,k+1}(t)) - b_{n,k}(t)] f(t) dt \]
\[ \leq \sum_{k=1}^{\infty} b_{n+1,k}(x) \int_0^{\infty} [b_{n,k+1}(t)) + b_{n,k}(t)] \cdot |f(t)| dt, \]
which implies Lemma 4.4. \hfill \Box

Lemma 4.5 For $f, f' \in L_p[0, \infty)$, $1 \leq p \leq \infty$, we have $\|B'_n(f, x)\|_p \leq C \|f'\|_p$.

Proof Recalling that $b'_{n,k}(t) = (n+1)(b_{n+1,k-1}(t)) - b_{n+1,k}(t)$, and $b_{n,0}(x) = 0$, one has the estimates for $\|B'_n(f, x)\|_p$,
\[ B'_n(f, x) = \frac{1}{n+1} \sum_{k=1}^{\infty} b'_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt \]
\[ \leq \frac{1}{n+1} \sum_{k=1}^{\infty} b_{n+1,k}(x) \int_0^{\infty} b_{n-1,k+1}(t)) f'(t) dt. \]
\hfill \Box

Proof of Theorem 1.2 Using the technique of Lemmas 4.1 and 4.2, for $x \in E_n$, one has
\[ \sum_{k=1}^{\infty} b_{n,k}(x) \left(\frac{k-1}{n} - x\right)^{2m} \leq C \varphi^{2m}(x). \]
For the case $p = 1$, from relations (4.8), (4.9) and Hölder inequality, and recalling that

$$\left| \int_0^\infty \frac{b_{n,k}(x)}{(1+x)^2} \, dx \right| \leq C,$$

we have

$$F_{11} \leq \frac{1}{n + 1} \int_{E_n} \left| x \sum_{k=1}^\infty b''_{n,k}(x) \int_0^\infty b_{n,k}(t)f(t) \, dt \right| \, dx \leq C \sum_{i=0}^2 n^{i+\frac{1}{2}} (n+1) \int_0^\infty |f(t)| \, dt \leq Cn \|f\|_1.$$

Using the expression of $B''_n(f, x)^{13, \text{Lemma 2.6}}$, for $n \geq 4$, $x < \frac{1}{n}$, we have

$$F_{12} \leq \frac{2}{n} \int_{E_n} \left| x \sum_{k=1}^\infty b_{n+2,k}(x) \int_0^\infty b''_{n-2,k+2}(t)f(t) \, dt \right| \, dx \leq \frac{2}{n^2} \int_{E_n} \sum_{k=1}^\infty b_{n+2,k}(x) \int_0^\infty |n(n-1)b_{n,k}(t) - 2n(n-1)b_{n,k+1}(t) + n(n-1)b_{n,k+2}(t)||f(t)| \, dt \, dx \leq Cn \|f\|_1.$$

Thus, $F_1 \leq Cn \|f\|_1$ for the case $p = 1$.

For the case $p = \infty$, by (4.7), we write

$$F_{11} \leq C \|f\|_\infty \sum_{i=0}^2 n^{i+1} \left( \sum_{k=1}^\infty b_{n,k}(x) \frac{(k-1-2x)^{2i}}{n} - x \right) \frac{k-1-2x}{n} \left( \sum_{k=1}^\infty b_{n,k}(x) \right) \frac{1}{\varphi^{-i}(x)(1+x)^{-2}} \leq Cn \|f\|_\infty.$$

Then,

$$F_{12} \leq \frac{2}{n^2} \sum_{k=1}^\infty b_{n+2,k}(x) \int_0^\infty |n(n-1)b_{n,k}(t) - 2n(n-1)b_{n,k+1}(t) + n(n-1)b_{n,k+2}(t)||f(t)| \, dt \, dx \leq Cn \|f\|_\infty.$$

Thus, $F_1 \leq Cn \|f\|_\infty$ for the case $p = \infty$.

Using the Riesz-Thorin Theorem, we obtain $F_1 \leq Cn \|f\|_p$ for $1 \leq p \leq \infty$.

Next we will estimate $F_2 = \|x^2 B_n''(f, x)\|_p \leq Cn \|f\|_p$. 

