On a Class of Analytic Functions Defined by Ruscheweyh Derivatives

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Abstract In the present paper a class of extended close-to-convex functions $Q_{k,\lambda}(\alpha, \beta, \rho)$ defined by making use of Ruscheweyh derivatives is introduced and studied. We provide integral representations, distortion theorem, radius of close-to-convexity and Hadamard product properties for this class.

Keywords Ruscheweyh derivatives; close-to-convex function; Hadamard product.

1. Introduction

Suppose that the parameters $\lambda$, $\alpha$, $\beta$, $\rho$ satisfy $\lambda > -1$, $\alpha \geq 0$, $0 \leq \beta \leq 1$, $0 \leq \rho < 1$. Let $H_k$ ($k = 1, 2, \ldots$) be the class of functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{k+n} z^{k+n}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let $P_k(\beta)$ denote the class of functions of the form $p(z) = 1 + p_k z^k + \cdots$ which are analytic in $U$ and satisfy $\text{Re} p(z) > \beta$. Let $S^*_k(\beta)$ and $K_k(\beta)$ stand for $\beta$ class starlike function and $\beta$ class convex function in $H_k$, respectively.

A function $f(z) \in H_k$ is said to be in the class $C_k(\beta, \rho)$ if and only if there exists $g(z) \in S^*_k(\beta)$ such that

$$\text{Re} \frac{zf'(z)}{g(z)} > \rho, \quad z \in U.$$

From [9], we know that

$$f(z) \in K_k(\beta) \Leftrightarrow zf'(z) \in S^*_k(\beta).$$

For fixed real number $\lambda > -1$, the operator $D^\lambda$ is defined by

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad f(z) \in H_k,$$

Received date: 2007-10-29; Accepted date: 2008-07-07

Foundation item: the Natural Science Foundation of Inner Mongolia (No. 2009MS0113); Higher School Research Foundation of Inner Mongolia (No. NJzy08510).
where the operation $\ast$ stands for Hadamard product. The operator $D^\lambda$ is the Ruscheweyh derivative introduced in [1,10] and is of the following properties:

$$D^\lambda f(z) = z + \sum_{n=1}^{\infty} \frac{(\lambda + 1)\cdots(\lambda + k + n - 1)}{(k + n - 1)!} a_{k+n} z^{k+n}; \quad (1.2)$$

$$z(D^\lambda f(z))' = (\lambda + 1)D^{\lambda+1} f(z) - \lambda D^\lambda f(z). \quad (1.3)$$

Next we introduce new functions class.

**Definition 1.1** If a function $f(z) \in H_k$ satisfies condition

$$\Re\left\{ (1 - \alpha) \frac{D^\lambda f(z)}{z} + \alpha(D^\lambda f(z))^\prime \right\} > \beta, \quad z \in U,$$

then we denote $f(z) \in V_{k,\lambda}(\alpha,\beta)$.

**Definition 1.2** Suppose $f(z) \in H_k$. If there exists a function $g(z) \in V_{k,\lambda}(\alpha,\beta)$ such that

$$\Re\frac{z(D^\lambda f(z))^\prime}{D^\lambda g(z)} > \rho, \quad z \in U,$$

then we denote $f(z) \in Q_{k,\lambda}(\alpha,\beta,\rho)$.

In [2], the functions class $Q_{1,0}(0, \frac{1}{2}, 0)$ was studied and distortion theorem, univalent radius and rotation theorem were obtained, but Hadamard product has not been solved. We will study the close-to-convex function class $Q_{k,\lambda}(\alpha,\beta,\rho)$ introduced above which is a great extension of [2].

