α-Resolvable Cycle Systems for Cycle Length 4

MA Xiu Wen¹, TIAN Zi Hong²
(1. State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China; 2. College of Mathematics and Information Science, Hebei Normal University, Hebei 050016, China) (E-mail: tianzh68@163.com)

Abstract An m-cycle system of order v and index λ, denoted by m-CS(v, λ), is a collection of cycles of length m whose edges partition the edges of λK_v. An m-CS(v, λ) is α-resolvable if its cycles can be partitioned into classes such that each point of the design occurs in precisely α cycles in each class. The necessary conditions for the existence of such a design are m|λ(v−1), m|α, α|λ(v−1)/2. It is shown in this paper that these conditions are also sufficient when m = 4.

Keywords cycle; cycle system; α-resolvable.

Document code A
MR(2000) Subject Classification 05B07
Chinese Library Classification O157.2

1. Introduction

Let m, v, λ be positive integers, X a v-set. An edge of X is an unordered pair \{x, y\} where x, y are distinct vertices of X. A complete multigraph of order v and index λ, denoted by λK_v, is a graph on X in which each pair of vertices x, y is joined by exactly λ edges \{x, y\}. A cycle of length m is a sequence of m distinct vertices u_1, u_2, ..., u_m, denoted by (u_1, u_2, ..., u_m), and its edge set is \{\{u_i, u_{i+1}\} : i = 1, 2, ..., m − 1\} ∪ \{\{u_1, u_m\}\}. If the edges of a λK_v can be decomposed into cycles of length m, then these cycles are called an m-cycle system, and denoted by m-CS(v, λ). An m-CS(v, λ) is said to be α-resolvable if its cycles can be partitioned into classes (called α-resolution classes) such that each point of the design occurs in precisely α cycles in each class. A 1-resolvable m-CS(v, λ) is simply called resolvable m-CS(v, λ). The existence of a resolvable m-CS(v, λ) had been solved completely.

Lemma 1.1 Let λ, d, m be positive integers with m ≥ 3. Then λK_dm has a resolvable m-CS(dm, λ) if and only if λ(dm−1) is even and except the following cases:

(1) λ ≡ 2 (mod 4), d = 2, m = 3;
(2) λ odd, d = 2, m = 3;
(3) λ = 1, d = 4, m = 3.

When m = 3, the existence of an α-resolvable 3-CS(v, λ) had also been solved.

Received date: 2007-12-05; Accepted date: 2008-10-07
Foundation item: the National Natural Science Foundation of China (No. 10971051).
Lemma 1.2\textsuperscript{[2]} An $\alpha$-resolvable 3-CS($v, \lambda$) exists if and only if
\[ \lambda(v - 1) \equiv 0 \pmod{2}, \quad \lambda v(v - 1) \equiv 0 \pmod{6}, \quad 3|\alpha v, \alpha \frac{\lambda(v - 1)}{2}, \]
and $(v, \alpha, \lambda) \notin \{(6, 1, 4i + 2) : i \geq 0\}$.

The purpose of this paper is to investigate the existence of $\alpha$-resolvable 4-CS($v, \lambda$)s. The necessary conditions for the existence of such a design are:
\[ 4|\alpha v(v - 1), 2|\lambda(v - 1), 4|\alpha v, \alpha \frac{\lambda(v - 1)}{2}. \]  

From condition (\textasteriskcentered), we can derive minimum values for $\alpha$ and $\lambda$, and call them $\alpha_0$ and $\lambda_0$. Similarly to the Lemmas 2.1–2.3 in \cite{3}, we have the following lemmas.

Lemma 1.3 If an $\alpha$-resolvable 4-CS($v, \lambda$) exists, then $\alpha_0|\alpha, \lambda_0|\lambda$.

Lemma 1.4 If an $\alpha$-resolvable 4-CS($v, \lambda$) exists, then a $\tau_0$-resolvable 4-CS($v, n\lambda$) exists for any positive integers $n, t$ with $t\frac{4\lambda(v - 1)}{2\alpha}$.

