General Induced Matching Extendability of $G^3$

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Abstract A graph $G$ is induced matching extendable if every induced matching of $G$ is included in a perfect matching of $G$. A graph $G$ is generalized induced matching extendable if every induced matching of $G$ is included in a maximum matching of $G$. A graph $G$ is claw-free, if $G$ does not contain any induced subgraph isomorphic to $K_{1,3}$. The $k$-th power of $G$, denoted by $G^k$, is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if the distance between them is at most $k$ in $G$. In this paper we show that, if the maximum matchings of $G$ and $G^3$ have the same cardinality, then $G^3$ is generalized induced matching extendable. We also show that this result is best possible. As a result, we show that if $G$ is a connected claw-free graph, then $G^3$ is generalized induced matching extendable.

Keywords near perfect matching; induced matching extendable; general induced matching extendability; power of graph.

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1. Introduction

The graphs considered in this paper are finite and simple. For a graph $G$, $V(G)$ and $E(G)$ denote, respectively, its vertex set and its edge set. For two vertex subsets $X$ and $Y$ in $G$, the distance between them, denoted by $d_G(X, Y)$, is the minimum length of a path connecting $X$ and $Y$. $d_G(\{x\}, \{y\})$ is written in shorter form as $d_G(x, y)$ for $x, y \in V(G)$. A component $H$ of $G$ is odd (even) if $|V(H)|$ is odd (even). The component number of $G$ is denoted by $c(G)$ and the odd component number of $G$ is denoted by $o(G)$. A graph $G$ is claw-free, if $G$ does not contain any induced subgraph isomorphic to $K_{1,3}$. The $k$-th power of $G$, denoted by $G^k$, is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they have distance at most $k$ in $G$. For two graphs $G$ and $H$, $G \cup H$ is used to denote the union of them. The join $G \vee H$ of disjoint graphs $G$ and $H$ is the graph obtained from the disjoint union $G \cup H$ by joining each vertex of $G$ to each vertex of $H$. For $X \subseteq V(G)$, the neighbor set $N_G(X)$ of $X$ is defined by

$$N_G(X) = \{y \in V(G) \setminus X : \text{there is } x \in X \text{ such that } xy \in E(G)\}.$$
If $I \subseteq V(G)$ such that $E(I) = \emptyset$, $I$ is called an independent set of $G$. For $M \subseteq E(G)$, set

$$E(S) = \{uv \in E(G) : u, v \in S\}.$$ 

$V(M) = \{v \in V(G) :$ there is $x \in V(G)$ such that $vx \in M\}$. $V(\{e\})$ is written in shorter form as $V(e)$ for $e \in E(G)$. $M \subseteq E(G)$ is a matching of $G$, if $V(e) \cap V(f) = \emptyset$ for every two distinct edges $e, f \in M$. A matching $M$ of $G$ is a maximum matching, if $|M| \geq |M'|$ for every matching $M'$ of $G$. A matching $M$ of $G$ is a perfect matching, if $V(M) = V(G)$. A matching $M$ of $G$ is a near perfect matching, if $|V(M)| = |V(G)| - 1$. A matching $M$ of $G$ is induced $[1, 2]$, if $E(V(M)) = M$. A graph $G$ is induced matching extendable [3] (shortly, IM-extendable), if every induced matching of $G$ is included in a perfect matching of $G$. A graph $G$ is strongly IM-extendable, if every spanning supergraph of $G$ is IM-extendable. A graph $G$ is nearly induced matching extendable (shortly, nearly IM-extendable), if $G \cup K_1$ is induced matching extendable. A graph $G$ is strongly nearly IM-extendable, if every spanning supergraph of $G$ is nearly IM-extendable. A graph $G$ is generalized induced matching extendable (shortly, generalized IM-extendable), if every induced matching of $G$ is included in a maximum matching of $G$. A graph $G$ is strongly generalized IM-extendable, if every spanning supergraph of $G$ is generalized IM-extendable. The following is the famous Tutte’s Theorem.

**Tutte’s Theorem** ([4, 5]) A graph $G$ has a perfect matching if and only if for every $S \subseteq V(G)$, $o(G - S) \leq |S|$.