As in [3], we introduce linear operator $L(a,c)$ which is more general than $D^\lambda$. Let

$$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad z \in U, \quad c \neq 0, -1, -2, \ldots$$

$$L(a,c)f(z) = \phi(a,c;z) \ast f(z), \quad f(z) \in H_k \quad (1.6)$$

where $(\zeta)_n = \frac{\Gamma(\zeta+n)}{\Gamma(\zeta)}$. From [4], we know that $L(a,c)$ is continuous mapping from $H_k$ to $H_k$. It is easy to see that

$$\phi(2(1-\alpha), 1; z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (1.7)$$

and for $c > a > 0$, we have

$$L(a,c)f(z) = \int_0^1 u^{a-1} f(uz) d\eta(a,c-a)(u), \quad (1.8)$$

where $\eta$ is $B$ distribution

$$d\eta(a,c-a)(u) = \frac{u^{a-1}(1-u)^{c-a-1}}{B(a,c-a)} du. \quad (1.9)$$

If $a \neq 0, -1, -2, \ldots$, then $L(c,a)$ is the inverse mapping of $L(a,c)$, so $L(a,c)$ is one-to-one mapping from $H_k$ to $H_k$. It is obvious that

$$L(a,c) = L(a,b)L(b,c) = L(b,c)L(a,b), \quad b, c \neq -1, -2, \ldots$$

If $g(z) = zf'(z)$, then $g(z) = L(2,1)f(z)$, $f(z) = L(1,2)g(z)$. 

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By (1.6) and (1.7), we have
\[ L(\lambda + 1, 1)f(z) = D^\lambda f(z). \] (1.10)

In view of the operator \( L(a, c) \) and (1.10), we may write (1.5) as:
\[
\Re \frac{L(2, 1)L(\lambda + 1, 1)f(z)}{L(\lambda + 1, 1)g(z)} > \rho, \quad z \in U.
\] (1.11)

In the present paper, we deduce integral representations of function in \( Q_{k, \lambda}(\alpha, \beta, \rho) \). Distortion theorems, radius of close-to-convexity and Hadamard product properties are obtained for functions belonging to this class. Then we solve the closeness of Hadamard product in [2].

### 2. Integral representations

If \( g(z) \in V_{k, \lambda}(\alpha, \beta) \), then it is not difficult to verify that there exists \( p(z) = 1 + p_k z^k + \cdots \in P_k(\beta) \) such that
\[
g(z) \in V_{k, \lambda}(\alpha, \beta) \Leftrightarrow zp(z) = L\left( \frac{1}{a}, \frac{1}{a} + 1 \right) \lambda(\lambda + 1, 1)g(z) \Leftrightarrow L(1, \lambda + 1)g(z)
\]
\[ = L\left( \frac{1}{a}, \frac{1}{a} + 1 \right) (zp(z)).
\]

In view of the Herglotz formula\(^5\) of positive real part and the property of \( L(\lambda + 1, 1) \), we prove the following result:

**Theorem 2.1** If \( g(z) \in V_{k, \lambda}(\alpha, \beta)(\alpha > 0) \), then there exists left continuous probability measure \( \eta(x) \) on \( X = \{ x : |x| = 1 \} \) such that
\[
g(z) = L(1, \lambda + 1)\left\{ \frac{1}{\alpha z^{\frac{1}{\beta} - 1}} \int_0^z (\int_{|x| = 1} \frac{1}{1 - tx} d\eta(x)) dt \right\}, \quad (2.1)
\]
or there exists \( p(z) \in P_k(\beta) \) such that
\[
g(z) = L(1, \lambda + 1)\left\{ \frac{1}{\alpha z^{\frac{1}{\beta} - 1}} \int_0^z p(t) dt \right\}.
\]

For fixed parameters \( \lambda, \alpha, \beta, V_{k, \lambda}(\alpha, \beta) \) and left continuous probability measure points \( \{ \eta(x) \} \) on \( X \) are one-to-one correspondence through the relation expressed by (2.1).