Lemma 1.5 If an $\alpha_0$-resolvable 4-CS($v, \lambda_0$) exists, and $\alpha, \lambda$ satisfy condition (\textasteriskcentered), then an $\alpha$-resolvable 4-CS($v, \lambda$) exists.

Thus, in order to show the necessary condition (\textasteriskcentered) for the existence of $\alpha$-resolvable 4-CS($v, \lambda$) is also sufficient, we only need to prove the existence of $\alpha_0$-resolvable 4-CS($v, \lambda_0$)s.

2. Direct constructions

In order to get the existence of $\alpha$-resolvable 4-CS($v, \lambda$)s, we need some definitions and marks. Let $m, v$ be positive integers and $\infty$ an infinite point. Let $Z_v$ be the residue ring of integers modulo $v$. Denote $Z_v^* = Z_v \setminus \{0\}$. Let $C$ be a set of cycles of length $m$ which are constructed on $Z_v$ or $Z_v \cup \{\infty\}$. For each cycle $C = (c_1, c_2, \ldots, c_m)$ and $j \in Z_v$, define $C + j$ to be $(c_1 + j, c_2 + j, \ldots, c_m + j)$ where $\infty + j = \infty$ if $\infty \in C$. Denote $C + j = \{C + j : C \in C\}$ for $j \in Z_v$. The differences of a cycle $C = (c_1, c_2, \ldots, c_m)$ mean $\pm(c_2 - c_1), \pm(c_3 - c_2), \ldots, \pm(c_m - c_{m-1}), \pm(c_1 - c_m)$, where $\infty - j = j - \infty = \infty$ for any $j \in Z_v$.

In what follows, we will get $\alpha$-resolvable 4-CS($v, \lambda$)s through direct constructions. According to condition (\textasteriskcentered), $\alpha_0$ and $\lambda_0$ are as follows
\[
\begin{cases}
\alpha_0 = 1, \; \lambda_0 = 2, \; v \equiv 0 \pmod{4}, \\
\alpha_0 = 4, \; \lambda_0 = 1, \; v \equiv 1 \pmod{8}, \\
\alpha_0 = 4, \; \lambda_0 = 2, \; v \equiv 5 \pmod{8}, \\
\alpha_0 = 2, \; \lambda_0 = 4, \; v \equiv 2 \pmod{4}, \\
\alpha_0 = 4, \; \lambda_0 = 4, \; v \equiv 3 \pmod{4}.
\end{cases}
\]

Lemma 2.1 There exists a resolvable 4-CS($v, 2$) for $v \equiv 0 \pmod{4}$.

Proof Since $\alpha_0 = 1$, the conclusion follows from Lemma 1.1. \hfill \Box

Lemma 2.2 There exists a 4-resolvable 4-CS($v, 1$) for $v \equiv 1 \pmod{8}$.
**Proof** Let the point set \( X = Z_{8k+1} \), \( k > 0 \). A 4-resolvable 4-CS\((v, 1)\) contains \( \frac{\lambda(v-1)}{2m} = (8k + 1) \times k \) cycles and \( \frac{\Lambda(v-1)}{2m} = k \) 4-resolution classes. Let \( C \) consist of the following \( k \) cycles:

\[
\begin{align*}
(1, 0, 2k, 4k + 1), \\
(2, 0, 2k - 1, 4k + 1), \\
(3, 0, 2k - 2, 4k + 1), \\
\vdots & \vdots \vdots \\
(k - 1, 0, k + 2, 4k + 1), \\
(k, 0, k + 1, 4k + 1).
\end{align*}
\]

It is easy to check that the differences of all cycles of \( C \) give every value of \( \mathbb{Z}^{*}_{8k+1} \) exactly once, which implies that \( \{C + i : i \in Z_{8k+1}\} \) forms a 4-CS\((v, 1)\). In addition, for every \( C \in C \), \( \{C + i : i \in Z_{8k+1}\} \) is a 4-resolution class of the 4-CS\((v, 1)\). So, we derive a 4-resolvable 4-CS\((v, 1)\).