For \( x \in E_n^c \), noticing that \( x^2 \leq x \), then \( F_2 \) can be written as follows

\[
F_2 \leq \| x^2 B''_n(f,x) \|_{p(E_n)} + \| x^2 B''_n(f,x) \|_{p(E_n^c)}
\]

\[
\leq \| \frac{x^2}{n+1} \sum_{k=1}^{\infty} b''_{n,k}(x) \int_{E_n} b_{n,k}(t)f(t)dt \|_{p(E_n)} + \\
\| \frac{x^2}{n+1} \sum_{k=1}^{\infty} b''_{n,k}(x) \int_{E_n^c} b_{n,k}(t)f(t)dt \|_{p(E_n)} + F_{12}
\]

\[
:= F_{21} + F_{22} + F_{12}.
\]

Again using the expression of \( B''_n(f,x) \)[13, Lemma 2.6] and Lemma 4.1, we have

\[
F_{21} \leq C \| \frac{x^2(n+2)}{n(n-1)} \sum_{k=1}^{\infty} b_{n+2,k}(x) \int_{E_n} b''_{n-2,k+2}(t)f(t)dt \|_{p(E_n)}
\]

\[
\leq C \left\{ \left| \sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{2} n \varphi^{i-1}(t)(\frac{k+1}{n})^i - t^i \cdot |f(t)|dt \|_{p(E_n)} + \\
\sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{2} n \varphi^{i-1}(t)(\frac{2(k+1)}{n(n-2)})^i + n \varphi^{i-1}(t)(\frac{2}{n(n-2)})^i \cdot |f(t)|dt \|_{p(E_n)} + \\
\sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{2} n \varphi^{i-1}(t)(\frac{2}{n})^i \cdot |f(t)|dt \|_{p(E_n)} \right\}
\]

\[
:= C \{ E_1 + E_2 + E_3 \}.
\]

By (4.8), one has

\[
F_{22} \leq C \left\{ \left| \sum_{k=1}^{2} n \varphi^{i-1}(x)(\frac{k+1}{n}) - x \right| \int_{E_n} \sum_{i=0}^{\infty} b_{n+1,k-1}(t) + b_{n,k}(t) f(t)dt \|_{p(E_n)} + \\
C \left| \sum_{k=1}^{2} n \varphi^{i-1}(x)(\frac{2}{n})^i \right| \int_{E_n} \sum_{i=0}^{\infty} b_{n+1,k-1}(t) + b_{n,k}(t) f(t)dt \|_{p(E_n)} + \\
C \left| \sum_{k=1}^{2} n \varphi^{i-1}(x)(\frac{2}{n})^i \right| \int_{E_n} \sum_{i=0}^{\infty} b_{n+1,k-1}(t) + b_{n,k}(t) f(t)dt \|_{p(E_n)} \right\}
\]

\[
:= E_4 + E_5 + E_6.
\]

Since the treatments for \( E_1 \sim E_6 \) are similar, we deal here only with \( E_1 \).

\[
E_1 = \left| \sum_{k=1}^{\infty} b_{n,k+2}(x) \int_{E_n} \sum_{i=0}^{\infty} n \varphi^{i-1}(t)(\frac{k+1}{n}) - t^i |f(t)|dt \|_{p(E_n)}.
\]

By Hölder inequality, we discuss two cases \( p = 1 \) and \( p = \infty \).

First, we will consider the cases \( p = \infty \).

\[
E_1 \leq C \| f \|_{\infty} \sum_{i=0}^{2} \left\{ \int_{E_n} \sum_{k=1}^{\infty} b_{n,k+2}(x) \left| \int_{E_n} b_{n,k+1}(t) \varphi^{-2i}(t)(\frac{k+1}{n})^i - t^{2i} dt \right| \right\}^{\frac{1}{2}} \times \\
\left\{ \int_{E_n} \sum_{k=1}^{\infty} b_{n,k+1}(t) dt \right\}^{\frac{1}{2}} \leq C \| f \|_{\infty}.
\]
Next, for \( p = 1 \), by [13, Lemma 2.1], and Hölder inequality, one has
\[
E_1 \leq \int_{E_n} \sum_{i=0}^{2} n^2 \sum_{k=1}^{\infty} b_{n,k+2}(x)b_{n,k+1}(t)\varphi^{-1}(t) t^{-1/n} \cdot |f(t)|dt\ dx
\]
\[
\leq \sum_{i=0}^{2} n^2 \left( \sum_{k=1}^{\infty} b_{n,k+1}(t)\varphi^{-2i}(t) \right)^{\frac{1}{2}} \cdot \left( \sum_{k=1}^{\infty} b_{n,k+1}(t) \right)^{\frac{k+1}{n}} \cdot \int_{E_n} |f(t)|dt
\]
\[
\leq Cn \| f \|_1.
\]
Therefore, we have \( 1 \leq p \leq \infty, \ F_2 \leq Cn \| f \|_p \).