**Theorem 2.2** A function \( f(z) \in Q_{k, \lambda}(\alpha, \beta, \rho)(\alpha > 0) \) if and only if there exists left continuous probability measures \( \eta(x), \mu(x) \) on \( X = \{ x : |x| = 1 \} \) such that
\[
f(z) = L(1, \lambda + 1)L(1, 2) \left\{ \frac{1}{\alpha z^{\frac{1}{\beta} - 1}} \int_0^z (\int_{|x| = 1} \frac{1}{1 - tx} d\eta(x)) dt \right\} \times
\[
\left[ \int_{|x| = 1} \frac{1 + (1 - 2\rho)z}{1 - zx} d\mu(x) \right], \quad (2.2)
\]
when \( \lambda = 0 \),
\[
f(z) = L(1, 2) \left\{ \frac{1}{\alpha z^{\frac{1}{\beta} - 1}} \int_0^z (\int_{|x| = 1} \frac{1}{1 - tx} d\eta(x)) dt \right\} \times
\[
\left[ \int_{|x| = 1} \frac{1 + (1 - 2\rho)z}{1 - zx} d\mu(x) \right]. \quad (2.3)
\]
For fixed parameters $\lambda, \alpha, \beta, \rho$, $Q_{k,\lambda}(\alpha, \beta, \rho)$ and left continuous probability measure points \{$(\eta(x), \mu(x))$\} on $X \times X$ are one-to-one correspondence through the relation expressed by (2.2).

**Proof** Let $f(z) \in Q_{k,\lambda}(\alpha, \beta, \rho)$. Then there exists $g(z) \in V_{k,\lambda}(\alpha, \beta)$ such that
\[
\text{Re} \frac{z(L(\lambda + 1)f(z))'}{L(\lambda + 1)g(z)} > \rho, \quad z \in U.
\]

By Theorem 2.1, we have
\[
g(z) = L(1, \lambda + 1) \left\{ \frac{1}{|z|^\frac{1}{\lambda} - 1} \int_0^z t^{\frac{1}{\lambda} - 1} \left( \int_{|x| = 1} \frac{1 + (1 - 2\beta)tx}{1 - tx} \eta(x) \right) dx \right\}, \quad (2.4)
\]
where $\eta(x)$ is left continuous probability measure on $X$. By Herglotz formula\[5\] for the functions in $P$ class, we get
\[
\frac{z(L(\lambda + 1)f(z))'}{L(\lambda + 1)g(z)} = \int_{|x| = 1} \frac{1 + (1 - 2\rho)zx}{1 - zx} \mu(x), \quad (2.5)
\]
where $\mu(x)$ is left continuous probability measure on $X$. From (2.4) and (2.5), we deduce that
\[
L(2, 1)L(\lambda + 1, 1)f(z) = \left\{ \frac{1}{|z|^\frac{1}{\lambda} - 1} \int_0^z t^{\frac{1}{\lambda} - 1} \left( \int_{|x| = 1} \frac{1 + (1 - 2\beta)tx}{1 - tx} \eta(x) \right) dx \right\} \times \left( \int_{|x| = 1} \frac{1 + (1 - 2\rho)zx}{1 - zx} \mu(x) \right).
\]

By using the property of $L(\lambda + 1, 1)$, we get (2.2). Conversely it is true too. When $\lambda = 0$, (2.2) reduce to (2.3). For fixed parameters $\lambda, \alpha, \beta, \rho$, since \{$(\eta(x), \mu(x))$\} and $P_{k}(\beta) \times P_{k}(\rho)$ are one-to-one correspondence, $P_{k}(\beta) \times P_{k}(\rho)$ and $Q_{k,\lambda}(\alpha, \beta, \rho)$ are one-to-one correspondence too, so the last result is true. This completes the proof of Theorem 2.2. \[\Box\]

3. Distortion theorems

**Lemma 3.1**\[6\] Let $p(z) = 1 + pkz^k + \cdots \in P_{k}(0)$ ($z \in U, k \geq 1$). Then for $|z| = r < 1$, we have
\[
\frac{1 - r^k}{1 + r^k} \leq \text{Re} p(z) \leq \frac{1 + r^k}{1 - r^k}.
\]

The result is sharp.

If $\text{Re} p(z) > \beta$, then by setting $q(z) = p(z) - \beta$, we have $\text{Re}(p(z) - \beta) > 0$. Hence it is easy to get from Lemma 3.1 and integral representation of positive real part functions\[5\] that

**Lemma 3.2** Let $q(z) = 1 + qkz^k + \cdots \in P_{k}(\beta)$ ($z \in U, k \geq 1$). Then for $|z| = r < 1$, we have
\[
\frac{1 - (1 - 2\beta)r^k}{1 + r^k} \leq \text{Re} q(z) \leq |q(z)| \leq \frac{1 + (1 - 2\beta)r^k}{1 - r^k}.
\]

The result is sharp.