Yuan proved in [6] that, for a connected graph $G$ with $|V(G)|$ even, $G^4$ is strongly IM-extendable. Qian proved in [7] that, for a 2-connected graph $G$ with $|V(G)|$ even, $G^3$ is strongly IM-extendable, and for a locally connected graph $G$ with $|V(G)|$ even, $G^2$ is strongly IM-extendable. It was shown in [8] that, for a connected graph $G$ with a perfect matching, $G^3$ is IM-extendable. In [9], it was shown that, if $G$ is a graph with $|V(G)|$ even and without independent vertex cut, then $G^2$ is strongly IM-extendable. These results solved three conjectures posed in [10].

We further study the IM-extendability of the 3-power of graphs. We show in this paper that, if the maximum matchings of $G$ and $G^3$ have the same cardinality, then $G^3$ is generalized induced matching extendable. We also show that this result is best possible. As a result, we show that if $G$ is a connected graph and has a near perfect matching, then $G^3$ is nearly IM-extendable. We also show that if $G$ is connected and claw-free, then $G^3$ is generalized induced matching extendable.

**2. Main results and proof**

The following Lemma was shown in [8].

**Lemma 1** ([8]) Suppose that $G$ is a connected graph with $|V(G)| \geq 3$. If $I \subseteq V(G)$ such that $|I| \leq 2$ and $|I|$ has same parity as $|V(G)|$, then $G^3 - I$ has a perfect matching.
Corollary 1 Suppose that $G$ is a connected graph with $|V(G)| \geq 3$. If $I \subset V(G)$ such that $1 \leq |I| \leq 2$ and $|I|$ has opposing parity as $|V(G)|$, then $G^3 - I$ has a near perfect matching.

Proof Arbitrarily select a vertex $u \in I$. Then $|I \setminus \{u\}|$ has the same parity as $|V(G)|$. Write $I' = I \setminus \{u\}$. From Lemma 1, we know that $G^3 - I'$ has a perfect matching $M$. Suppose that $uv \in M$. Then $M \setminus \{uv\}$ is a near perfect matching of $G^3 - I$. The result follows. □

Corollary 2 Suppose that $G$ is a connected graph with $|V(G)| \geq 3$ and $|V(G)|$ odd. Then $G^3$ has a near perfect matching.

Proof For an edge $uv \in E(G)$, from Corollary 1, we know that $G^3 - \{u, v\}$ has a near perfect matching $M$. Then $M \cup \{uv\}$ is a near perfect matching of $G^3$. The result follows. □

Theorem 1 If the maximum matchings of $G$ and $G^3$ have the same cardinality, then $G^3$ is generalized induced matching extendable.

Proof Let $M$ be a maximum matching of $G$. It is clear that $G - V(M)$ is an independent set of $G$. Let $N$ be an induced matching of $G^3$. For $e = xy \in N$, let $P_e$ be a shortest $(x, y)$-path in $G$. Set

$$E_e = E(P_e) \cup \{f \in M : \text{there is } u \in V(P_e) \text{ such that } f \text{ is incident to } u \text{ in } M\}.$$ 

Let $H_e$ be the edge induced subgraph of $G$ induced by $E_e$. Then $H_e$ is connected. From the fact that $d_G(x, y) \leq 3$, it is easy to see that for every edge $zw \in M \cap E(H_e)$,

$$d_G({x, y}, \{z, w\}) \leq 1.$$

Suppose that $e = xy$ and $f = uv$ are two distinct edges in $N$ such that $V(H_e) \cap V(H_f) \neq \emptyset$. Let $z$ be a vertex in $V(H_e) \cap V(H_f)$. Let $zw$ be the edge such that $zw \in M$. By the definition of $H_e$ and $H_f$, we know that $zw \in E(H_e) \cap E(H_f)$. Because $d_G({x, y}, \{z, w\}) \leq 1$ and $d_G({u, v}, \{z, w\}) \leq 1$, we must have $d_G({x, y}, \{u, v\}) \leq 3$. This contradicts the fact that $N$ is an induced matching in $G^3$. So we must have for every two distinct edges $e = xy$ and $f = uv$ in $N$, $V(H_e) \cap V(H_f) = \emptyset$.

Now we distinguish the following two cases.

Case 1 $V(H_e) \subseteq V(M)$. In this case, $E_e \cap M$ is a perfect matching of $H_e$. Then $|V(H_e)|$ must be even. By Lemma 1, $(H_e)^3 - \{x, y\}$ has a perfect matching. Now for each edge $e = xy \in N$ with $|V(H_e)|$ even, $V(H_e) \subseteq V(M)$ and $|V(H_e)| \geq 4$, let $M_e$ be a perfect matching in $(H_e)^3 - \{x, y\}$. For $e = xy \in N$ with $|V(H_e)| = 2$, we know that $V(H_e) = \{x, y\} \subseteq V(M)$ and we define $M_e = \emptyset$.