**Lemma 2.3** There exists a 4-resolvable 4-CS\((v, 2)\) for \( v \equiv 5 \pmod{8} \).

**Proof** Let the point set \( X = Z_{8k+5} \), \( k \geq 0 \). A 4-resolvable 4-CS\((v, 2)\) contains \((8k + 5) \times (2k + 1)\) cycles and \( 2k + 1 \) 4-resolution classes. Let \( C \) consist of the following \( 2k + 1 \) cycles:

**Part 1:** Construct \( k \) cycles and repeat them twice:

\[
\begin{align*}
(1, 0, 2, 4k + 3), \\
(3, 0, 4, 4k + 3), \\
(5, 0, 6, 4k + 3), \\
\vdots & \vdots \vdots \\
(2k - 3, 0, 2k - 2, 4k + 3), \\
(2k - 1, 0, 2k, 4k + 3).
\end{align*}
\]

**Part 2:** Construct 1 cycle:

\((2k + 1, 0, 2k + 2, 4k + 3)\).

Since the differences of all cycles of \( C \) give every value of \( \mathbb{Z}^{*}_{8k+5} \) exactly twice, \( \{C + i : i \in Z_{8k+5}\} \) forms a 4-CS\((v, 2)\). On the other hand, for every \( C \in C \), \( \{C + i : i \in Z_{8k+5}\} \) is a 4-resolution class of the 4-CS\((v, 1)\). Hence, we get a 4-resolvable 4-CS\((v, 2)\).

**Lemma 2.4** There exists a 2-resolvable 4-CS\((v, 4)\) for \( v \equiv 2 \pmod{4} \).

**Proof** Let the point set \( X = Z_{4k+1} \cup \{\infty\}, k > 0 \). A 2-resolvable 4-CS\((v, 4)\) contains \((4k + 1)(2k + 1)\) cycles and \( 4k + 1 \) 2-resolution classes. Let \( C \) consist of the following \( 2k + 1 \) cycles:

**Part 1:** Construct \( k \) cycles:
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\[
\begin{align*}
&\ ( k, \ 3k - 1, \ 3k, \ k - 2 ), \\
&\ (k + 1, \ 3k - 2, \ 3k + 1, \ k - 3 ), \\
&\ (k + 2, \ 3k - 3, \ 3k + 2, \ k - 4 ), \\
&\ \vdots \quad \vdots \quad \vdots \\
&\ (2k - 2, \ 2k + 1, \ 4k - 2, \ 0 ), \\
&\ (2k - 1, \ 2k, \ 4k - 1, \ 4k ). \\
\end{align*}
\]

Part 2: Construct \(k - 1\) cycles:

\[
\begin{align*}
&\ ( 1, \ 2k - 1, \ 2k + 1, \ 4k - 1), \\
&\ ( 2, \ 2k - 2, \ 2k + 2, \ 4k - 2), \\
&\ ( 3, \ 2k - 3, \ 2k + 3, \ 4k - 3), \\
&\ \vdots \quad \vdots \quad \vdots \\
&\ (k - 2, \ k + 2, \ 3k - 2, \ 3k + 2), \\
&\ (k - 1, \ k + 1, \ 3k - 1, \ 3k + 1). \\
\end{align*}
\]

Part 3: Construct 2 cycles:

\[
\begin{align*}
&\ (\infty, \ 0, \ 2k, \ 4k), \ (\infty, \ 3k, \ k, \ k - 1). \\
\end{align*}
\]

The differences of all cycles of \(\mathcal{C}\) give every value of \(Z_{4k+1}^* \cup \{\infty\}\) exactly biquadratic, so \(\{\mathcal{C} + i : i \in Z_{4k+1}\}\) forms a 4-CS\((v, 4)\). Furthermore, \(\mathcal{C}\) is a 2-resolution class of the 4-CS\((v, 4)\), and \(\mathcal{C}, \mathcal{C} + 1, \ldots, \mathcal{C} + 4k\) are all 2-resolution classes. So, a 2-resolvable 4-CS\((v, 4)\) is given. \(\square\)