**Theorem 3.1** Let $\alpha > 0$, $f(z) \in Q_{k,\lambda}(\alpha, \beta, \rho)$. Then for $|z| = r < 1$, we have
\[
\frac{1 - (1 - 2\rho)r^k}{r\alpha(1 + r^k)} \int_0^1 t^{\frac{1}{\alpha} - 1} \frac{1 - (1 - 2\beta)(rt)^k}{1 + (rt)^k} dt \leq |(L(\lambda + 1, 1)f(z))'| \leq \frac{1 + (1 - 2\rho)r^k}{r\alpha(1 - r^k)} \int_0^1 r^{\frac{1}{\alpha} - 1} \frac{1 + (1 - 2\beta)(rt)^k}{1 - (rt)^k} dt. \quad (3.1)
\]
The result is sharp.

Proof Let \( f(z) \in Q_{k,\lambda}(\alpha, \beta, \rho) \). Then there exists \( g(z) \in V_{k,\lambda}(\alpha, \beta) \) such that

\[
\operatorname{Re} \frac{z(L(\lambda + 1, 1)f(z))'}{L(\lambda + 1, 1)g(z)} > \rho, \quad z \in U.
\]

Set \( \frac{z(L(\lambda + 1, 1)f(z))'}{L(\lambda + 1, 1)g(z)} = q(z) \), \( z \in U \). Then \( \text{Re} q(z) > \rho \). Firstly, we prove the distortion property of \( |L(\lambda + 1, 1)g(z)| \). By Lemma 3.2 and Since \( g(z) \in V_{k,\lambda}(\alpha, \beta) \), there exists \( \text{Re} p(z) > \beta \) such that

\[
|L(\lambda + 1, 1)g(z)| \geq \text{Re}(L(\lambda + 1, 1)g(z)) = \text{Re} \left\{ \frac{1}{\alpha z^{\frac{1}{\alpha} - 1}} \int_0^1 t^{\frac{1}{\alpha} - 1} \frac{1 - (1 - 2\beta)(rt)^k}{1 + (rt)^k} \, dt \right\}
\]

\[
\geq \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha} - 1} \frac{1 - (1 - 2\beta)(rt)^k}{1 + (rt)^k} \, dt; \tag{3.2}
\]

\[
|L(\lambda + 1, 1)g(z)| \leq \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha} - 1} \frac{1 + (1 - 2\beta)(rt)^k}{1 - (rt)^k} \, dt. \tag{3.3}
\]

Since \( z(L(\lambda + 1, 1)f(z))' = L(2, 1)L(\lambda + 1, 1)f(z) = q(z)L(\lambda + 1, 1)g(z), \ z \in U \). By (3.2), (3.3) and Lemma 3.2, we have

\[
\frac{1 - (1 - 2\rho)r^k}{\alpha(1 + r^k)} \int_0^1 t^{\frac{1}{\alpha} - 1} \frac{1 - (1 - 2\beta)(rt)^k}{1 + (rt)^k} \, dt \leq |q(z)L(\lambda + 1, 1)g(z)|
\]

\[
\leq \frac{1 + (1 - 2\rho)r^k}{\alpha(1 - r^k)} \int_0^1 t^{\frac{1}{\alpha} - 1} \frac{1 + (1 - 2\beta)(rt)^k}{1 - (rt)^k} \, dt.
\]

We get (3.1). Equality in (3.1) is obtained by function

\[
f(z) = L(1, \lambda + 1)L(1, 2) \left[ \frac{1 + (1 - 2\rho)z^k}{\alpha(1 - z^k)} \right] z^{\frac{1}{\alpha} - 1} \int_0^z t^{\frac{1}{\alpha} - 1} \frac{1 + (1 - 2\beta)t^k}{1 - t^k} \, dt \tag{3.4}
\]

at \( z = r e^{i \theta} \).