**Proof of Theorem 1.3** We will prove 1° first.

By the definition of \( K \)-functional, we can choose \( g = B_n(f, x) \) such that
\[
K_2(f, t^2)p \leq \| f(x) - B_n(f, x) \|_p + t^2 \| \varphi^2 B_n''(f, x) \|_p.
\]

The condition \( \| B_n(f, x) - f(x) \|_p = O(n^{-\frac{\alpha}{2}}) \), Theorem 1.2 and Lemma 2.6 imply
\[
K_2(f, t^2)p \leq Cn^{-\frac{\alpha}{2}} + t^2 \{ \| \varphi^2 B_n''(f - g, x) \|_p + \| \varphi^2 B_n''(g, x) \|_p \}
\]
\[
\leq Cn^{-\frac{\alpha}{2}} + t^2 \{ \| f - g \|_p + \| \varphi^2 g'' \|_p \} \leq Cn^{-\frac{\alpha}{2}} + t^2 nK_2(f, \frac{1}{n})p.
\]

By the well-known Berens-Lorentz Lemma[5, p. 122], we have
\[
K_2(f, t^2)p \leq Ct^\alpha.
\]
Combining the equivalence relation (1.2), one has \( \omega_2^p(f, t)_p = O(t^{\alpha}) \).

Next we will prove 2°. We introduce a new \( K \)-functional:
\[
K_1(f, t)_p = \inf_{g \in D_1} \{ \| f - g \|_p + t\| g' \|_p \},
\]
where \( D_1 = \{ g \mid g, g' \in L_p[0, \infty) \} \).

By the definition of the new \( K \)-functional, for \( t = \frac{1}{n} \), we can choose \( g \in D_1 \) such that
\[
\| f - g \|_p + \frac{1}{n}\| g' \|_p \leq CK_1(f, \frac{1}{n})p.
\]

On the other hand, by Lemmas 4.4 and 4.5, one has
\[
K_1(f, t)_p \leq \| f - B_n f \|_p + t\| B_n' f \|_p
\]
\[
\leq Cn^{-\frac{\alpha}{2}} + t\{ \| f - g \|_p + \| g' \|_p \} \leq Cn^{-\frac{\alpha}{2}} + tnK_1(f, \frac{1}{n})p.
\]

Again using Berens-Lorentz Lemma[5, Lemma 9.3.4], we have \( K_1(f, t)_p \leq Cn^{\frac{\alpha}{2}} \).

Finally we will estimate \( \omega_1(f, t)_p = O(t^{\frac{\alpha}{2}}) \). Write
\[
\| \Delta_1^1 f(x) \|_p \leq \| \Delta_1^1 f - g \|_p + \| \Delta_1^1 g \|_p := T_1 + T_2,
\]
where \( T_1 = \| (f - g)(x + \frac{1}{2}) - (f - g)(x - \frac{1}{2}) \|_p \leq 2\| f - g \|_p \).

Using Hölder inequality, one has
\[
T_2 = \| \int_{-\frac{1}{2}}^{\frac{1}{2}} g'(x + u)du \|_p = \{ \int_0^{\infty} | \int_{-\frac{1}{2}}^{\frac{1}{2}} g'(x + u)du |^p dx \}^{\frac{1}{p}}
\]
\[
\leq \left\{ \int_0^\infty \left( \int_{-\frac{t}{2}}^{\frac{t}{2}} |g'(x+u)|^p du \right) \cdot \left( \int_{-\frac{t}{2}}^{\frac{t}{2}} du \right)^{\frac{p}{q}} dx \right\}^\frac{1}{p} \leq Ct\|g'\|_p.
\]

Therefore, \( \|\Delta_1 t f(x)\|_p \leq 2\|f - g\|_p + Ct\|g'\|_p \leq C K_1(f, t)_p \leq C t^{\frac{p}{2}} \), which implies \( \omega_1(f, t)_p = O(t^{\frac{p}{2}}). \)

References