**Lemma 2.5** There exists a 4-resolvable 4-CS\((v, 4)\) for \(v \equiv 3 \pmod{4}\).

**Proof** \((1)\) \(v \equiv 3 \pmod{8}\). Let the point set \(X = Z_{8k + 3}, \ k > 0\). A 4-resolvable 4-CS\((v, 4)\) contains \((8k + 3)(4k + 1)\) cycles and \(4k + 1\) 4-resolution classes. Let \(\mathcal{C}\) consist of the following \(4k + 1\) cycles:

Part 1: Construct 2\(k\) cycles:

\[
\begin{align*}
&\ ( 1, \ 0, \ 2, \ 5 ), \\
&\ ( 5, \ 0, \ 6, \ 13 ), \\
&\ ( 9, \ 0, \ 10, \ 21 ), \\
&\ \vdots \quad \vdots \quad \vdots \\
&\ (8k - 7, \ 0, \ 8k - 6, \ 8k - 14 ), \\
&\ (8k - 3, \ 0, \ 8k - 2, \ 8k - 6 ). \\
\end{align*}
\]

Part 2: Construct 2\(k\) cycles:

\[
\begin{align*}
&\ ( 3, \ 0, \ 4, \ 9 ), \\
&\ ( 7, \ 0, \ 8, \ 17 ). \\
\end{align*}
\]
(11, 0, 12, 25),
\vdots \vdots \vdots 
(8k - 5, 0, 8k - 4, 8k - 10),
(8k - 1, 0, 8k, 8k - 2).

Part 3: Construct 1 cycle:
(1, 0, 2, 3).

It is easy to check that \( \{C + i : i \in Z_{8k+3}\} \) forms a 4-CS\((v, 4)\). In addition, for every \( C \in \mathcal{C} \),
\( \{C + i : i \in Z_{8k+3}\} \) is a 4-resolution class of the 4-CS\((v, 4)\). So we derive a 4-resolvable 4-CS\((v, 4)\).

(2) \( v \equiv 7 \pmod{8} \). Let the point set \( X = Z_{8k+7}, k \geq 0 \). A 4-resolvable 4-CS\((v, 4)\) contains \((8k + 7)(4k + 3)\) cycles and \(4k + 3\) 4-resolution classes. Let \( \mathcal{C} \) consist of the following \(4k + 3\) cycles:

Part 1: Construct \( k \) cycles and repeat them biquadratic:
(1, 0, 2, 4k + 4),
(3, 0, 4, 4k + 4),
(5, 0, 6, 4k + 4),
\vdots \vdots \vdots 
(2k - 3, 0, 2k - 2, 4k + 4),
(2k - 1, 0, 2k, 4k + 4).

Part 2: Construct 1 cycle and repeat them twice:
(2k + 1, 0, 2k + 2, 4k + 4).

Part 3: Construct 1 cycle:
(2k + 1, 0, 2k + 3, 4k + 4).

It is easy to check that \( \{C + i : i \in Z_{8k+7}\} \) forms a 4-CS\((v, 4)\). In addition, for every \( C \in \mathcal{C} \),
\( \{C + i : i \in Z_{8k+7}\} \) is a 4-resolution class of the 4-CS\((v, 4)\). Therefore, we get a 4-resolvable 4-CS\((v, 4)\).

\[ \Box \]

3. Main result

Combining Lemmas 2.1–2.5, we obtain the main result:

**Theorem 3.1** There exists an \( \alpha \)-resolvable 4-CS\((v, \lambda)\) if and only if

\[ 4\lambda v(v - 1) = 2|\lambda(v - 1), 4|\alpha v, \alpha|\lambda(v - 1). \]

References