4. Radius of close-to-convexity

Lemma 4.1\(^7\) If \( q(z) = 1 + q_kz^k + \cdots \in P_k(\beta) \) \( (z \in U, k \geq 1) \), then for \( |z| = r < 1 \), we have

\[
\left| \frac{\frac{zq'(z)}{q(z)} - 2k(1 - 2\beta)r^k}{(1 - r^k)(1 + (1 - 2\beta)r^k)} \right| < \frac{2k(1 - 2\beta)r^k}{(1 - r^k)(1 + (1 - 2\beta)r^k)}.
\]

The result is sharp.

Theorem 4.1 Let \( \alpha > 0 \), \( f(z) \in Q_{k,\lambda}(\alpha, \beta, \rho) \). Then \( D^\lambda f(z) \) is close-to-convex in disk \( |z| < r_1 \), where \( r_1 \) is the minimum positive root of the following equation:

\[
1 - 2[m + k(1 - m)]r^k - (1 - 2m)r^{2k} = 0 \tag{4.1}
\]

and

\[
m = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha} - 1} \frac{1 - (1 - 2\beta)t^k}{1 + t^k} \, dt < 1. \tag{4.2}
\]

Proof Let \( f(z) \in Q_{k,\lambda}(\alpha, \beta, \rho) \). It suffices to prove that \( D^\lambda g(z) \) is starlike function. Let \( F(z) = \frac{D^\lambda g(z)}{z^2} \). Then \( F(z) \) is analytic in \( U \). By Theorem 2.1, Lemma 3.2 and since \( g(z) \in V_{k,\lambda}(\alpha, \beta) \),
there exists \( p(z) \in P(\beta) \) such that
\[
\Re \left\{ \frac{L(\lambda + 1, 1)g(z)}{z} \right\} = \Re \left\{ \frac{1}{\alpha z^\pi} \int_0^1 t^{\pi-1} p(t) dt \right\} > m = \frac{1}{\alpha} \int_0^1 t^{\pi-1} \frac{1 - (1 - 2\beta)t^k}{1 + t^k} dt.
\]

Noticing the definition of \( F(z) \) and by making use of Lemma 4.1, we have
\[
\Re \left\{ \frac{z(D^\lambda g(z))'}{D^\lambda g(z)} \right\} = 1 + \Re \left\{ \frac{zF'(z)}{F(z)} \right\} \geq 1 - \left| \frac{zF'(z)}{F(z)} \right| = 1 - \frac{1 - 2[m + k(1 - m)]r^k - (1 - 2m)r^{2k}}{(1 - r^k)[1 + (1 - 2m)r^k]}.
\]

Let \( \varphi(r) = 1 - 2[m + k(1 - m)]r^k - (1 - 2m)r^{2k} \). Then \( \varphi(r) \) is continuous on \([0, 1]\) and \( \varphi(0) = 1 > 0, \varphi(1) = -2k(1 - m) < 0 \). So (4.1) has minimum positive root in \((0, 1)\) denoted by \( r_1 \).

For \( |z| < r_1 \), we have \( \Re \left\{ \frac{z(D^\lambda g(z))'}{D^\lambda g(z)} \right\} > 0 \). So \( D^\lambda g(z) \) is starlike function, namely, \( D^\lambda f(z) \) is close-to-convex function in disk \(|z| < r_1\).

**Corollary 4.1** Let \( \alpha > 0, f(z) \in Q_k,0(\alpha, \beta, \rho) \). Then \( f(z) \) is close-to-convex in disk \(|z| < r_1\), where \( r_1 \) is the minimum positive root of (4.1).

5. Hadamard product

**Lemma 5.1** \([8]\) Let \( \varphi(z) \) and \( h(z) \) be analytic in \( U \) and satisfy \( \varphi(0) = h(0) = 0, \varphi'(0) \neq 0, h'(0) \neq 0 \) and suppose for all complex numbers \( \sigma, \tau \) satisfying \( |\sigma| = |\tau| = 1 \), there holds
\[
\varphi(z) \ast \frac{1 + \tau \sigma z}{1 - \sigma z} h(z) \neq 0 \ (0 < |z| < 1).
\]

Let \( F(z) \) be analytic in \( U \) and satisfy \( \Re F(z) > 0 \) \((0 < |z| < 1)\). Then
\[
\Re \left\{ \frac{\varphi(z) \ast (F(z)h(z))}{\varphi(z) \ast h(z)} \right\} > 0, \ 0 < |z| < 1.
\]