Case 2 $H_e$ contains a vertex $u \in G - V(M)$. Then $|V(H_e)|$ must be odd. Note that $H_e$ can only contain one such vertex $u$. Otherwise, suppose there is a vertex $v \in G - V(M)$, $v \neq u$ and $v \in V(H_e)$. We must have $u, v \in V(P_e)$ and $uv \in E(G^3)$, a contradiction to the fact that the maximum matchings of $G$ and $G^3$ have the same cardinality. We have the following two subcases.
Case 2.1  $u \in V(P_e) \setminus \{x, y\}$. Then by Corollary 1, $(H_e)^3 - \{x, y\}$ has a near perfect matching. Let $M_e$ be a near perfect matching in $(H_e)^3 - \{x, y\}$.

Case 2.2  $u = x$ or $u = y$. Without loss of generality, suppose that $u = x$. Then $(H_e)^3 - u$ has a perfect matching. Let $M_e'$ be a perfect matching in $(H_e)^3 - u$ and suppose that $e' = yt \in M_e'$. Let $M_e = M_e' - e'$.

It can be seen that $(M \setminus (\cup_{e \in N} E(H_e))) \cup (\cup_{e \in N} M_e) \cup N$ is a maximum matching in $G^3$.

This completes the proof. □

From Theorem 1, we can easily have

**Theorem 2**  If $G$ is a connected graph and has a near perfect matching, then $G^3$ is nearly IM-extendable.

**Lemma 2** ([11])  If $G$ is a connected claw-free graph with even number of vertices, then $G$ has a perfect matching.

**Theorem 3**  If $G$ is a connected claw-free graph, then $G^3$ is generalized induced matching extendable.

**Proof**  We distinguish the following two cases.

**Case 1**  $V(G)$ is even. By Lemma 2, $G$ has a perfect matching and so $G^3$ also has a perfect matching. By Theorem 1, $G^3$ is generalized induced matching extendable.

**Case 2**  $V(G)$ is odd. We have the following two subcases.

**Case 2.1**  $G$ is 2-connected. Then $G$ has no cut vertex. For any vertex $u \in V(G)$, $G - u$ is connected and $|V(G - u)|$ is even. It is easy to see that $G - u$ is also claw-free. By Lemma 2, $G - u$ has a perfect matching, and so $G$ has a near perfect matching. By Theorem 2, $G^3$ is nearly IM-extendable.

**Case 2.2**  $G$ has a cut vertex $v$. From the fact that $G$ is claw-free, we know that $G - v$ has exactly two components $G_1$ and $G_2$. If both $|V(G_1)|$ and $|V(G_2)|$ are even, from Lemma 2, $G_1$ and $G_2$ have perfect matchings $M_1$ and $M_2$, respectively. So $M_1 \cup M_2$ is a near perfect matching of $G$. By Theorem 2, $G^3$ is nearly IM-extendable. If both $|V(G_1)|$ and $|V(G_2)|$ are odd, then $G_2 + v$ has a perfect matching $N_2$. Repeat the above analysis on $G_1$, we will finally deduce that $G_1$ has a near perfect matching $N_1$. So $N_1 \cup N_2$ is a near perfect matching of $G$. By Theorem 2, $G^3$ is nearly IM-extendable.

This completes the proof. □

### 3. Examples

Our result in Theorem 2 is best possible in three aspects. Firstly, there is a $k$-connected $(k \geq 2)$ graph $G$ having a near perfect matching such that $G^2$ is not nearly IM-extendable. This
will be shown in Example 1. Secondly, there is a connected graph $H$ with $|V(H)|$ odd such that $H^3$ is not nearly IM-extendable. This will be shown in Example 2. Thirdly, there is a connected graph $D$ having a near perfect matching such that $D^3$ is not strongly nearly IM-extendable. This will be shown in Example 3. Note that the nearly IM-extendable graph is a special case of the generalized IM-extendable graph, we can also use these three Examples to explain the best possibility of the result in Theorem 1.