**Theorem 5.1** Let \( \sigma, \tau \) satisfy \( |\sigma| = |\tau| = 1, \alpha \geq 0, f(z) \in Q_k,\lambda(\alpha, \beta, \rho), \varphi(z) = z + \sum_{n=k}^\infty a_{k+n}z^{k+1} \) be analytic in \( U \) and
\[
\varphi(z) \ast \frac{1 + \tau \sigma z}{1 - \sigma z} = 0, \ 0 < |z| < 1.
\]

Then
\[
f(z) \ast \varphi(z) \in Q_k,\lambda(\alpha, \beta, \rho).
\]

**Proof** (i) Firstly, we prove that \( g \ast \varphi(z) \in V_k,\lambda(\alpha, \beta) \). Let
\[
F(z) = (1 - \alpha) \frac{D^\lambda g(z)}{z} + \alpha(D^\lambda g(z))' - \beta, \ h(z) = z.
\]

Then \( F(z) \) is analytic in \( U \) and \( \Re F(z) > 0 \) and \( \varphi \ast h(z) = z \). Since
\[
\varphi(z) \ast (F(z)h(z)) = \varphi \ast [(1 - \alpha)D^\lambda g(z) + \alpha z(D^\lambda g(z))'] - \beta z = (1 - \alpha)\varphi \ast D^\lambda g(z) + \alpha \varphi \ast z(D^\lambda g(z))' - \beta z = (1 - \alpha)D^\lambda (\varphi \ast g)(z) + \alpha z(D^\lambda (\varphi \ast g))'(z) - \beta z,
\]
by Lemma 5.1, we get
\[
\Re \left\{ \frac{\varphi(z) \ast (F(z)h(z))}{\varphi(z) \ast h(z)} \right\} = \Re \left\{ (1 - \alpha) \frac{D^\lambda (\varphi \ast g)(z)}{z} + \alpha(D^\lambda (\varphi \ast g))'(z) \right\} - \beta > 0.
\]
that is,
\[
\Re\left\{ (1 - \alpha) \frac{D^\lambda (\varphi \ast g)(z)}{z} + \alpha (D^\lambda (\varphi \ast g))'(z) \right\} > \beta, \quad z \in U.
\]
So \( g \ast \varphi(z) \in V_{k,\lambda}(\alpha, \beta) \).

(ii) Next we prove that \( f \ast \varphi(z) \in Q_{k,\lambda}(\alpha, \beta, \rho) \). Let \( f(z) \in Q_{k,\lambda}(\alpha, \beta, \rho) \), \( p(z) = \frac{z(D^\lambda f(z))'}{D^\lambda g(z)} - \rho \) and \( h(z) = z \). Then \( p(z) \) is analytic in \( U \) and \( \Re p(z) > 0 \) \( (z \in U) \) and \( \varphi \ast h(z) = z \). Since
\[
\varphi \ast D^\lambda g(z) \cdot p(z) = \varphi \ast z(D^\lambda f(z))' - \rho \varphi \ast D^\lambda g(z),
\]
noticing that
\[
\varphi \ast D^\lambda g(z) = D^\lambda (\varphi \ast g)(z); \quad \varphi \ast z(D^\lambda f(z))' = z(D^\lambda (\varphi \ast f))'(z),
\]
by (5.3), we get
\[
\Re p(z) = \Re\left\{ \frac{z(D^\lambda (\varphi \ast f))'(z)}{D^\lambda (\varphi \ast g)(z)} \right\} > \rho.
\]
From (i), \( \varphi \ast g(z) \in V_{k,\lambda}(\alpha, \beta) \). So \( f \ast \varphi(z) \in Q_{k,\lambda}(\alpha, \beta, \rho) \). This completes the proof of Theorem 5.1.

Remark 5.1 Setting \( \lambda = 0, \alpha = 0, \beta = \frac{1}{2} \) and \( \rho = 0 \) in Theorem 5.1, respectively, we get the corresponding product properties of functions in \( Q_{1,0}(0, \frac{1}{2}, 0) \).

References