**Example 1** Let $k \geq 2$ be an integer. Let $G_1$, $G_2$, $G_3$ and $G_4$ be four complete graphs with $|V(G_1)| = |V(G_2)| - 1 = |V(G_3)| = |V(G_4)|$ and such that $|V(G_1)| \geq k^2$ and $|V(G_1)|$ is odd. Let $(H_1, U_2, \ldots, U_k)$ be a $k$-partition of $V(G_1)$, $(R_1, R_2, \ldots, R_k)$ be a $k$-partition of $V(G_2)$, $(S_1, S_2, \ldots, S_k)$ be a $k$-partition of $V(G_3)$ and $(S_1, S_2, \ldots, S_k)$ be a $k$-partition of $V(G_4)$ such that $|V_i|, |U_i|, |R_i|, |S_i| \geq k$ for $1 \leq i \leq k$. Let $M$ be the set of $2k$ edges with $M = \{v_i u_i, r_i s_i : 1 \leq i \leq k\}$ and let $M_1$ be the set of $k$ edges with $M_1 = \{u_2 r_1, \ldots, u_k r_k\}$, where $v_i, u_i, u_2, r_1, r_2, s_i \notin V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4)$ for $1 \leq i \leq k$. The graph $G$ is constructed as follows.

$$V(G) = V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4) \cup V(M) \cup V(M_1),$$

$$E(G) = E(G_1) \cup E(G_2) \cup E(G_3) \cup E(G_4) \cup E(M) \cup M_1 \cup \{v_i u_i, v_i u_j : v_i \in V_i\} \cup \{u_i u_j : u_i \in U_i\} \cup \{u_i u_j : u_i \in U_i\} \cup \{r_i r_j : r_i \in R_i\} \cup \{s_i s_j : s_i \in S_i\}.$$ 

$M$ is an induced matching of $G$. It is easy to check that $G$ is a $k$-connected graph and has a near perfect matching. Now, $M$ is still an induced matching in $G^2$. But $G^2 - V(M)$ has three odd components. Hence, $G^2$ is not nearly IM-extendable.

**Example 2** Let $P = x_1 x_2 x_3 x_4 x_5$, $Q = y_1 y_2 y_3 y_4 y_5$, $R = z_1 z_2 z_3 z_4 z_5$ and $S = w_1 w_2 w_3 w_4 w_5$ be four 5-pathes. Let $v$ be a vertex which is different from $x_i, y_i, z_i, w_i$, $1 \leq i \leq 5$. The graph $H$ is constructed as follows.

$$V(H) = \{v\} \cup V(P) \cup V(Q) \cup V(R) \cup V(S),$$

$$E(H) = E(P) \cup E(Q) \cup E(R) \cup E(S) \cup \{vx_3, vy_3, vz_3, vw_3\}.$$ 

Let $M = \{x_2 x_4, y_2 y_4, z_2 z_4, w_2 w_4\}$. It is easy to see that $M$ is an induced matching of $H^3$. For a vertex $u \notin V(H)$, $\{x_1, x_5, y_1, y_5, z_1, z_5, w_1, w_5\}$ is an independent set in $H^3$ and $H^3 \cup u$. This means that $H^3 \cup u - V(M)$ has eight odd components. By Tutte’s Theorem, $H^3 \cup u - V(M)$ has no perfect matching. Hence, $H^3$ is not nearly IM-extendable.

**Example 3** Let $P = x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$, $Q = y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8$, $R = z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8$ and $S = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8$ be four 8-pathes. Let $v$ be a vertex which is different from $x_i, y_i, z_i, w_i$, $1 \leq i \leq 8$. The graph $D$ is constructed as follows.

$$V(D) = \{v\} \cup V(P) \cup V(Q) \cup V(R) \cup V(S),$$

$$E(D) = E(P) \cup E(Q) \cup E(R) \cup E(S) \cup \{vx_3, vy_3, vz_3, vw_3\}.$$
Let \( M = \{x_2x_4, y_2y_4, z_2z_4, w_2w_4, x_8y_8, z_8w_8\} \). For a vertex \( u \notin V(D) \), it is easy to see that \( M \) is an induced matching of \( D^3 + x_8y_8 + z_8w_8 \) and \( (D^3 + x_8y_8 + z_8w_8) \lor u \). But \( (D^3 + x_8y_8 + z_8w_8) \lor u - V(M) - \{u, v, x_3, y_3, z_3, w_3\} \) has eight odd components. By Tutte’s Theorem, \( (D^3 + x_8y_8 + z_8w_8) \lor u - V(M) \) has no perfect matching. Hence \( D^3 + x_8y_8 + z_8w_8 \) is not nearly IM-extendable, and so, \( D^3 \) is not strongly nearly IM-extendable.

References